Congratulations, Peter!

Based on

- A. Aptekarev, S. Denisov, M. Yattselev, Self-adjoint Jacobi matrices on trees and multiple orthogonal polynomials, 2020.
- A. Aptekarev, S. Denisov, M. Yattselev, Jacobi matrices on trees generated by Angelesco system: asymptotics of coefficients and essential spectrum, 2021.
- S. Denisov, M. Yattselev, Spectral theory of Jacobi matrices on trees whose coefficients are generated by multiple orthogonality, 2021.

Extensive existing literature and I can not give credit to all. However, the results by *Fidalgo-Lopes Lagomasino, Gonchar-Rakhmanov-Sorokin, Van Assche, Stahl* played crucial role.

Plan of the talk:

- Some basic facts of spectral theory
- Multiple orthogonal polynomials
- Jacobi matrices on binary trees generated by MOP
- Angelesco system
- Some results for Angelesco systems
- Nikishin systems
- MOP of the second type and the Jacobi matrices on finite rooted trees.

1. Some basic facts

Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{A}$ be a bounded self-adjoint operator acting on it. We can study the spectrum of this operator by obtaining a decomposition of $\mathfrak{H}$ into an orthogonal sum of cyclic subspaces of $\mathfrak{A}$. That is, take any $\mathfrak{g}_1 \in \mathfrak{H}$ with unit norm, i.e., $\|\mathfrak{g}_1\| = 1$, and generate the cyclic subspace

$$\mathfrak{C}_1 := \overline{\text{span}\{\mathfrak{A}^m \mathfrak{g}_1 : \ m = 0, 1, \ldots\}}.$$

We shall call $\mathfrak{g}_1$ the first generator and $\mathfrak{C}_1$ the first cyclic subspace. One can show that $\mathfrak{C}_1$ is invariant with respect to $\mathfrak{A}$. If $\mathfrak{C}_1 \subset \mathfrak{H}$, we take $\mathfrak{g}_2 \in \mathfrak{H}$, that satisfies $\|\mathfrak{g}_2\| = 1$ and $\mathfrak{g}_2 \perp \mathfrak{C}_1$. We denote by $\mathfrak{C}_2$ the cyclic space generated by $\mathfrak{g}_2$. It is also invariant under $\mathfrak{A}$ and satisfies $\mathfrak{C}_1 \perp \mathfrak{C}_2$. Continuing this way, we obtain the following representation of $\mathfrak{H}$ as a sum of orthogonal cyclic subspaces:

$$\mathfrak{H} = \oplus_{m=1}^{N} \mathfrak{C}_m, \qquad (1)$$

where $N \in \mathbb{N} \cup \infty$.

Since $\mathfrak{A}$ is self-adjoint, the operator $(\mathfrak{A}-z)^{-1}$ is bounded on $\mathfrak{H}$ for every $z \in \mathbb{C}_+$, the upper half-plane. For each $f \in \mathfrak{H}$, the function $\langle(\mathfrak{A}-z)^{-1}f, f\rangle$ is in Herglotz-Nevanlinna class, i.e., it is analytic in $\mathbb{C}_+$ and has non-negative imaginary part there. Moreover, since $\mathfrak{A}$ is bounded, we have an integral representation

$$\langle(\mathfrak{A}-z)^{-1}f, f\rangle = \int_{\mathbb{R}} \frac{d\rho_f(x)}{x-z}, \quad z \in \mathbb{C}_+\,,$$

where the measure ρ_f is called the *spectral measure* of f.
It holds that

$$\sigma(\mathfrak{A}) = \overline{\bigcup_{m=1}^{N} \operatorname{supp} \rho_{\mathfrak{g}_m}}\,,$$

where $\rho_{\mathfrak{g}_m}$ is the spectral measure of the generator $\mathfrak{g}_m$ for the cyclic subspace $\mathfrak{C}_m$ from decomposition (1).

Decomposition (1) can be used as follows. Fix $\mathfrak{C}_m$. Taking a sequence of vectors

$$\{\mathfrak{g}_m, \mathfrak{A}\mathfrak{g}_m, \mathfrak{A}^2\mathfrak{g}_m, \ldots\}$$

and running Gramm-Schmidt orthogonalization procedure gives the orthonormal basis in $\mathfrak{C}_m$ in which the restriction of $\mathfrak{A}$ to $\mathfrak{C}_m$ takes the form of either an infinite or a finite (depending on $\dim \mathfrak{C}_m$) one-sided Jacobi matrix.

Let $\{a_j\}, \{b_j\} \in \ell^\infty(\mathbb{Z}_+)$ and $a_j > 0, b_j \in \mathbb{R}$, hereafter $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ and $\mathbb{N} := \{1, 2, \ldots\}$. An infinite one-sided Jacobi matrix is a matrix of the form

$$\mathcal{J} := \begin{bmatrix} b_0 & \sqrt{a_0} & 0 & 0 & \ldots \\ \sqrt{a_0} & b_1 & \sqrt{a_1} & 0 & \ldots \\ 0 & \sqrt{a_1} & b_2 & \sqrt{a_2} & \ldots \\ 0 & 0 & \sqrt{a_2} & b_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}, \qquad (2)$$

and an N–dimensional Jacobi matrix is the upper-left $N \times N$ corner of (2). We define two sets of measures on the real line

$$\mathfrak{M} := \left\{\mu : \ \mathrm{supp}\mu \subset [-R_\mu, R_\mu], \ R_\mu < \infty, \ \#\mathrm{supp}\mu = \infty\right\}$$

and

$$\mathfrak{M}_1 := \left\{\mu \in \mathfrak{M} : \ \mu(\mathbb{R}) = 1\right\},$$

One-sided infinite Jacobi matrices with uniformly bounded entries are known to be in one-to-one correspondence with $\mathfrak{M}_1$, the set of probability measures on $\mathbb{R}$ whose support is compact and has infinite cardinality. This bijection is realized via polynomials orthogonal on the real line. On the one hand, since $\mathcal{J}$ defines a bounded self-adjoint operator on the Hilbert space $\ell^2(\mathbb{Z}_+)$, we can consider the spectral measure of the vector $(1,0,0,\ldots)$. We will call it $\rho(\mathcal{J})$. On the other hand, given $\mu \in \mathfrak{M}_1$, one can produce a Jacobi matrix in the following way. Let $p_n(x,\mu)$ be the n-th orthonormal polynomial with respect to μ, i.e., $p_n(x,\mu)$ is a polynomial of degree n such that

$$\int_{\mathbb{R}} p_n(x,\mu)x^m d\mu(x) = 0, \quad m = 0,\ldots,n-1\,,$$

that is normalized so that

$$\mathrm{coeff}_n p_n > 0, \quad \int_{\mathbb{R}} p_n^2(x,\mu)d\mu(x) = 1\,,$$

where $\mathrm{coeff}_n Q$ is the coefficient in front of x^n of the polynomial $Q(x)$.

It is known that polynomials $p_n(x,\mu)$ satisfy the three-term recurrence relations

$$xp_n(x,\mu) = \sqrt{a_n}p_{n+1}(x,\mu)+b_np_n(x,\mu)+\sqrt{a_{n-1}}p_{n-1}(x,\mu), \quad n = 0,1,\dots\,, \tag{3}$$

where $a_n > 0$, $b_n \in \mathbb{R}$ and $p_{-1} := 0, a_{-1} := 0$. The coefficients $\{a_n\},\{b_n\}$ are defined uniquely by μ and one can show that

$$\{a_n\},\{b_n\} \in \ell^\infty(\mathbb{Z}_+)\,.$$

Let $\mathcal{J}$ be defined via (2) with these coefficients. It is a general fact of the theory that

$$\rho(\mathcal{J}) = \mu \quad \text{and therefore} \quad \sigma(\mathcal{J}) = \text{supp}\mu\,.$$

The above correspondence is one-to-one: one can start with a bounded self-adjoint Jacobi matrix (2), compute $\rho(\mathcal{J})$, the spectral measure of $(1, 0, 0, \ldots)$, via (4), take $\rho(\mathcal{J})$ as a measure of orthogonality μ and, finally, define the orthogonal polynomials whose recurrence coefficients will give rise to the same $\mathcal{J}$.

It follows from (3) that the sequence $\{p_n(x, \mu)\}$, with $\mu = \rho(\mathcal{J})$, represents the generalized eigenfunction of $\mathcal{J}$. That can be made explicit by the following statement, which, together with (3), can be taken as a definition of a generalized eigenfunction.

Lemma

Suppose $\mu \in \mathfrak{M}_1$*. The map*

$$\alpha(x) \mapsto \widehat{\alpha} = \{\widehat{\alpha}(n)\}_{n\in\mathbb{Z}_+}, \quad \widehat{\alpha}(n) := \int \alpha(x) p_n(x,\mu) d\mu(x),$$

is a unitary map from $L^2(\mu)$ *onto* $\ell^2(\mathbb{Z}_+)$ *such that*

$$\|\alpha\|^2_{L^2(\mu)} = \|\widehat{\alpha}\|^2_{\ell^2(\mathbb{Z}_+)}.$$

This map establishes unitary equivalence of the operator $\mathcal{J}$ *on* $\ell^2(\mathbb{Z}_+)$ *and the operator of multiplication by* x *on* $L^2(\mu)$*. In particular,*

$$x\alpha(x) \mapsto \mathcal{J}\widehat{\alpha}.$$

2. Multiple orthogonal polynomials

The system of polynomials orthogonal on the real line can be generalized to the case of orthogonality with respect to several measures. This multiple orthogonality, being a classical area of approximation theory, has connections to number theory (e.g., Hermite's proof that e is transcendental), numerical analysis (simultaneous rational approximation of analytic vector-function, simultaneous Gaussian quadrature for numerical calculation of integrals), Random Matrix Theory, etc.. To define it, consider

$$\vec{\mu} := (\mu_1, \mu_2), \quad \mathrm{supp}\mu_k \subseteq \mathbb{R}, \quad \text{and} \quad \vec{n} := (n_1, n_2) \in \mathbb{Z}_+^2, \quad |\vec{n}| := n_1 + n_2,$$

where we assume that all the moments of the measures μ_1, μ_2 are finite.

Definition. Polynomials $A_{\vec{n}}^{(1)}(x)$ and $A_{\vec{n}}^{(2)}(x)$, $\deg A_{\vec{n}}^{(k)} \leqslant n_k - 1$, $k \in \{1, 2\}$, that satisfy

$$\int_{\mathbb{R}} x^m \big(A_{\vec{n}}^{(1)}(x) d\mu_1(x) + A_{\vec{n}}^{(2)}(x) d\mu_2(x)\big) = 0, \quad m \in \{0, \ldots, |\vec{n}| - 2\}\,, \tag{4}$$

are called type I multiple orthogonal polynomials (type I MOP). We assume that $A_{\vec{n}}^{(k)}(x) \not\equiv 0$ unless $n_k - 1 < 0$. Furthermore, non-identically zero polynomial $P_{\vec{n}}(x)$ is called type II multiple orthogonal polynomial (type II MOP) if it satisfies

$$\deg P_{\vec{n}} \leqslant |\vec{n}|, \quad \int_{\mathbb{R}} P_{\vec{n}}(x) x^m d\mu_k(x) = 0 \tag{5}$$

for all $m \in \{0, \ldots, n_k - 1\}$ and $k \in \{1, 2\}$.

Polynomials of the first and second type always exist. The question of uniqueness is more involved. The index $\vec{n}$ is called *normal* if *monic* $P_{\vec{n}} = x^{|\vec{n}|} + \cdots$ exists and is unique. It turns out that $\vec{n}$ is normal if and only if the linear form

$$Q_{\vec{n}}(x) := A_{\vec{n}}^{(1)}(x)d\mu_1(x) + A_{\vec{n}}^{(1)}(x)d\mu_2(x)$$

that satisfies (4) and

$$\int_{\mathbb{R}} x^{|\vec{n}|-1}Q_{\vec{n}}(x) = 1$$

exists and is unique.

If $d = 1$, type II polynomials $P_{\vec{n}}(x)$ are the standard monic polynomials orthogonal on the real line with respect to the measure μ_1 and the polynomials $A_{\vec{n}}^{(1)}(x)$ are proportional to $p_{n-1}(x, \mu_1)$ with the coefficient of proportionality that can be computed explicitly.

Definition. The vector $\vec{\mu}$ is called perfect if all the multi-indices $\vec{n} \in \mathbb{Z}_+^2$ are normal.

We will consider only perfect systems (most of the systems studied in the literature are perfect).

Besides the orthogonal polynomials, we will need the functions of the second kind.

Definition. The functions

$$L_{\vec{n}}(z) := \int_{\mathbb{R}} \frac{Q_{\vec{n}}(x)}{z-x} \quad \text{and} \quad R_{\vec{n},k}(z) := \int_{\mathbb{R}} \frac{P_{\vec{n}}(x)d\mu_k(x)}{z-x}, \quad k \in \{1,2\},$$

are called functions of the second kind associated to the linear forms $Q_{\vec{n}}(x)$ and to polynomials $P_{\vec{n}}(x)$, respectively.

In the literature on orthogonal polynomials, the following Cauchy-type integral

$$\widehat{\mu}(z) := \int_{\mathbb{R}} \frac{d\mu(x)}{z-x}\,, \quad z \notin \operatorname{supp}\mu\,, \quad \mu \in \mathfrak{M}, \tag{6}$$

is often referred to as a Markov function. If $\mu_1, \mu_2 \in \mathfrak{M}$, we can rewrite $L_{\vec{n}}(z)$ as

$$L_{\vec{n}}(z) = A_{\vec{n}}^{(1)}(z)\widehat{\mu}_1(z) + A_{\vec{n}}^{(2)}(z)\widehat{\mu}_2(z) - A_{\vec{n}}^{(0)}(z),$$

where $A_{\vec{n}}^{(0)}(z)$ is a polynomial given by

$$A_{\vec{n}}^{(0)}(z) := \int_{\mathbb{R}} \frac{A_{\vec{n}}^{(1)}(z) - A_{\vec{n}}^{(1)}(x)}{z-x} d\mu_1(x) + \int_{\mathbb{R}} \frac{A_{\vec{n}}^{(2)}(z) - A_{\vec{n}}^{(2)}(x)}{z-x} d\mu_2(x).$$

Type I Hermite-Pade approximation of $\{\widehat{\mu}_j\}, j \in \{1,2\}$ reads

$$L_{\vec{n}}(z) = O(|z|^{-|\vec{n}|}), \quad |z| \to \infty\,.$$

Type II approximation gives

$$P_{\vec{n}}\,\widehat{\mu}_j - R_{\vec{n},j} = O(|z|^{-n_j-1}), \quad |z| \to \infty,\, j \in \{1,2\}.$$

Similarly to classical orthogonal polynomials on the real line, the above MOP also satisfy nearest-neighbor lattice recurrence relations. Denote by $\vec{e}_1 := (1,0)$ and $\vec{e}_2 := (0,1)$ the standard basis vectors in $\mathbb{R}^2$. Assume that

$$\vec{\mu} = (\mu_1, \mu_2) \ \text{ is perfect}\,.$$

In this case, there exist real constants $\{a_{\vec{n},1}, a_{\vec{n},2}, b_{\vec{n},1}, b_{\vec{n},2}\}_{\vec{n}\in\mathbb{Z}_+^2}$, which we call the *recurrence coefficients* corresponding to the system $\vec{\mu}$, such that the linear forms $Q_{\vec{n}}(x)$ satisfy

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_i}(x)+b_{\vec{n}-\vec{e}_i,i}Q_{\vec{n}}(x)+a_{\vec{n},1}Q_{\vec{n}+\vec{e}_1}(x)+a_{\vec{n},2}Q_{\vec{n}+\vec{e}_2}(x)\,,\ \vec{n}\in\mathbb{N}^2, \tag{7}$$

for each $i \in \{1,2\}$, while it holds for type II polynomials that

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_i}(x)+b_{\vec{n},i}P_{\vec{n}}(x)+a_{\vec{n},1}P_{\vec{n}-\vec{e}_1}(x)+a_{\vec{n},2}P_{\vec{n}-\vec{e}_2}(x)\,, \quad \vec{n}\in\mathbb{Z}_+^2, \tag{8}$$

again, for each $i \in \{1,2\}$, where we let $P_{\vec{n}-\vec{e}_l}(x) \equiv 0$ when the l-th components of $\vec{n}-\vec{e}_l$ is negative.

It is known that

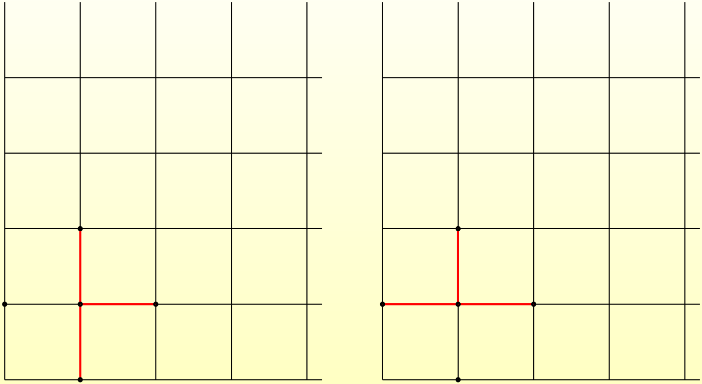
$$a_{\vec{n},i} \neq 0, \ \ \vec{n} \in \mathbb{N}^2, \ \ i \in \{1,2\}, \quad \text{and} \quad \begin{cases} a_{(n,0),1}, \ a_{(0,n),2} > 0, & n \in \mathbb{N}, \\ a_{(0,n),1} = a_{(n,0),2} := 0, & n \in \mathbb{Z}_+, \end{cases}$$

where the first conclusion follows from perfectness and an explicit integral representation for $a_{\vec{n},i}$, and the second one is part definition and part a consequence of positivity of parameters $\{a_n\}$ in (3).

For perfect systems $\vec{\mu}$, one can show that (7) implies the recursion for the type I polynomials themselves:

$$xA_{\vec{n}}^{(j)}(x) = A_{\vec{n}-\vec{e}_i}^{(j)}(x) + b_{\vec{n}-\vec{e}_i,i}A_{\vec{n}}^{(j)}(x) + a_{\vec{n},1}A_{\vec{n}+\vec{e}_1}^{(j)}(x) + a_{\vec{n},2}A_{\vec{n}+\vec{e}_2}^{(j)}(x)\,, \quad (9)$$

where $\vec{n} \in \mathbb{N}^2$, $i, j \in \{1,2\}$.



Two equations for Q at point $\vec{n} = (1, 1)$ from (7)

The recurrence coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$ are uniquely determined by $\vec{\mu}$. However, when $d > 1$, unlike in the one-dimensional case, we can not prescribe them arbitrarily. In fact, coefficients in (7) and (8) satisfy the so-called "consistency conditions", which is a system of nonlinear difference equations:

$$b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n},j} = b_{\vec{n}+\vec{e}_j,i} - b_{\vec{n},i},$$

$$\sum_{k=1}^{2} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{2} a_{\vec{n}+\vec{e}_i,k} = b_{\vec{n}+\vec{e}_j,i} b_{\vec{n},j} - b_{\vec{n}+\vec{e}_i,j} b_{\vec{n},i},$$

$$a_{\vec{n},i}(b_{\vec{n},j} - b_{\vec{n},i}) = a_{\vec{n}+\vec{e}_j,i}(b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}),$$

where $\vec{n} \in \mathbb{N}^2$ and $i, j \in \{1, 2\}$. Conversely, solution to this nonlinear system is unique and uniquely defines $\vec{\mu}$ (μ_k's are the spectral measures of the Jacobi operators corresponding to the boundary values) provided the boundary values are properly defined.

3. Jacobi matrices on binary trees generated by MOP

The finite binary trees and Jacobi matrices on them correspond to MOP of the second type $\{P_{\vec{n}}\}$ and infinite binary trees and Jacobi matrices on them correspond to MOP of the first type $\{A_{\vec{n}}^{(j)}\}$.

Let us focus on the latter case and consider (7) and untwine it to the infinite rooted tree.

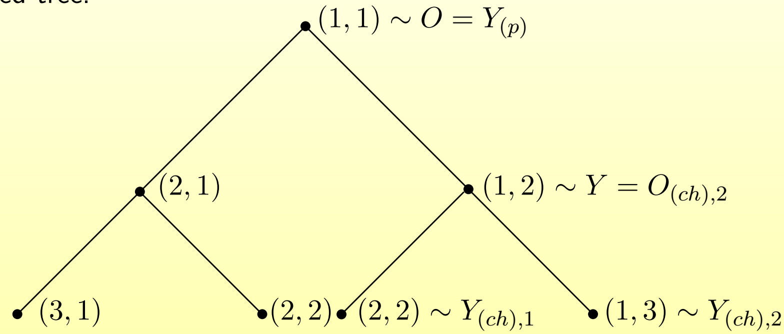


Figure: Three generations of $\mathcal{T}$.

Specifically, let $\mathcal{T}$ be an infinite 2-homogeneous rooted tree (rooted Cayley tree) and $\mathcal{V}$ be the set of its vertices with O being the root. On the lattice $\mathbb{N}^2$, consider an infinite path

$$\{\vec{n}^{(1)}, \vec{n}^{(2)}, \dots\}, \quad \vec{n}^{(1)} = \vec{1} := (1,1) \text{ and}$$
$$\vec{n}^{(l+1)} = \vec{n}^{(l)} + \vec{e}_{k_l}, \quad k_l \in \{1,2\}, \quad l \in \mathbb{N}.$$

These are paths for which, as we move from $\vec{1}$ to infinity, the multi-index of each next vertex is increasing by 1 at exactly one position. Each such path can be mapped bijectively to a non-self-intersecting path on $\mathcal{T}$ that starts at O. This construction defines a projection $\Pi : \mathcal{V} \to \mathbb{N}^2$ as follows: given $Y \in \mathcal{V}$ we consider the non-self-intersecting path from O to Y, map it to a path on $\mathbb{N}^2$ and let $\Pi(Y)$ be the endpoint of the mapped path.

Every vertex $Y \in \mathcal{V}$, which is different from O, has a unique parent, which we denote by $Y_{(p)}$. That allows us to define the following index function:

$$\imath : \mathcal{V} \to \{1, 2\},\ Y \mapsto \imath_Y \ \text{ such that } \ \Pi(Y) = \Pi(Y_{(p)}) + \vec{e}_{\imath_Y}.$$

This way, if $Z = Y_{(p)}$, then we write that $Y = Z_{(ch),\iota_Y}$. For a function f on $\mathcal{V}$, we denote its value at a vertex $Y \in \mathcal{V}$ by f_Y. Fix $\vec{\kappa} \in \mathbb{R}^2$ such that $|\vec{\kappa}| = 1$ and define the potentials $V = V^{\vec{\mu}}, W = W^{\vec{\mu}} : \mathcal{V} \to \mathbb{R}$ by

$$V_O := \kappa_1 b_{(0,1),1} + \kappa_2 b_{(1,0),2}, \quad W_O := 1,$$
$$V_Y := b_{\Pi(Y_{(p)}),\iota_Y}, \quad W_Y := a_{\Pi(Y_{(p)}),\iota_Y}, \quad Y \neq O.$$

4. Angelesco system

Consider **Angelesco system**, a system for which $\{\Delta_j\}$ are disjoint where Δ_j is the convex hull of $\operatorname{supp}\mu_j$. It is known to be perfect, $a_{\vec{n},j} > 0$ and $\{a_{\vec{n},j}\}, \{b_{\vec{n},j}\}$ are uniformly bounded.
That follows from two facts:

Lemma

We have representations

$$a_{\vec{n},j} = \frac{\displaystyle\int_{\mathbb{R}} P_{\vec{n}}(x)\, x^{n_j}\, d\mu_j(x)}{\displaystyle\int_{\mathbb{R}} P_{\vec{n}-\vec{e}_j}(x)\, x^{n_j-1} d\mu_j(x)}, \quad \vec{n} \in \mathbb{Z}_+^d, \quad j \in \{1, \ldots, d\}, \quad n_j - 1 \geq 0,$$

and

$$b_{\vec{n}-\vec{e}_j,j} = \int_{\mathbb{R}} x^{|\vec{n}|} Q_{\vec{n}}(x) - \int_{\mathbb{R}} x^{|\vec{n}|-1} Q_{\vec{n}-\vec{e}_j}(x), \quad \vec{n} \in \mathbb{N}^d, \quad j \in \{1, \ldots, d\}.$$

and the second fact that $P_{\vec{n}}$ has n_1 simple roots on Δ_1 and n_2 roots on Δ_2 that interlace when $\vec{n}$ is increased. Putting these two facts together with the variational properties of monic orthogonal polynomials, we get boundedness and positivity of $\{a_{\vec{n},j}\}$. The boundedness of $\{b_{\vec{n},j}\}$ can be obtained in several ways.

Now we can define Jacobi matrix $\mathcal{J}_{\vec{\kappa}} = \mathcal{J}_{\vec{\kappa}}^{\vec{\mu}}$ on $\mathcal{T}$:

$$(\mathcal{J}_{\vec{\kappa}} f)_Y := V_Y f_Y + W_Y^{1/2} f_{Y_{(p)}} + W_{Y_{(ch),1}}^{1/2} f_{Y_{(ch),1}} + W_{Y_{(ch),2}}^{1/2} f_{Y_{(ch),2}}.$$

It is self-adjoint and bounded operator on $\ell^2(\mathcal{T})$.

We can relate the spectral quantities of Jacobi matrix to MOPS. Define

$$L_{\vec{n}}(z) := \int_{\mathbb{R}} \frac{Q_{\vec{n}}(x)}{z-x}, \, z \notin \mathbb{R} \quad \text{and}$$

$$l_Y(z) = m_Y^{-1} L_Y(z),$$

$$L_Y(z) := L_{\Pi(Y)}(z), \quad \text{and} \quad m_Y := \prod_{Z \in \text{path}(Y,O)} W_Z^{-1/2}.$$

Then, one can show

$$l_Y(z) = -L_{\vec{\kappa}}(z) G(O,Y,z), \quad L_{\vec{\kappa}} = \left(\varkappa_1 \|\mu_1\|^{-1}\right)\widehat{\mu}_1(z) + \left(\varkappa_2 \|\mu_2\|^{-1}\right)\widehat{\mu}_2(z)$$

which establishes the connection between Green's function of $\mathcal{J}_{\vec{\kappa}}$ and $L_{\vec{n}}$.

5. Some results for Angelesco systems

• We obtain a decomposition (1) of Hilbert space $\ell^2(\mathcal{T})$ into the orthogonal sum of cyclic invariant subspaces and the generators are described in terms of MOP and measures μ_1 and μ_2.

Theorem

We have

$$\sigma(\mathcal{J}_{\vec{\kappa}}) \subseteq \Delta_1 \cup \Delta_2 \cup E_{\vec{\kappa}}$$

where $E_{\vec{\kappa}}$ is a single point outside $\Delta_1 \cup \Delta_2$ or an empty set (can be found exactly). In general, this can be proper inclusion but it becomes equality if $\operatorname{supp} \mu_j = \Delta_j, j \in \{1, 2\}$.

Warning: it is not true in general that $\sigma(\mathcal{J}_{\vec{\kappa}}) = \operatorname{supp} \mu_1 \cup \operatorname{supp} \mu_2$.

- The analysis of spectral type can be performed in many cases.

Theorem

Suppose that $\{\mu_j\}, j \in \{1,2\}$ *are Steklov measures, i.e.,* $d\mu_j = w_j dx, w_j^{-1} \in L^\infty(\Delta_j)$. *Then the spectrum of* $\mathcal{J}_{\vec{e}_i}$ *is purely absolutely continuous for each* $i \in \{1,2\}$.

- The inverse spectral problem can be solved.

Theorem

If $\vec{\kappa}$, $\{\|\mu_j\|\}$, $\int_{\mathbb{R}} x(d\mu_1/\|\mu_1\| - d\mu_2/\|\mu_2\|)$, $\langle(\mathcal{J}_{\vec{\kappa}} - z)^{-1}\delta_O, \delta_O\rangle$ *are all known, then* $\mathcal{J}_{\vec{\kappa}}$ *is uniquely defined.*

Remark. The procedure of recovering $\mathcal{J}_{\vec{\kappa}}$ is "constructive".

• The methods of Complex Analysis (e.g., Riemann-Hilbert technique) are applied to obtain strong results when the measures μ_1 and μ_2 are real analytic. Define

$$\mathcal{N}_{\vec{c}} = \{\vec{n}\} \ : \quad n_i = c_i|\vec{n}| + o(\vec{n}), \quad i \in \{1,2\}, \quad |\,\vec{c}\,| := \sum_{i=1}^{2} c_i = 1. \quad (10)$$

Theorem

Suppose the measure $\mu_i, i \in \{1,2\}$ is absolutely continuous with respect to the Lebesgue measure on Δ_i and the density $\mu_i'(x) := \mathrm{d}\mu_i(x)/\mathrm{d}x$ extends to a holomorphic and non-vanishing function in some neighborhood of Δ_i. Then the ray limits (10) of coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$ exist for any $\vec{c} \in (0,1)^2$:

$$\lim_{\mathcal{N}_{\vec{c}}} a_{\vec{n},i} = A_{\vec{c},i} \quad \textit{and} \quad \lim_{\mathcal{N}_{\vec{c}}} b_{\vec{n},i} = B_{\vec{c},i}, \quad i \in \{1,2\}.$$

Remark. The numbers $A_{\vec{c},i}$ and $B_{\vec{c},i}$ can be obtained from a certain vector-valued potential theory problem.

Define

$$E(\mu, \nu) := -\int \log|x-y| d\mu(x) d\nu(y),$$

for any probability Borel measures ν, μ on Δ. Now, given $c \in (0,1)$, define

$$M_c := \big\{(\nu_1, \nu_2): \operatorname{supp}(\nu_i) \subseteq \Delta_i,\ \|\nu_1\| = c,\ \|\nu_2\| = 1-c\big\}. \qquad (11)$$

It is known, that there exists the unique pair of measures $(\omega_{c,1}, \omega_{c,2}) \in M_c$ (**equilibrium measures**) such that

$$I(\omega_{c,1}, \omega_{c,2}) \le I(\nu_1, \nu_2),$$
$$I(\nu_1, \nu_2) := 2E(\nu_1, \nu_1) + 2E(\nu_2, \nu_2) + E(\nu_1, \nu_2) + E(\nu_2, \nu_1),$$

for all pairs $(\nu_1, \nu_2) \in M_c$. Moreover, $\operatorname{supp}(\omega_{c,1}) = [\alpha_1, \beta_{c,1}] =: \Delta_{c,1}$ and $\operatorname{supp}(\omega_{c,2}) = [\alpha_{c,2}, \beta_2] =: \Delta_{c,2}$.

Let $\mathfrak{R}_c$, $c \in (0,1)$, be a 3-sheet Riemann surface realized as follows: cut a copy of $\overline{\mathbb{C}}$ along $\Delta_{c,1} \cup \Delta_{c,2}$, which henceforth is denoted by $\mathfrak{R}_c^{(0)}$, the second copy of $\overline{\mathbb{C}}$ is cut along $\Delta_{c,1}$ and is denoted by $\mathfrak{R}_c^{(1)}$, while the third copy is cut along $\Delta_{c,2}$ and is denoted by $\mathfrak{R}_c^{(2)}$. These copies are then glued to each other crosswise along the corresponding cuts.

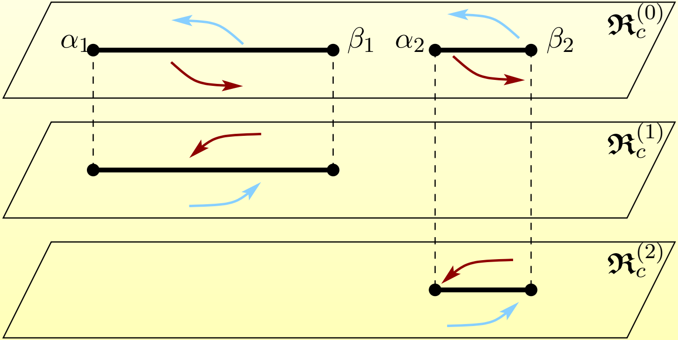


Figure: Surface $\mathfrak{R}_c$ when $\beta_{c,1} = \beta_1$ and $\alpha_{c,2} = \alpha_2$.

Let $\mathfrak{R}_c$, $c \in (0,1)$, be as above and $\chi_c(\boldsymbol{z})$ be the conformal map of $\mathfrak{R}_c$ onto $\overline{\mathbb{C}}$ such that

$$\chi_c\big(z^{(0)}\big) = z + \mathcal{O}\big(z^{-1}\big), z \to \infty. \tag{12}$$

Then, the numbers $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ are defined by

$$\chi_c\big(z^{(i)}\big) =: B_{c,i} + A_{c,i} z^{-1} + \mathcal{O}\big(z^{-2}\big),\ z \to \infty,\ i \in \{1,2\}. \tag{13}$$

For analytic weights in Angelesco model, the asymptotics of

$$A_{\vec{n}}^{(j)}, P_{\vec{n}}, L_{\vec{n}}, \quad |\vec{n}| \to \infty$$

is established off the spectrum and on the spectrum. This provides the asymptotics of Green's function of the Jacobi matrix.
We also compute all "right limits" that happen to be "periodic".

6. Nikishin system

A vector $\vec{\mu} = (\mu_1, \mu_2)$ defines a *Nikishin system* if there exists a measure τ such that

$$d\mu_2(x) = \widehat{\tau}(x) d\mu_1(x) \quad \text{and} \quad \Delta_1 \cap \Delta_\tau = \varnothing, \tag{14}$$

where $\widehat{\tau}(z)$ is the Markov function of τ, see (6), $\Delta_1 := \mathrm{ch}(\operatorname{supp} \mu_1)$, and $\Delta_\tau := \mathrm{ch}(\operatorname{supp} \tau)$ (here, $\mathrm{ch}(\cdot)$ stands for the convex hull). Given two sets E_1 and E_2, we write $E_1 < E_2$ if $\sup E_1 < \inf E_2$. Suppose

$$\Delta_\tau < \Delta_1. \tag{15}$$

The case when $\Delta_\tau > \Delta_1$ can be handled similarly. The recurrence coefficients $\{a_{\vec{n},1}, a_{\vec{n},2}\}_{\vec{n} \in \mathbb{N}^2}$, see (7)–(8), of Nikishin systems have a definite sign pattern.

Theorem

For all $\vec{n} \in \mathbb{N}^2$ *and* $j \in \{1,2\}$ *it holds that*

$$\operatorname{sgn} a_{\vec{n},j} = (-1)^{j-1}, \ \ n_2 \le n_1, \quad \textit{and} \quad \operatorname{sgn} a_{\vec{n},j} = (-1)^{j}, \ \ n_2 \ge n_1 + 1.$$

That gives rise to Jacobi matrices on rooted trees that are symmetric with respect to properly defined indefinite metric. In general, the Jacobi matrix has unbounded coefficients and the theory is not developed. However, in the recent paper

A. Aptekarev, V. Lysov, Multilevel interpolation of a Nikishin system and boundedness of the Jacobi matrices on a binary tree, 2021

the Jacobi matrix for another type of interpolation is proved to be both bounded and self-adjoint in indefinite metric.

7. MOP of the second type and Jacobi matrices on finite rooted tree

Consider, e.g., the Angelesco system (this construction can be further generalized). We consider $\vec{N} = (N_1, N_2), N_j \geq 1$ and the corresponding rectangle $0 \leq n_j \leq N_j, j \in \{1, 2\}$. Then, we build the finite binary tree $\mathcal{V}_{\vec{N}}$ and consider the self-adjoint Jacobi matrix on it generated by the recurrence (8) for $\{P_{\vec{n}}\}$ and vector $\vec{\kappa}$ for the vertex $\vec{N}$. The Hilbert space is $\ell^2(\mathcal{V}_{\vec{N}})$.
Then, the spectrum of $\mathcal{J}_{\vec{\kappa},\vec{N}}$ can be associated with the roots of polynomials.

Denote

$$P_{\Pi(O_{(p)})}(z) := \kappa_1 P_{\vec{N}+\vec{e}_1}(z) + \kappa_2 P_{\vec{N}+\vec{e}_2}(z)$$

and

$$\mathcal{E}_{\vec{\kappa},\vec{N}} := E_{\Pi(O_{(p)})} \cup \bigcup_{Y \in \mathcal{V}_{\vec{N}}:\ \#ch(Y)=2} E_{\Pi(Y)}.$$

Theorem

If $\vec{\kappa} = \vec{e}_j, j \in \{1,2\}$, *then*

$$\sigma(\mathcal{J}_{\vec{\kappa},\vec{N}}) = \mathcal{E}_{\vec{\kappa},\vec{N}}.$$

The condition $\#ch(Y) = 2$ is equivalent to $\Pi(Y) \in \mathbb{N}^2$. Hence, the set $\mathcal{E}_{\vec{\kappa},\vec{N}}$ consists of $E_{\Pi(O_{(p)})}$ and the zeroes of type II MOP that are "truly" multiple orthogonal, i.e., they satisfy orthogonality conditions on both intervals.

The orthonormal basis of eigenfunctions can be found explicitly via polynomials $\{P_{\vec{n}}\}$ and its roots.

THANK YOU!