L^p bounds for orthogonal polynomials and applications

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Continuity of Weighted Operators, Muckenhoupt A_p Weights, and Steklov Problem for Orthogonal Polynomials

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We consider weighted operators acting on $L^p(\mathbb{R}^d)$ and show that they depend continuously on the weight $w \in A_p(\mathbb{R}^d)$ in the operator topology. Then, we use this result to estimate $L^p_w(\mathbb{T})$ norm of polynomials orthogonal on the unit circle when the weight w belongs to Muckenhoupt class $A_2(\mathbb{T})$ and $p > 2$. The asymptotics of the polynomial entropy is obtained as an application.

To Peter Yuditskii on the occasion of his 65th birthday.

1 Introduction

Suppose μ is a probability measure on the unit circle $\mathbb{T}$ and $\{\varphi_n(z, \mu)\}$ is the sequence of

Outline

Section 1. Introduction

Aplication of L^p norms for orthogonal polynomials

Background, the Steklov problem and conjecture

Section 2. The general Steklov problem

W – weighted space, X – norm space

Section 3. The Steklov problem $X = L^p_w$

Section 4. Theorem: the Steklov problem $X = L^p_w$, $W = A_2$

Section 5. On the proof:

Continuity of singular operators

1. Application of L^p norms: information entropy

- $\surd$ Discrete probability distribution $\{q_j\}_{j=1}^n \; : \quad \sum_{j=1}^n q_j = 1, \; q_j > 0.$
- $\surd$ Shannon discrete entropy : $\mathbb{S}_n \; = \; -\sum_{j=1}^n \log(q_j) q_j.$
- $\surd$ Examples:

$$\{q_j\}_{j=1}^n \; := \left\{\frac{1}{n}, \ldots, \frac{1}{n}\right\} \qquad \Rightarrow \qquad \mathbb{S}_n = \log(n)$$

$$\{q_j\}_{j=1}^n \; := \{1, ..., 0, 1, 0, ..., 0\} \quad \Rightarrow \quad \mathbb{S}_n = 0$$

- $\surd$ In general: $\quad \log(n) \geq \mathbb{S}_n \geq 0 \quad$ (measure of uncertainness).
- $\surd$ Boltzmann entropy for continues measure $d\mu(r) := \rho(r) d^n r$:

$$\mathbb{S}_\rho := -\langle \log \rho \rangle := -\int \log \rho(r) \rho(r) \, d^n r$$

- $\surd$ Reny entropy: $\qquad R_{p,\rho} := -\int \rho^p(r) \rho(r) \, dr^n$

1. Aplication of L^p norms: uncertainty principles

$$\begin{cases} \rho(r) = |\Psi(r)|^2, \\ \tilde\rho(k) = |\tilde\Psi(k)|^2, \end{cases} \qquad \tilde\Psi(k) := \frac{1}{(2\pi)^{n/2}} \int \exp\{-i\,k\,r\}\Psi(r)\,dr^n$$

$$\mathbb{S}_\rho := -\langle \log\rho\rangle := -\int \log\rho(r)\rho(r)\,d^n r \qquad \Rightarrow \qquad \mathbb{S}_\rho + \mathbb{S}_{\tilde\rho} \geq \log(e\,\pi)\,(!)$$

Proof. Babenko-Beckner inequality $\downarrow$ $\quad (q \geq 2,\ \frac{1}{p} + \frac{1}{q} = 1)$

$$W(q) := K(p,q)\,\|\Psi\|_p - \|\tilde\Psi\|_q \geq 0, \quad K(p,q) := \left(\frac{2\pi}{q}\right)^{n/2q}\left(\frac{2\pi}{p}\right)^{-n/2p}$$

$$W(2) = 0, \qquad \frac{d\,W(q)}{d\,q}\Big|_{q=2+0} = \ldots \geq 0 \quad \Rightarrow \quad (!)\ \ q.e.d.$$

Uncertainty relations. 1) Solve $\mathbb{S}_\rho \uparrow \max:\ \{\langle 1\rangle, \langle (r-\langle r\rangle)^2\rangle\} = r_0\}$;

2) Solution : $e\pi\exp\{-\frac{2}{n}\mathbb{S}_\rho\} \geq \frac{n}{2}(\langle (r-\langle r\rangle)^2\rangle)^{-1}$;

3) Analogously: $\frac{2}{n}\langle (k-\langle k\rangle)^2\rangle \geq e\pi\exp\{\frac{2}{n}\mathbb{S}_{\tilde\rho}\}$

4) Combining with (!) $\Rightarrow$ Shannon sharper than Heisenberg !

1. Background (the Steklov problem)

For $\{P_n(x)\}_{n=0}^{\infty}$, orthonormal polynomials

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, \rho(x)\, dx = \delta_{n,m}\ , \quad n, m = 0, 1, 2\ \ldots$$

relative to positive weight ρ:

$$\rho(x) \geq \delta > 0\ , \quad x \in [-1, 1]\ ,$$

to find the bounds on $[-1, 1]$

V.A. Steklov —> S.N. Bernshtein, M.G. Krein, N.I. Akhiezer,
Ya.B. Geronimus, B.L. Golinskiy, P.K. Suetin

P.K. Suetin,
The Steklov problem in theory orthogonal polynomials,
Journal of Soviet Mathematics, 1979, 12(6), 631–682.

1. Background (the Steklov Conjecture)

$$(*) \quad \limsup_{n\to\infty} |P_n(x)| < \infty \qquad \Leftarrow \quad \rho \neq 0 \quad \text{on} \quad [-1,1].$$

In the paper:
Une method de la solution du problem de development des fonctions en series de polynomes de Tchebychef independante de la theorie de fermeture. II. Izvestia RAS, 1921, 15, 303–326,

V.A. Steklov wrote (adapted translation from French):
"I believe that inequality (*) is the common property of all polynomials whose orthogonality weight $\rho(x)$ does not vanish inside the given interval, but so far I haven't succeeded in finding either the rigorous proof to that statement or an example when this estimate does not hold at each interior point of the given interval....... ."

In 1979, E.A. Rakhmanov refuted this conjecture by constructing a weight from the Steklov class, for which

$$\limsup_{n\to\infty} |P_n(0)| = \infty \,.$$

1. Introduction (segment → circle)

√ Orthonormal polynomials on the circle (ONPUC):

$$\{\phi_n(z,\sigma)\},\ z = e^{i\theta}: \quad \int_0^{2\pi} \phi_n \overline{\phi}_m \, d\sigma(\theta) = \delta_{n,m}\,,\ n,m = 0,1,2\ldots$$

√ Steklov class:

$$S_\delta \ : \quad w := \sigma' \geqslant \delta/(2\pi)\,.$$

√ Steklov Conjecture:

$$\sigma \in S_\delta \qquad ?\ \sup_n \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} < \infty\,?$$

√ Rakhmanov's counterexample:

$$\limsup_{n\to\infty} \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} = \infty.$$

2. The general Steklov problem

$W := \{w(z) > 0 \text{ a.e.}\}, \quad z \in \mathbb{T}$ – classes and spaces of weights

Examples: $W = S_\delta,\ S,\ L^p,\ L^\infty,\ \text{BMO},\ A_p$.

X – norm space, $X = L^\infty,\ L^P_w$

The general Steklov problem:

$$w \in W \implies B := \sup_n \|\varphi_n(z, w)\|_X \leq \ ?$$

2. $W \to C^{0+},\ C,\ L^p; \qquad X = L^\infty$

1. Theorem [S.N. Bernstein–1930]. Let $\sigma \in C^{0+}$:

$$\frac{w(t) - w(x)}{|\ln|t-x||^{-\gamma}}, \quad \gamma > 1, \quad w := \sigma' .$$

Then $\sigma \in S_\delta \cap C^{0+} \quad \Rightarrow \quad \|\phi_n\|_{L^\infty(\mathbb{T})} \leq \text{Const}, \qquad \forall n \geqslant 0.$

2. Theorem [M. Ambroladze–1991]. Let $\sigma \in S_\delta \cap C$. Then

$$C \ln n \leqslant \sup_\sigma \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} \qquad \forall n \geqslant 0.$$

3. $\sigma \in S_\delta \cap L^p, \quad \delta \in (0,1),\ \ p \in [1,\infty).$

Theorem [A., Denisov, Tulyakov–2016]. $\exists\ C_j(p,\delta),\ j = 0,\ 1,\ 2$:

$$C_1(p,\delta)\sqrt{n} \leqslant \sup_{\|\sigma'\|_{L^p} \leqslant C} \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} \leqslant C_2(p,\delta)\sqrt{n}, \quad \forall C > C_0.$$

2. $W \to L^\infty$, BMO and A_p weights; $X = L^\infty$

The series of the papers, preprints and folkloric [2015–18]: S. Denisov, F. Nazarov, K. Rush, ...

1. Upper bounds:
Example [Denisov–Nazarov 2015]. $S_\delta \cap L^\infty$ for $1 \leqslant \sigma' \leqslant 1 + C\varepsilon$,

$$\sup_\sigma \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} \quad \leqslant \quad n^\varepsilon, \qquad \forall \varepsilon > 0.$$

and for $1/(2\pi) \leqslant \sigma' \leqslant T$:

$$\|\phi_n\|_{L^\infty(\mathbb{T})} \leqslant C(T)\, n^{1/2-\frac{C}{T}}, \quad T \gg 1,$$

2. Lower bounds: Upper bounds are exact.

3. The Steklov problem in $X = L^p_w$

Motivation:

1) $X = L^p_w \quad \Longrightarrow \quad \forall w$, we have $B_{L^2_w} \equiv \|\varphi_n(z,w)\|_{L^2_w(T)} = 1$,

in the same time $\quad \exists w \in W \ : \ B_{L^\infty_w} = \infty \quad p_{\mathrm{cr}} = ?$

2) $X = L^p \quad \underset{\text{Nikolskii inequality}}{\leftrightarrows} \quad X = L^\infty$

3) Entropy OP

$$\int_{\mathrm{T}} \varphi_n^2(z) \log |\varphi_n^2(z)| w(z)\,|dz| \leftarrow \int_{\mathrm{T}} |\varphi_n(z)|^p\, w(z)\,|dz|$$

Stated in:

A. I. A., V. S. Buyarov, and I. S. Degeza, Asymptotic behavior of L_p-norms and entropy for general orthogonal polynomials. Mat. Sb., 185(8):3–30, 1994.

Paul Nevai and Ying Guang Shi, Notes on Steklov's conjecture in L_p and on divergence of Lagrange interpolation in L_p. J. Approx. Theory, 90(1):147–152, 1997.

3. The Steklov problem in $X = L^p_w$, $W := L^{\tilde p}$

$W := L^{\tilde p} \cap S_\delta$, $\tilde p \in [1, \infty)$

Theorem [A., Denisov, Tulyakov–2016] + Nikolskii inequality

$$\Longrightarrow \quad \exists w \in W: \ \|\varphi_n(z, w)\|_{L^p_w} > C(p) n^{1/2-1/p}$$

i.e., $X = L^p_w$, $p \in (2, \infty]$,

$$\exists w \in W \quad : \quad B := \sup_n \|\varphi_n(z, w)\|_X = \infty$$

Thus $p_{\text{crit}} = 2$, so

weights from $W := L^{\tilde p} \cap S_\delta$ for $X = L^p_w$, $p \in (2, \infty]$ are not interesting!

3. The Steklov problem in $X = L^p_w$, $W \not\equiv L^{\tilde p}$, $\tilde p \in [1, \infty)$

$$p_{\text{cr}} := \sup\{p : B < \infty\} \in [2, \infty], \quad B := \sup_n \|\varphi_n(z, w)\|_X$$

1) $W = S \cap C^{0+}$, Bernstein $\Longrightarrow p_{\text{cr}} = \infty$

2) $W = (1-x)^\alpha (1+x)^\beta$, $\alpha \geq \beta$, Theorem [A., Buyarov, Degeza–1994]:

$$\Longrightarrow \qquad p_{\text{cr}} = \begin{cases} +\infty, & \alpha \in (-1, -1/2); \\ 2 + \frac{1}{\alpha + 1/2}, & \alpha > -1/2. \end{cases}$$

3) $W = L^\infty \cap S_\delta \ni w : \ 1 \leq w \leq 1 + \varepsilon$, Theorem [Denisov-Nazarov, 2016]:

$$\Longrightarrow p_{\text{cr}} = O\left(\frac{1}{\varepsilon}\right)$$

4) $W = \text{BMO}$, Theorem [Denisov-Rush, 2017]: $\qquad \langle f \rangle_J := \frac{1}{|T|} \int_J f\,|dz|$,

$$s := \|w\|_{\text{BMO}} := \sup_{J \subseteq \text{T}} \langle w - \langle w_J \rangle \rangle_J, \qquad t := \|w^{-1}\|_{\text{BMO}}$$

$$\Longrightarrow \quad \exists\, p_{\text{cr}}(s, t) > 2$$

4. The Steklov problem in $X = L^p_w$, $W := A_2(\mathrm{T}) \subset S$

$$[w]_{A_p} := \sup_{\mathrm{I}} \left(\langle f \rangle_{\mathrm{I}} \left(\langle f^{\frac{1}{1-p}} \rangle_{\mathrm{I}} \right)^{p-1} \right), \; w \in A_2 \Rightarrow w^{-1} \in L^1 \Rightarrow \ln w \in L^1$$

$$P_{\mathrm{cr}}(t) := \sup \left\{ p : \; B := \sup_n \|\varphi_n\|_X < \infty, \; [w]_{A_2} \le t \right\}$$

Theorem (Alexis, A., Denisov–2019)

$P_{\mathrm{cr}}(t)$ does not increase $t \in (1, \infty)$, $P_{\mathrm{cr}}(t) > 2$:

$$\lim P_{\mathrm{cr}}(t) = \begin{cases} +\infty, & t \to 1 \\ 2, & t \to \infty \end{cases}$$

4. Corollary (I) of Theorem 1

Definitions:

$$q_{\text{cr}}(w) = \sup\{q : \|w^{-1}\|_{L^q(T)} < \infty\}\,.$$

Szegö function ($|D|^2 = 2\pi w$ on T):

$$D(z) = \exp\left(\frac{1}{2\pi}\int_T \frac{1+\bar{\xi}z}{1-\bar{\xi}z}\log\sqrt{2\pi w(\theta)}d\theta\right),\ \xi = e^{i\theta}\,, |z| < 1\,.$$

Corollary. Let $[w]_{A_2} < \infty$. Then

$$\lim_{n\to\infty}\|\phi_n^* - D^{-1}\|_{L^p_w(T)} = 0$$

for any $p \in [2, \min\Big(p_{\text{cr}}([w]_{A^2}), 2(1+q_{\text{cr}}(w))\Big))$.

4. Corollary (II) of Theorem 1

Entropy OP:

$$E(n,\mu) = -\int_{\mathbb{T}} |\phi_n(\xi,\mu)|^2 \log|\phi_n(\xi,\mu)| d\mu\,,$$

where $\xi = e^{i\theta}$, $\theta \in [-\pi,\pi)$.

Corollary. Let $w \in A_2(T)$. Then

$$\lim_{n\to\infty} E(n,w) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log(2\pi w) d\theta\,.$$

Entropy OP and Szego class S

Entropy OP ($n \to \infty$):

$$E(n,\mu) = -\int_{\mathbb{T}} |\phi_n(\xi,\mu)|^2 \log|\phi_n(\xi,\mu)| d\mu \to \frac{1}{4\pi}\int_{-\pi}^{\pi} \log(2\pi w) d\theta\,.$$

$\surd$ S. Denisov, S. Kupin, On the growth of the polynomial entropy integrals for the measures in the Szego class, (2011)

$\surd$ S. B. Beckermann, A. Martinez-Finkelshtein, E.A. Rakhmanov and F. Wielonsky, Asymptotic upper bounds for the entropy of orthogonal polynomials in the Szego class, Journal of Mathematical Physics, 2004, 45(11), 4239 -4254.

$$\Delta_n(M) := \{x \in [-1,1] : p_n^2(x)\rho_0(x) \geq M\}.$$

$$\int_{\Delta_n(M)} \log^+(p_n^2(x))\ p_n^2(x)\rho(x)dx \quad \to \quad ?$$

6. On the proof

Proof. $1 \le w \le 1+\varepsilon$; $\Phi_n(z) = z^n + \sum_{j=0}^{n-1} z^j \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_n(e^{i\theta}) e^{-j\theta} \underbrace{(1-w)}_{\mu:=1-w} d\theta$,

$$\Phi_n(z) = z^n + \mathcal{P}_{n-1}(\mu \Phi_n)$$

We iterate N times (notation $\mathcal{P}^{(\mu)}_{n-1} f := \mathcal{P}_{n-1}(\mu f)$)

$$\Phi_n(z) = \sum_{k=0}^{N} \left(\mathcal{P}^{(\mu)}_{n-1}\right)^k (e^{in\theta}) + \left(\mathcal{P}^{(\mu)}_{n-1}\right)^{N+1} (\Phi_n)$$

Go to $\| \, \|$, sum up

$$\|\Phi_n\|_{L^p(\pi)} \le \frac{1 + (\varepsilon \|\mathcal{P}_{n-1}\|_{p,p})^N}{1 - (\varepsilon \|\mathcal{P}_{n-1}\|_{p,p})} + (\varepsilon \|\mathcal{P}_{n-1}\|_{p,p})^{N+1}$$

It is known,

$$\mathcal{P}_{n-1} = \mathcal{P}^+ - z^n \mathcal{P}^+ z^{-n} = \frac{1}{2}\left(H - z^{n+1} H z^n\right)$$

$$c(p) = \|H\|_{p,p} = \operatorname{ctg}\left(\frac{\pi}{2p}\right), \quad p \in [2, \infty) \quad (H - \text{Hilbert operator})$$

$$\text{(Pichorides)} \quad \|\mathcal{P}_{n-1}\|_{p,p} \le c(p).$$

Choose $p_{\text{cr}} = \frac{1}{2\varepsilon}$, $\|\Phi_n\|_{L^p(\mathrm{T})} \le 2$, $p \in [2, p_{\text{cr}}]$

Continuity of weighted singular operators

H – a singular integral operator

Lerner, Nazarov, Volberg, $w \in A_p$

$$\|w^{1/p} H w^{-1/p}\|_{p,p} \le \mathfrak{F}\left([w]_{A_p}, p\right)$$

Perturbation:

$$w_\delta = w\, e^{\delta f}, \quad f \in \mathrm{BMO}$$

Theorem 2 (Alexis, A., Denisov–2019)

$\exists \delta_0 \left(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}}\right) > 0 : \forall \delta : \ |\delta| < \delta_0$

$$\|w_\delta^{1/p} H w_\delta^{-1/p} - w^{1/p} H w^{-1/p}\| < |\delta| C(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}}, \mathfrak{F})$$