Szegő measures and vibration of Krein strings

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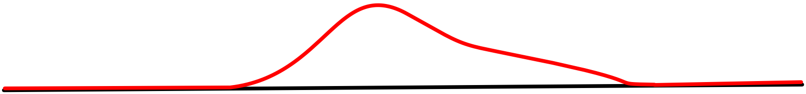
Complex Analysis, Spectral Theory and Approximation meet in Linz, 2022

Joint work with Sergey Denisov

String equation (1D wave equation)

$$u_{tt}(x,t) = u_{xx}(x,t), \quad u(x,0) = u_0(x), \quad u_t(x,0) = 0, \quad x, t \in \mathbb{R}.$$

$u(x,t)$ is the displacement of the string at x, it changes with time t
u_0 is the initial form of the string;
$u_t(x,0) = 0$ means that the velocity of string at time $t = 0$ is zero.



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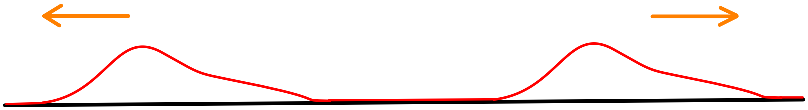

d'Alembert solution

$$u(x,t) = \frac{u_0(x+t) + u_0(x-t)}{2}, \quad x, t \in \mathbb{R}.$$

$u(x,t)$ is a linear combination of two **travelling waves**: $u_0(x \pm t)$

Non-homogeneous strings

$$\rho(x)u_{tt}(x,t) = u_{xx}(x,t),\ u(x,0) = u_0(x),\ u_t(x,0) = 0,\ x,t \in \mathbb{R}.$$

ρ is the **density** of the material of the string. If $\rho \not\equiv const$, the string is called non-homogeneous.

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For which strings the waves $u(x,t)$ look like travelling waves at large times $t \to \pm\infty$?

"Answer"

This occurs if and only if the spectral measure of the string has a finite logarithmic integral (belongs to the Szegő class). Densities ρ of such strings can be explicitly described.

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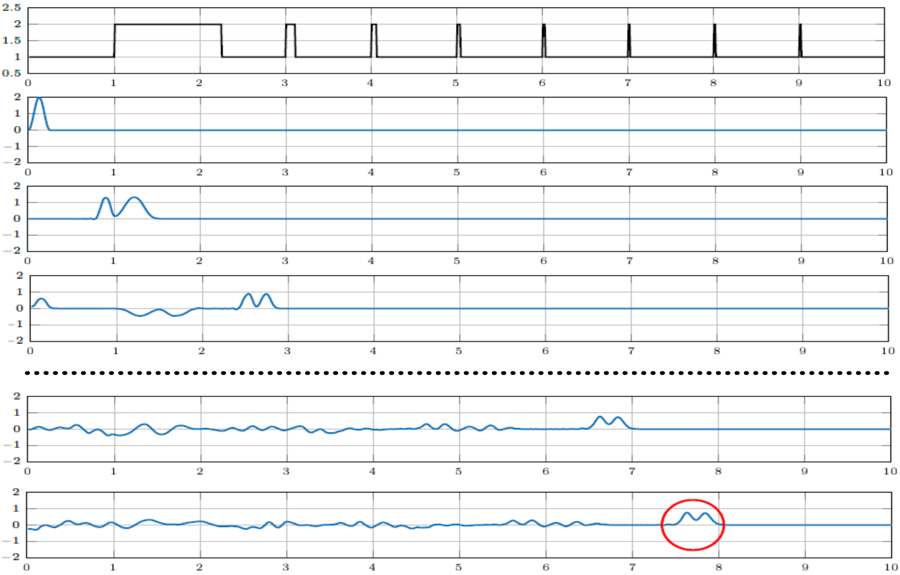
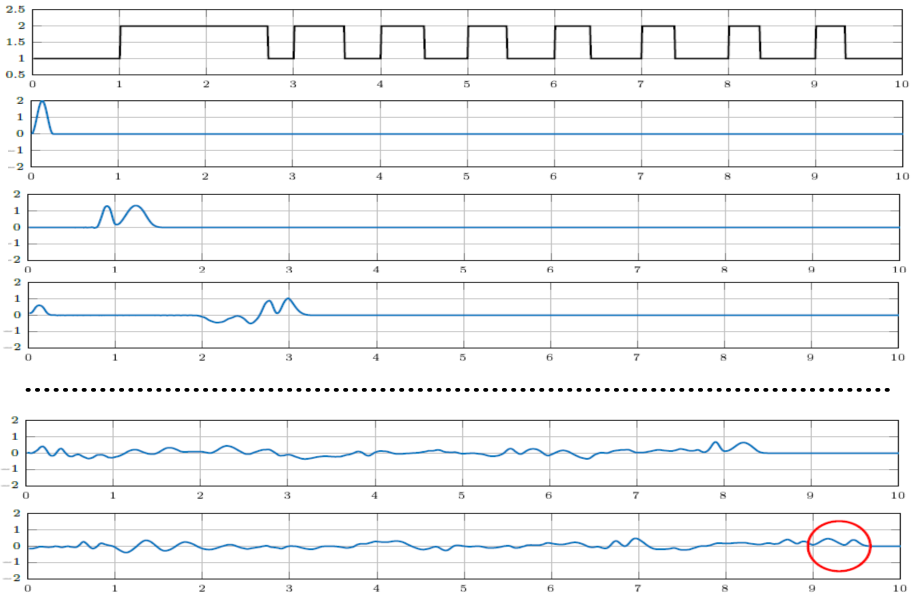
Propagation of waves, example 1



Propagation of waves, example 2



Non-homogeneous strings

Parameters

$L \in (0, +\infty]$ is the length of the string;

m is the density measure: a Borel measure such that $m([0,x])$ is the mass of the piece $[0,x]$ of the string, $x \in [0, L)$.

Assumptions

$m = \rho\, dx + m_s$ is nonnegative, supported on $[0, L)$;

$m([0,x]) \in (0, +\infty)$ for every $x \in (0, L)$;

$L + m([0, L)) = +\infty$;

$m \neq \delta_0$.

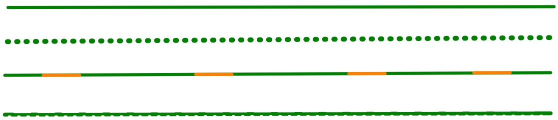
Examples

homogeneous

point masses

two materals

singular continuous



The string equation, non-homogeneous case

$$m(x)u_{tt}(x,t) = u_{xx}(x,t),$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = 0, \qquad x \in [0,L), \quad t \in \mathbb{R}_+,$$
$$u_x(0,t) = 0 \qquad \text{(N) Neumann boundary condition.}$$

For a general density measure m and general $u_0 \in L^2(m)$, the mathematical interpretation (and solution) is via operator calculus:

$$u(x,t) = \cos(t\sqrt{S_m})u_0$$

where $S_m = -\frac{1}{m}\frac{d}{dx^2}$ is the Krein string operator (a self-adjoint nonnegative operator densely defined (N) on $L^2(m)$).

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Szego class on the unit circle $\mathbb{T}$

Probability measures $\mu = w\,dm + \mu_s$ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \log w\,dm > -\infty$$

Szego class on the real line $\mathbb{R}$

Measures $\mu = w\,d\lambda + \mu_s$ on $\mathbb{R}$ such that $(1+\lambda^2)^{-1} \in L^1(\mu)$ and

$$\int_{\mathbb{R}} \frac{\log w(\lambda)}{1+\lambda^2}\,d\lambda > -\infty$$

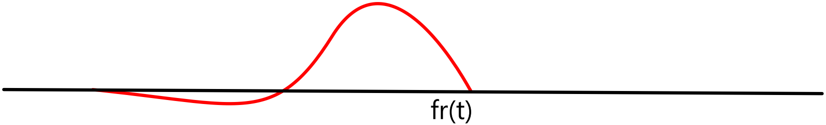
Szego class on the half-line $\mathbb{R}_+$

Measures $\sigma = v\,d\lambda + \sigma_s$ on $\mathbb{R}_+$ such that $(1+\lambda)^{-1} \in L^1(\sigma)$ and

$$\int_{\mathbb{R}_+} \frac{\log v(\lambda)}{\sqrt{\lambda}(1+\lambda)}\,d\lambda > -\infty$$

For a compactly supported $u_0 \in L^2(m)$, define the front of the propagating wave by

$$fr(t) = \inf\{y \geq 0 : u(x,t) = 0 \ \text{ for } m\text{-a.e. } x > y\}.$$



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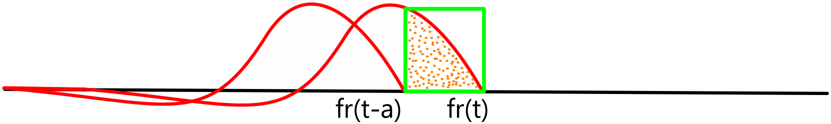

Theorem 1

Let $[m, L]$ be a string, let $a > 0$, and let σ be the spectral measure of the string operator $S_m = -\frac{1}{m}\frac{d^2}{dx^2}$ densely defined (N) on $L^2(m)$. Then we have

$$\liminf_{t \to L} \int_{fr(t-a)}^{fr(t)} |u(\cdot,t)|^2 \, dm > 0$$

for every (for some) nonzero $u_0 \in L^2_{comp}(m)$ iff $\sigma \in Sz(\mathbb{R}_+)$.

Proof of Theorem 1: idea

Let μ be an even measure on $\mathbb{R}$ such that $\mu([x_1, x_2]) = \sigma([x_1^2, x_2^2])$. It turns out that condition

$$\liminf_{t \to L} \int_{fr(t-a)}^{fr(t)} |u(\cdot, t)|^2 \, dm > 0, \qquad a > 0$$

implies that some function in $L^2(\mu)$ cannot be approximated in norm of $L^2(\mu)$ by smooth functions with positive Fourier spectrum. Then Krein-Wiener theorem yields $\mu \in Sz(\mathbb{R})$, hence $\sigma \in Sz(\mathbb{R}_+)$.

This idea works if you know how to calculate $fr(t)$. In literature, the formula

$$fr(t) = L(fr(0) + t), \quad L(y) = \inf\left\{ x \geq 0 : \int_0^x \sqrt{\rho'(s)} \, ds = y \right\}$$

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Theorem 2 (Ref. ?)

Let $[m = \rho dx + m_s, L]$ be a string, and let $u(x,t)$ be the solution of the string equation corresponding to a compactly supported real initial profile $u_0 \in L^2(m)$. Then

$$fr(t) = L(fr(0) + t), \qquad t \in \mathbb{R} \setminus \{0\}.$$

Beurling-Malliavin theorem

Let E be an entire function of finite exponential type such that

$$\int_{\mathbb{R}} \frac{\log^+ |E(x)|}{1+x^2}\, dx < \infty.$$

Then there is an entire function φ of an arbitrarily small exponential type such that φ is not identically zero and $(1+|E|)\varphi$ is bounded on $\mathbb{R}$.

$$L(y) = \inf\{x \geq 0 : T(x) = y\}, \quad T(x) = \int_0^x \sqrt{\rho'(s)}\, ds.$$

Theorem 3

Let $[m = \rho dx + m_s, L]$ be a string, $a > 0$. Assume that the spectral measure σ of $[m, L]$ is in the Szegő class $Sz(\mathbb{R}_+)$. Let $u(x,t)$ be the solution of the string equation, $u(x,0) = u_0$, $u_0 \in L^2(m)$. Then there exists $F_{u_0} \in L^2(\mathbb{R})$ such that

$$u(x,t) = \rho(x)^{-1/4} F_{u_0}(T(x) - t) + o(1), \qquad t \to +\infty,$$

with $o(1)$ in $L^2(m, \Delta_t)$, $\Delta_t = [L_{t-a}, L_{t+a}]$.

In other words, the Szego case occurs if and only if we have a stable propagation near the front of the wave. F_{u_0} is a "travelling wave".

If $u_0 \in H_{ac}(S_m)$, then $o(1)$ in Theorem 3 is with respect to $L^2(m)$ norm. We always have $\|F_{u_0}\|_{L^2(\mathbb{R})} = \|P_{ac} u_0\|_{L^2(m)}$.

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Steps in the proof of Theorem 3

1. Define an entropy function of a string (or canonical system) and prove its monotonicity and additivity properties;
2. Define regularized Krein's orthogonal entire functions and prove Khrushchev formula from OPUC for them
3. Use Khrushchev's idea of weak/strong convergence and properties of regularized Krein's functions to find long-time asymptotics of generalized eigenvectors of a string
4. Compare the free dynamics (pure travelling waves) with perturbed one near the front of waves.

Theorem 4 (2017)

The spectral measure $\sigma = vdx + \sigma_s$ of a string $[m = \rho\, dx + \sigma_s, L]$ lies in the Szegő class $Sz(\mathbb{R}_+)$, i.e.,

$$\int_{\mathbb{R}_+} \frac{\log v(\lambda)}{\sqrt{\lambda}(1+\lambda)}\, d\lambda > -\infty,$$

if and only if

$$\sum_{k\geq 0} (t_{n+2} - t_n) m[t_n, t_{n+2}] - \left(\int_{t_n}^{t_{n+2}} \sqrt{\rho(x)}\, dx \right)^2 < \infty$$

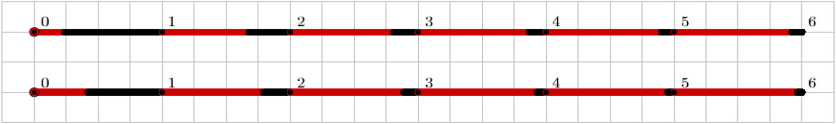
for some (for every) sequence $t_n \uparrow L$ such that $\int_{t_n}^{t_{n+1}} \sqrt{\rho(x)}\, dx \sim 1$

Example: strings that are made from 2 materials

Consider a string with density

$$\rho(x) = \begin{cases} a, & x \in E \\ b, & x \in F \end{cases}$$

for some measurable partition $E \cup F = \mathbb{R}$.



Corollary

We have $\sigma \in Sz(\mathbb{R}_+)$ if and only if ether $a = b$ (homogeneous case) or $\min(|E|, |F|) < \infty$. In particular, the geometry of the partition does not affect the character of propagation of waves.

Example: almost homogeneous strings

Here $L = +\infty$, $m = \chi_{\mathbb{R}_+} dx + m_s$, $m_s \perp dx$, $u_0 \in L^2_{comp}(m)$.

Front of the wave: $\operatorname{ess\,sup} u(x,t) = \operatorname{ess\,sup} u_0 + |t|$, $t \in \mathbb{R}$.

Asymptotic behaviour, non-Szegő case

If $m_s(\mathbb{R}_+) = +\infty$, then for every $a > 0$ we have

$$\lim_{t \to +\infty} \|u(x,t)\|_{L^2(m,[t-a,t+a])} = 0.$$

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$$\lim_{t \to +\infty} \|u(x,t)\|_{L^2(m_s,[t-a,t+a])} = 0,$$

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Example: Dirac operators, Wiegner-von Neumann potentials

For $\alpha, \beta \in \mathbb{R}$, set $q = \frac{\sin x^\alpha}{x^\beta}$. Let $Q_{\alpha,\beta} = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}$ or $Q_{\alpha,\beta} = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}$.

$$D_{Q_{\alpha,\beta}} : X \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X' + Q_{\alpha,\beta} X$$

is the Dirac operator densely defined (D) on $L^2(\mathbb{R}_+, \mathbb{C}^2)$. Let $\mu_{\alpha,\beta}$ denote its main spectral measure, and let D_0 be the free Dirac operator $(Q = 0)$.

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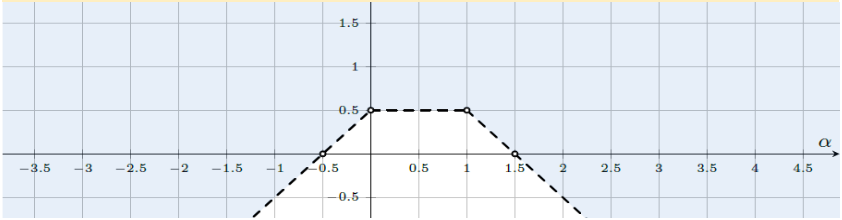
Corollary

The wave operators $W_\pm(D_{Q_{\alpha,\beta}}, D_0) = \lim_{t \to \pm\infty} e^{-itD_{Q_{\alpha,\beta}}} e^{itD_0}$ exist iff $\mu_{\alpha,\beta} \in Sz(\mathbb{R})$ iff $(\alpha, \beta) \in \Omega$ (the set Ω is below).



Main tool: entropy function

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☐ **MR4425804** Bessonov, Roman; Denisov, Sergey; A new life of the classical Szegő formula. *Extended abstracts fall 2019—spaces of analytic functions: approximation, interpolation, sampling,* 31–36, Trends Math. Res. Perspect. CRM Barc., 12, *Birkhäuser/Springer, Cham,* [2021], ©2021. 30C40 (33C45 42C05)

☐ **MR1952927 (2003m:47019)** Christ, M.; Kiselev, A. Scattering and wave operators for one-dimensional Schrödinger operators with slowly decaying nonsmooth potentials. *Geom. Funct. Anal.* 12 (2002), no. 6, 1174–1234. (Reviewer: Christoph M. Thiele) 47A40 (34L25 34L40 47B80 47E05 47N50 81U99)

☐ **MR4322036** Damanik, David; Eichinger, Benjamin; Yuditskii, Peter Szegő's theorem for canonical systems: the Arov gauge and a sum rule. *J. Spectr. Theory* 11 (2021), no. 3, 1255–1277. (Reviewer: Leonid B. Golinskiĭ) 42C05 (30H10 47A48)

☐ **MR1815919 (2002b:42032)** Khrushchev, Sergei Schur's algorithm, orthogonal polynomials, and convergence of Wall's continued fractions in $L^2(\mathbb{T})$. *J. Approx. Theory* 108 (2001), no. 2, 161–248. (Reviewer: Leonid B. Golinskiĭ) 42C05 (30C10 30D50 33C47)

☐ **MR0080735 (18,291b)** Kreĭn, M. G. Continuous analogues of propositions on polynomials orthogonal on the unit circle. (Russian) *Dokl. Akad. Nauk SSSR (N.S.)* **105** (1955), 637–640. (Reviewer: F. V. Atkinson) 30.0X

☐ **MR2105089 (2006a:42002b)** Simon, Barry Orthogonal polynomials on the unit circle. Part 2. Spectral theory. American Mathematical Society Colloquium Publications, 54, Part 2. *American Mathematical Society, Providence, RI,* 2005. pp. i–xxii and 467–1044. ISBN: 0-8218-3675-7 (Reviewer: P. L. Duren) 42-02 (30C85 33C45 42C05 47B36 47N50)

Thank you!