Fun and Games With Gap Labelling

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Complex Analysis, Spectral Theory and Approximation meet in Linz

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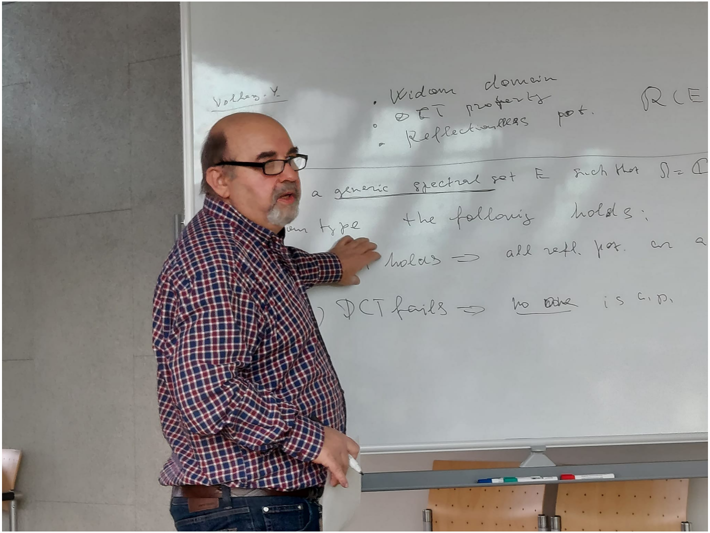
Thank you, Peter!

in Sankt Gilgen (2018)

Thank you, Peter!

in Peter's office (with a photo of Franz Peherstorfer), Linz (2019)

Thank you, Peter!



Peter's retirement lecture, Linz (2021)

gap labelling

the gift that keeps on giving

The One-Dimensional Anderson Model

Choose a compactly supported Borel probability measure ν on $\mathbb{R}$. Form the product measure $\mu = \nu^{\mathbb{Z}}$. For each $\omega \in \Omega := (\operatorname{supp} \nu)^{\mathbb{Z}}$, consider the potential $V_\omega(n) = \omega_n$, $n \in \mathbb{Z}$, and the operator

$$[H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$.

The family $\{H_\omega\}_{\omega \in \Omega}$ is called the one-dimensional Anderson model.

Theorem (Kunz-Souillard)

For μ-almost every $\omega \in \Omega$, we have

$$\sigma(H_\omega) = \Sigma := [-2, 2] + \operatorname{supp} \nu$$

The critical aspects of this result are (i) the almost sure constancy of the spectrum, (ii) the ability to determine the almost sure spectrum explicitly, and (iii) the absence or scarcity of spectral gaps.

The Almost Mathieu Operator

Choose a coupling constant $\lambda > 0$, a frequency $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and a phase $\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Consider the potential $V_\omega(n) = 2\lambda \cos(2\pi(\omega + n\alpha))$, $n \in \mathbb{Z}$, and the operator

$$[H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$.

The family $\{H_\omega\}_{\omega \in \Omega}$ is called the almost Mathieu operator.

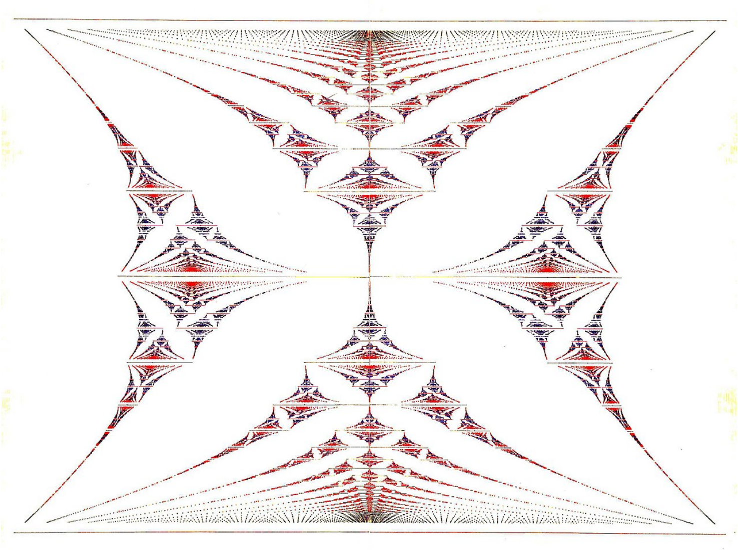
Theorem (Avila-Jitomirskaya)

There is a perfect nowhere dense set $\Sigma \subset \mathbb{R}$ such that for every $\omega \in \mathbb{T}$, we have

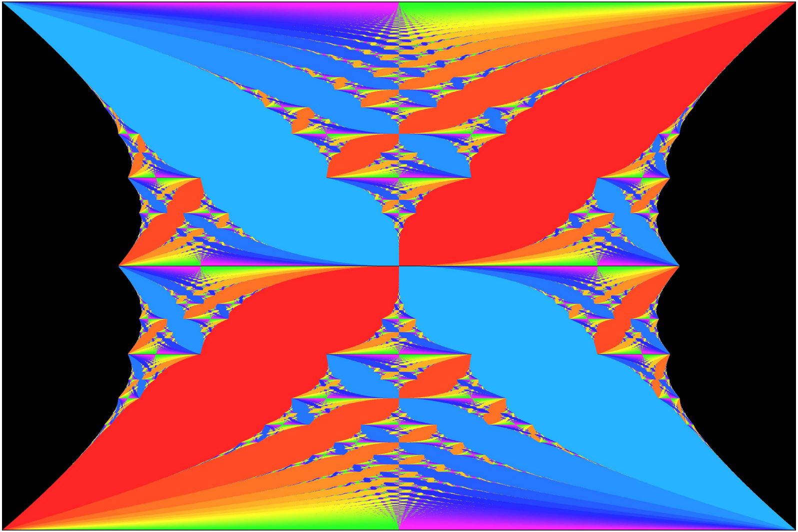
$$\sigma(H_\omega) = \Sigma$$

The critical aspects of this result are (i) the constancy of the spectrum and (ii) the persistent denseness of spectral gaps.

The Spectrum of the Almost Mathieu Operator



The Spectrum of the Almost Mathieu Operator



(Almost Sure) Constancy of the Spectrum

Given a compact metric space Ω, a homeomorphism $T : \Omega \to \Omega$, $f \in C(\Omega, \mathbb{R})$, we consider the potentials

$$V_{f,\omega}(n) = f(T^n \omega), \quad n \in \mathbb{Z},\ \omega \in \Omega$$

and the operators

$$[H_{f,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + V_{f,\omega}(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$.

Theorem (Pastur)

If μ is a T-ergodic Borel probability measure on Ω, then there exists a compact $\Sigma_{f,\mu} \subset \mathbb{R}$ such that for μ-almost every $\omega \in \Omega$, we have $\sigma(H_\omega) = \Sigma_{f,\mu}$.

Theorem

If T is minimal, then there exists a compact $\Sigma_f \subset \mathbb{R}$ such that for every $\omega \in \Omega$, we have $\sigma(H_\omega) = \Sigma_f$.

The Almost Sure Spectrum

Let us consider an ergodic family of Schrödinger operators $\{H_\omega\}_{\omega\in\Omega}$ in $\ell^2(\mathbb{Z})$ as above and, for some T-ergodic measure μ, the associated almost sure spectrum Σ.

The density of states measure $\kappa = \kappa_{\mu,f}$ associated with the family $\{H_\omega\}_{\omega\in\Omega}$ is given by

$$\int g \, d\kappa = \int_\Omega \langle \delta_0, g(H_\omega)\delta_0 \rangle \, d\mu(\omega)$$

In other words, κ is the μ-average of the spectral measure corresponding to the pair (H_ω, δ_0). The accumulation function of κ,

$$k(E) = k_{\mu,f}(E) = \int \chi_{(-\infty,E]} \, d\kappa$$

is called the integrated density of states (IDS).

It is not hard to see that κ is a probability measure whose topological support coincides with Σ. Thus k is an increasing function that grows precisely at points in Σ and takes the value 0 below Σ and the value 1 above Σ.

In particular, k is constant on each gap of Σ and each gap can be labeled uniquely by the value k takes on it.

Gap Labelling via Schwartzman à la Johnson

Russell Johnson
(1947–2017)

Gap Labelling in One Dimension via the Schwartzman Group

Gap labelling theory attempts to characterize these gaps labels in useful ways. The classical approach to gap labelling, developed by Bellissard et al., is based on K-theory of C^*-algebras. The advantage of this approach lies in its scope, as it applies in arbitrary dimensions and to operators that are more general than Schrödinger operators. The disadvantage is that actual computations of gap labels are very difficult.

For Schrödinger operators in one dimension, there is an alternative approach due to Johnson. While its scope is much more restricted, actual computations of gap labels are far easier.

Given a topological dynamical system (Ω, T) as above, we define the suspension of the dynamics $(X, \overline{T})$ as follows: X is the quotient of $\Omega \times [0, 1]$ modulo the equivalence relation $(\omega, 1) \sim (T\omega, 0)$ and $\overline{T}^t \cdot [\omega, s] = [\omega, s + t]$.

If μ is a T-ergodic Borel probability measure on Ω, we define the suspension of the measure $\overline{\mu}$ on X by

$$\int_X f \, d\overline{\mu} = \int_0^1 \int_\Omega f([\omega, s]) \, d\mu(\omega) \, ds$$

Gap Labelling in One Dimension via the Schwartzman Group

Let $C^\sharp(X, \mathbb{T})$ be the set of equivalence classes in $C(X, \mathbb{T})$ modulo homotopy; $C^\sharp(X, \mathbb{T})$ is a countable abelian group (with group operation $[\phi_1] + [\phi_2] = [\phi_1 + \phi_2]$).

Given $\phi \in C(X, \mathbb{T})$ and $x \in X$, we obtain a continuous function $\phi_x : \mathbb{R} \to \mathbb{T}$ by following the image of ϕ along the orbit of x, $\phi_x(t) = \phi(\overline{T}^t x)$.

With the canonical projection $\pi : \mathbb{R} \to \mathbb{T}$, we observe that for each $a \in \pi^{-1}\{\phi_x(0)\}$, there is a unique continuous lift $\widetilde{\phi}_x : \mathbb{R} \to \mathbb{R}$ that satisfies $\pi \circ \widetilde{\phi}_x = \phi_x$ and $\widetilde{\phi}_x(0) = a$.

Proposition (Johnson)

(a) *For each $\phi \in C(X, \mathbb{T})$, the limit*

$$\mathrm{rot}(\phi; x) = \lim_{t \to \infty} \frac{\widetilde{\phi}_x(t)}{t}$$

exists for $\overline{\mu}$-almost every $x \in X$ and does not depend on the choice of lift. Moreover, $\mathrm{rot}(\phi; x)$ is $\overline{\mu}$-almost surely independent of x and hence may be denoted by $\mathfrak{A}_{\overline{\mu}}(\phi)$.

(b) *If ϕ and ϕ' are homotopic, then $\mathfrak{A}_{\overline{\mu}}(\phi) = \mathfrak{A}_{\overline{\mu}}(\phi')$.*

Gap Labelling in One Dimension via the Schwartzman Group

The induced map

$$\mathfrak{A}_{\overline{\mu}} : C^\sharp(X, \mathbb{T}) \to \mathbb{R}$$

is called the Schwartzman homomorphism and its range

$$\mathfrak{A} = \mathfrak{A}(\Omega, T, \mu) := \mathfrak{A}_{\overline{\mu}}(C^\sharp(X, \mathbb{T}))$$

is called the Schwartzman group.

Theorem (Johnson's Gap-Labelling Theorem)

Assume in the setting above that $\operatorname{supp} \mu = \Omega$. *Then, for every* $f \in C(\Omega, \mathbb{R})$,

$$k_{\mu,f}(E) \in \mathfrak{A} \cap [0, 1]$$

for all $E \in \mathbb{R} \setminus \Sigma_{\mu,f}$.

This result naturally arises from an alternative perspective on the IDS (via normalized eigenvalue counting), oscillation theory (which relates eigenvalue counting and sign changes/rotations) of solutions, and Johnson's theorem (which provides continuous sections to which the Schwartzman homomorphism can be applied).

Gap Labelling: A Survey and Some Novel Applications

Jake Fillman
(Texas State University)

Examples of Schwartzman Groups

Let us consider a few examples:

- (quasi-periodic) $\Omega = \mathbb{T}^d$, $T_\alpha : \mathbb{T}^d \to \mathbb{T}^d$, $\omega \mapsto \omega + \alpha$, where $\alpha \in \mathbb{T}^d$ has rationally independent entries, $\mu = \mathrm{Leb}$:

$$\mathfrak{A}(\mathbb{T}^d, T_\alpha, \mu) = \mathbb{Z}^d \alpha + \mathbb{Z}$$

- (skew-shift) $\Omega = \mathbb{T}^2$, $T_{\mathrm{ss}} : \mathbb{T}^2 \to \mathbb{T}^2$, $(\omega_1, \omega_2) \mapsto (\omega_1 + \alpha, \omega_1 + \omega_2)$, where $\alpha \in \mathbb{T}$ is irrational, $\mu = \mathrm{Leb}$:

$$\mathfrak{A}(\mathbb{T}^2, T_{\mathrm{ss}}, \mu) = \mathbb{Z}\alpha + \mathbb{Z}$$

- (cat map) $\Omega = \mathbb{T}^2$, $T_{\mathrm{cm}} : \mathbb{T}^2 \to \mathbb{T}^2$, $(\omega_1, \omega_2) \mapsto (2\omega_1 + \omega_2, \omega_1 + \omega_2)$, $\mu = \mathrm{Leb}$:

$$\mathfrak{A}(\mathbb{T}^2, T_{\mathrm{cm}}, \mu) = \mathbb{Z}$$

The underlying general result is the following:

Theorem (D.-Fillman)

Consider $T_{A,b} : \mathbb{T}^d \to \mathbb{T}^d$, $\omega \mapsto A\omega + b$, where $A \in \mathrm{SL}(d, \mathbb{Z})$, $b \in \mathbb{T}^d$. Suppose μ is $T_{A,b}$-ergodic with $\mathrm{supp}(\mu) = \mathbb{T}^d$. Then,

$$\mathfrak{A}(\mathbb{T}^d, T_{A,b}, \mu) = \{kb + n : n \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^d \cap \ker(I - A^*)\}$$

Gap Labelling: Generic Gap Opening

Artur Avila
(Universität Zürich)

Jairo Bochi
(Penn State University)

Gap Labelling: Generic Gap Opening

Johnson's gap-labelling theorem shows that for any continuous $f \in C(\Omega, \mathbb{R})$,

$$k_{\mu,f}(E) \in \mathfrak{A} \cap [0,1]$$

for all $E \in \mathbb{R} \setminus \Sigma_{\mu,f}$. Two natural questions:

1. Is this collection of gap labels minimal, that is, are they all needed?
2. If a computation of $\mathfrak{A}$ yields a dense set, can one use this to show Cantor spectrum, that is, that the gaps are dense for some $f \in C(\Omega, \mathbb{R})$?

Theorem (Avila-Bochi-D.)

Suppose T is strictly ergodic and has a non-periodic finite-dimensional factor. Then for each $\ell \in \mathfrak{A} \cap [0,1]$, the set

$$\{f \in C(\Omega, \mathbb{R}) : \Sigma_{\mu,f} \text{ has an open gap with label } \ell\}$$

is open and dense. In particular, for a generic $f \in C(\Omega, \mathbb{R})$, all gaps allowed by the gap labelling theorem are open.

Remark

Among the three examples above, this theorem applies to torus translations and the skew-shift. It would be interesting to clarify if a result of this type holds for more general base dynamics.

Gap Labelling: Bellissard's Question

Artur Avila
(Universität Zürich)

Anton Gorodetski
(University of California at Irvine)

The One-Dimensional Anderson Model

Let us recall the following fundamental result mentioned earlier:

Theorem (Kunz-Souillard)

If $\{H_\omega\}$ is the one-dimensional Anderson model with single-site measure ν, then for μ-almost every $\omega \in \Omega$, we have

$$\sigma(H_\omega) = \Sigma := [-2, 2] + \operatorname{supp} \nu$$

As pointed out before, two critical aspects are the explicit formula and the forced finiteness of the number of gaps. To appreciate the former, let us briefly sketch the proof.

Proof.
"$\Sigma \subseteq [-2, 2] + \operatorname{supp} \nu$": $\operatorname{supp} \nu$ is the a.s. spectrum of $\{V_\omega\}$ and they are perturbed by Δ, which has $\|\Delta\| \leq 2$.
"$\Sigma \supseteq [-2, 2] + \operatorname{supp} \nu$": For each $a \in \operatorname{supp} \nu$, we have $\sigma(\Delta + a) = [a - 2, a + 2]$. For a.e. ω, V_ω has arbitrarily long stretches on which it is arbitrarily close to a. Now use a Weyl sequence argument via trial vectors. □

The key to the proof is that we know two types of spectra explicitly, $\sigma(V_\omega)$ and $\sigma(\Delta + a)$. In particular, it is absolutely essential to rely on the presence of constant realizations.

Bellissard's Question: The Question

Suppose we pass to more general random operators, for example random perturbations of a regular background. Specifically, we could take the sum of the two key examples:

$$[H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + [V_\omega(n) + 2\lambda \cos(2\pi n \alpha)]\psi(n).$$

One terms wants to force few spectral gaps, while the other wants to force a dense set of spectral gaps. Who wins?

Bellissard has asked whether one can show that the random terms always wins. That is, in the model above and in models like it, the almost sure spectrum only has finitely many gaps.

The proof of the result in the zero background case does not seem to extend, because we have no constant realizations and indeed the unperturbed spectrum is a Cantor set.

A naive application of gap labelling yields no answer, because the Schwartzman group of the product system still yields a dense set of labels.

Are we stuck entirely or is there something that gap labelling can tell us about Bellissard's question?

Bellissard's Question: The Setup

Given a compact metric space X, a homeomorphism $T : X \to X$, an ergodic Borel probability measure μ with full topological support, $\operatorname{supp} \mu = X$, and a sampling function $f \in C(X, \mathbb{R})$, we generate potentials

$$V_x(n) = f(T^n x), \; x \in X, \; n \in \mathbb{Z}$$

and Schrödinger operators

$$[H_x \psi](n) = \psi(n+1) + \psi(n-1) + V_x(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$. There is a compact set Σ_0 such that

$$\Sigma_0 = \sigma(H_x) \text{ for } \mu\text{-almost every } x \in X.$$

The random perturbation is given by

$$W_\omega(n) = \omega_n, \; \omega \in \Omega, \; n \in \mathbb{Z},$$

where $\Omega = (\operatorname{supp} \nu)^{\mathbb{Z}}$ and ν is a compactly supported probability measure on $\mathbb{R}$ with topological support $S := \operatorname{supp} \nu$ satisfying $\#S \geq 2$.
Since the product of μ and $\tilde{\mu} := \nu^{\mathbb{Z}}$ is ergodic, there is a compact set Σ_1 such that

$$\Sigma_1 = \sigma(H_x + W_\omega) \text{ for } \mu \times \tilde{\mu}\text{-almost every } (x, \omega) \in X \times \Omega.$$

Bellissard's Question: The Result

Definition

Suppose A and B are compact subsets of $\mathbb{R}$. We define $A \bigstar B$ as follows. If $\mathrm{diam}(A) \geq \mathrm{diam}(B)$, then $A \bigstar B = A + \mathrm{ch}(B)$, and if $\mathrm{diam}(A) < \mathrm{diam}(B)$, then $A \bigstar B = \mathrm{ch}(A) + B$.

Theorem (Avila-D.-Gorodetski)

Consider the setting described above and assume that X is connected. Then, we have

$$\Sigma_1 = \Sigma_0 \bigstar S.$$

This theorem provides an affirmative answer to Bellissard's question:

Corollary (Avila-D.-Gorodetski)

If X is connected, then the almost sure spectrum Σ_1 has only finitely many gaps. Equivalently, Σ_1 is given by a finite union of non-degenerate compact intervals.

Bellissard's Question: The Result

Theorem (Avila-D.-Gorodetski)

Consider the setting described above and assume that X is connected. Then, we have

$$\Sigma_1 = \Sigma_0 \bigstar S.$$

The proof uses Johnson's approach to gap labelling in an essential way.

Moreover, the result extends the classical Kunz-Souillard formula.

Consider the two cases in question:

(a) If $\mathrm{diam}(S) \le 4 = \mathrm{diam}(\Sigma_0)$, then $\Sigma_1 = \Sigma_0 \bigstar S = [-2,2] + \mathrm{ch}(S) = \Sigma_0 + S$.

(b) If $\mathrm{diam}(S) > 4$, then $\Sigma_1 = \Sigma_0 \bigstar S = \mathrm{ch}([-2,2]) + S = \Sigma_0 + S$.