LONG-TIME ASYMPTOTICS OF STEPLIKE SOLUTIONS OF KDV EQUATION

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Complex Analysis, Spectral Theory and Approximation meet in Linz, 2022

FWF Der Wissenschaftsfonds.

universität wien

100-th anniversary of Vladimir Aleksandrovich Marchenko

- Spectral analysis of differential operators.
- Spectral theory of random matrices.
- Inverse scattering theory.
- Homogenization theory.
- Theory of nonlinear integrable equations.

Books:

[*Inverse problem of the scattering theory* (with Z.S. Agranovich), 1960, 1963, 2020]
[*Sturm-Liouville operators and their applications*, 1977, 1986, 2011]
[*Nonlinear equations and operator algebras*, 1986, 1987]
[*Homogenization of partial differential equations* (with E. Ya. Khruslov), 2006]
[*Inverse problems in the theory of small oscillations* (with V.V. Slavin), 2015, 2018]

[V.A. Marchenko, E.Ya. Khruslov, *Boundary-value problems with a fine-grained boundary* , 1964.]
[V.A. Marchenko, L.A. Pastur, *Distribution of eigenvalues in certain sets of random matrices*, 1967.]

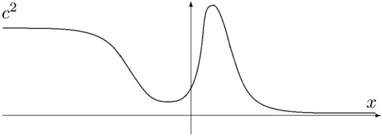
The shock-rarefaction KdV Problems

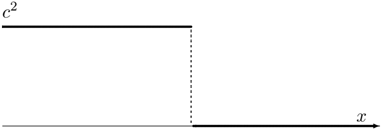
We discuss rigorous long-time asymptotics of IVP solutions

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

$$q(x,0) = q(x) \to c_\pm, \quad x \to \pm\infty.$$

Without loss of generality one can put $c_+ = 0$.
If $c_- = -c^2$, where $c \in \mathbb{R}$, we talk about the KdV shock problem. For $c_- = c^2$ we deal with the KdV rarefaction problem.



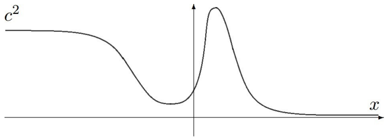


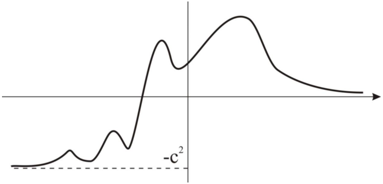
Methods: classical Inverse Scattering Transform and Nonlinear Steepest Descent.

Based on joint works with Gerald Teschl, Johanna Michor and Mateusz Piorkowski.

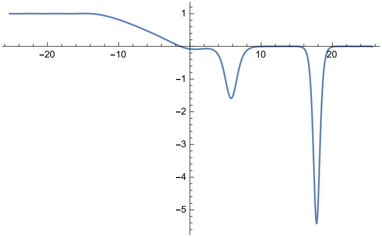
Rarefaction problem   Shock problem

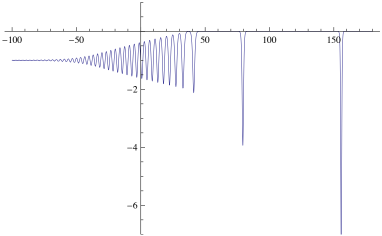
Initial profile





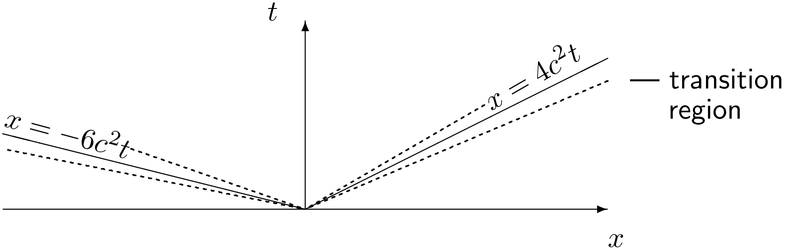
Large time





Whitham's method: A. Gurevich, L. Pitaevskii (1973);
V. Novokshenov, R. Bikbaev, R. Sharipov (80-th).
Matching asymptotics: J. Leach, D. Needham (2009, 2014)

KdV shock problem: Asymptotical solitons



E. Khruslov (1976)

In the domain $4c^2 t > x > 4c^2 t - (2c)^{-1} \ln t^M$ the solution has a form

$$q(x,t) = \sum_{n=1}^{[\frac{M}{2}]} \frac{-2c^2}{\cosh^2\{cx - 4c^3 t + \frac{1}{2}\ln t^{2n-1/2} + \phi_n\}} + O(t^{-1/2+\varepsilon})$$

as $t \to +\infty$, where the phases ϕ_n are determined by the initial scattering data.

Well-posedness of IVP

Requirements of applicability of IST and NSD methods: initial data $q(x)$ should be chosen from a class there exists the unique classical solution of the initial-value problem, satisfying

$$\int_0^{+\infty} (1+x)\left(|q(-x,t)+c^2| + |q(x,t)|\right) dx < \infty.$$

Existence of the classical solution:

- T. Kappeler (1986):

$$q \in L_{1,loc}(\mathbb{R}),\ x(q(x)+c^2) \in L_1(\mathbb{R}_-),\ x^N q(x) \in L_1(\mathbb{R}_+),\ N \geq 3.$$

- S. Grudsky, A. Rybkin (2020):

$$\sup_{|I|=1} \int_I \max(-q(x),0)dx < \infty, \quad x^N q(x) \in L_1(\mathbb{R}_+), \quad N \geq 5/2.$$

- T. Laurens (2022) (well posedness):

$$W(x) = c_1 \tanh x + c_2, \quad q(x) \in W + H^{-1}(\mathbb{R}).$$

Well posedness by IST

Definition. Let $m_0, n_0 \in \mathbb{N}$. We say that $f \in \mathcal{L}_{m_0}^{n_0}$ if $f(x) \in C^{n_0}(\mathbb{R})$ and for $n = 0, 1, ..., n_0$ satisfies

$$\int_0^{+\infty} \left(1 + |x|^{m_0}\right) \left(\left| \frac{d^n}{dx^n} f(x) \right| + \left| \frac{d^n}{dx^n} (f(-x) + c^2) \right| \right) dx < \infty.$$

I.E., J. Michor, G. Teschl, 22'

Let $q \in \mathcal{L}_{m_0}^{n_0}$ for some $m_0 \geq 3$, and $n_0 \geq m_0 + 3$. Then there exists the unique classical KdV solution $q(x,t)$ such that for all $t \in \mathbb{R}$

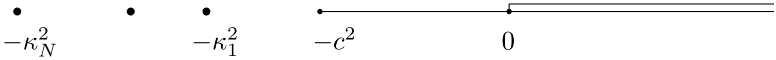
$$\int_0^{+\infty} \left(1 + |x|^{[\frac{m_0}{2}]-1}\right) \left(\left| \frac{\partial^n}{\partial x^n} q(x,t) \right| + \left| \frac{\partial^n}{\partial x^n} (q(-x,t) + c^2) \right| \right) dx < \infty,$$

for all $0 \leq n \leq n_0 - m_0$, and

$$\int_{\mathbb{R}} \left(1 + |x|^{[\frac{m_0}{2}]-1}\right) \left| \frac{\partial}{\partial t} q(x,t) \right| dx < \infty.$$

Spectrum, Jost solutions, Wronskian

Spectrum of the underlying Schrödinger operator $L(t) = -\frac{d^2}{dx^2} + q(x,t)$:



Equation $L(t)\phi = \lambda\phi$ has the Jost solutions $\phi(k,x,t)$ and $\phi_1(k,x,t)$,

$$\lim_{x\to+\infty} \mathrm{e}^{-\mathrm{i}kx}\phi(k,x,t) = \lim_{x\to-\infty} \mathrm{e}^{\mathrm{i}k_1 x}\phi_1(k,x,t) = 1.$$

Here $k^2 = \lambda$, $k_1 = \sqrt{\lambda + c^2}$. Continuous spectrum of $L(t)$ is the set $k_1 \in \mathbb{R}$, spectrum of multiplicity two - $k \in \mathbb{R}$.
Wronskian

$$W(k) := \phi_1(k,x,0)\phi'(k,x,0) - \phi_1'(k,x,0)\phi(k,x,0), \quad k \in \overline{\mathbb{C}^+ \setminus (0, \mathrm{i}c)}$$

has simple zeros at $\mathrm{i}\varkappa_j$, and, possibly at $\mathrm{i}c$.

Right scattering data

- If $W(\mathrm{i}c) = 0$ then we deal with the resonance case ($\ell = -1$), otherwise with a nonresonance one ($\ell = 1$).
- The solution $\phi(\mathrm{i}\varkappa_j, x, t) \in \mathbb{R}$ is an eigenfunction of $L(t)$. The value

$$\gamma_j = \left(\int_{\mathbb{R}} \phi^2(\mathrm{i}\varkappa_j, x, 0) dx \right)^{-1},$$

is called the right normalizing constant.

- Function

$$\chi(k) = -\frac{k|k_1|}{|W(k)|^2} = -T_1(k+0)\overline{T(k+0)}, \quad k \in [0, \mathrm{i}c]$$

- The solution $q(x,t)$ of the Cauchy problem can be uniquely restored from the right scattering data of the initial profile

$$\{\chi(k),\ k \in [0, \mathrm{i}c]; \quad R(k), \quad k \in \mathbb{R}; \ \ -\varkappa_j^2,\ \gamma_j > 0,\ j = 1, \dots, N\}.$$

The right Marchenko equation

Transformation operator for the right Jost solution:

$$\phi(k,x,t) = \mathrm{e}^{\mathrm{i}kx}\left(1 + \int_0^\infty B(x,y,t)\mathrm{e}^{2\mathrm{i}ky}dy\right), \quad q(x,t) = -\frac{\partial B(x,0,t)}{\partial x}.$$

Time-dependent Marchenko equation:

$$B(x,y,t) + F(x+y,t) + \int_0^\infty B(x,s,t)F(x+y+s,t)ds = 0,$$

where $F(x,t) = F_\chi(x,t) + F_R(x,t) + F_d(x,t):$

$$F_\chi(x,t) = \frac{2}{\pi}\int_c^0 \chi(\mathrm{i}h)\mathrm{e}^{8h^3t-2hx}dh, \quad F_d(x,t) = 2\sum_{j=1}^N \gamma_j^2 \mathrm{e}^{-2\kappa_j x + 8\kappa_j^3 t},$$

$$F_R(x,t) = \frac{2}{\pi}\,\mathrm{Re}\int_0^{+\infty} R(k)\mathrm{e}^{8\mathrm{i}k^3t+2\mathrm{i}kx}dk.$$

The right Marchenko equation

Transformation operator for the right Jost solution:

$$\phi(k,x,t)=\mathrm{e}^{\mathrm{i}kx}\left(1+\int_0^\infty B(x,y,t)\mathrm{e}^{2\mathrm{i}ky}dy\right),\ q(x,t)=-\frac{\partial B(x,0,t)}{\partial x}.$$

Time-dependent Marchenko equation:

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$$F_\chi(x,t)=\frac{2}{\pi}\int_c^0\chi(\mathrm{i}h)\mathrm{e}^{8h^3t-2hx}dh,\quad F_d(x,t)=2\sum_{j=1}^N\gamma_j^2\mathrm{e}^{-2\kappa_j x+8\kappa_j^3t},$$

$$F_R(x,t)=\frac{2}{\pi}\,\mathrm{Re}\int_0^{+\infty}R(k)\mathrm{e}^{8\mathrm{i}k^3t+2\mathrm{i}kx}dk.$$

Let δ_{ij} be the Kronecker symbol and let $A(x,t)$ be a $N\times N$ matrix with elements

$$A_{ij}(x,t)=\delta_{ij}+\frac{\gamma_j^2\mathrm{e}^{8\kappa_j^3t}}{\kappa_i+\kappa_j}\mathrm{e}^{-(\kappa_i+\kappa_j)x}.$$

Reflectionless case

The Marchenko equation with kernel $F_d(x,t)$,

$$B_d(x,y,t) + F_d(x+y,t) + \int_0^\infty B_d(x,s,t)F_d(x+y+s)ds = 0,$$

has a unique solution $B_d(x,y,t)$ such that

$$u(x,t) =: -\frac{\partial B_d(x,0,t)}{\partial x} = -2\frac{\partial^2}{\partial x^2}\log\det A(x,t).$$

For $x > \varepsilon t$, as $t \to +\infty$:

$$u(x,t) = q^{sol}(x,t) + O(\mathrm{e}^{-Ct}),$$

$$q^{sol}(x,t) = -\sum_{j=1}^N \frac{2\kappa_j^2}{\cosh^2\left(\kappa_j x - 4\kappa_j^3 t - \frac{1}{2}\log\frac{\gamma_j^2}{2\kappa_j} - \sum_{i=j+1}^N \log\frac{\kappa_j - \kappa_i}{\kappa_i + \kappa_j}\right)}.$$

Let $q \in \mathcal{L}_{m_0}^{n_0}(\mathbb{R})$, $m_0 \geq 3$, $n_0 \geq 2$, and let $\varepsilon > 0$ be arbitrary small. Put

$$G(y,t) = F_\chi(y,t) + F_R(y,t), \quad x = 4c^2 t + \frac{m_0 - \frac{3}{2} - \varepsilon}{c} \log t + \zeta, \quad \zeta \geq 0.$$

The following estimate is valid:

$$\big|G(x+y,t)\big| + \left|\frac{\partial}{\partial \zeta} G(x+y,t)\right| \leq C \frac{1}{t^{m_0 - \frac{3}{2} - \varepsilon}} \, \frac{1}{(\zeta + y + 1)^{\frac{1}{2} + \varepsilon}}.$$

In $L_2(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ introduce operators

$$[\mathcal{F}\phi](y) = \int_{\mathbb{R}_+} F_d(x+y+s,t)\phi(s)ds, \quad [\mathcal{G}\phi](y) = \int_{\mathbb{R}_+} G(x+y+s,t)\phi(s)ds.$$

The Marchenko equation can be represented as:

$$\phi + \mathcal{R}\mathcal{G}\phi = \mathcal{R}f + \mathcal{R}g, \quad \text{where}$$

$$\mathcal{R} = (\mathbb{I} + \mathcal{F})^{-1}, \ f(\cdot) = -F_d(x+\cdot,t), \ g(\cdot) = -G(x+\cdot,t), \ \phi(\cdot) = B(x,\cdot,t),$$
$$[\mathcal{R}f](\cdot) = B_d(x,\cdot,t).$$

Asymptotics in the soliton region

I.E., J. Michor, G. Teschl, 22'

Let $q(x,t)$ be the solution of KdV, with $q \in \mathcal{L}_{m_0}^{n_0}$, $m_0 \geq 3$, $n_0 \geq m_0 + 3$. Then the asymptotics of $q(x,t)$ in the region

$$x \geq 4c^2 t + \frac{m_0 - \frac{3}{2} - \varepsilon}{2c} \log t$$

is the following as $t \to \infty$:

$$q(x,t) = q^{sol}(x,t) + O\left(\frac{1}{t^{m_0 - \frac{3}{2} - \varepsilon}}\right).$$

Previous results given by RHP:

$$\int_0^{+\infty} \mathrm{e}^{(c+\delta)x} \left(|q(x)| + |q(-x) + c^2|\right) dx < \infty, \quad x^4 q^{(n)}(x) \in L_1(\mathbb{R}), 0 < n \leq 8.$$

Then for $x \geq (4c^2 + \varepsilon)t$ and $t \to \infty$:

$$q(x,t) = q^{sol}(x,t) + O(\mathrm{e}^{-C(\varepsilon)t}).$$

Riemann-Hilbert problem approach

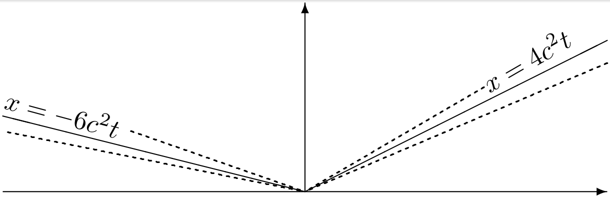
I.E., J. Michor, G. Teschl, 22'

Assume that $q(x) \in \mathcal{L}_{m_0}^{n_0}$, $m_0 \geq 4$, $n_0 \geq m_0 + 3$ and there is no resonance at $-c^2$. Assume that $x \to \infty$, $t \to \infty$ such that

$$(x,t) \in \mathcal{D} := \left\{ x \geq 4c^2 t + \frac{\beta}{c} \log t, \quad t \gg 1, \quad \beta \geq 0 \right\}.$$

Then in the domain $\mathcal{D}$ we have

$$q(x,t) = q^{sol}(x,t) + O\left(\frac{1}{t^{\nu}}\right), \quad \nu = \min\{m_0 - 3, \beta + 1\} \geq 1.$$



Vector RHP for steplike KdV

- We study asymptotics in the regime $x \to \infty$, $t \to \infty$, $\frac{x}{12t} = \xi$ is slow varying.
- We always assume that $q(\cdot, t) \in \mathcal{L}_1^3$.
- RHP can be formulated via the left (in variable k_1) and the right (in variable k) scattering data, x and t are parameters.
- We deal with a vector solution $m(k) = (m_1(k), m_2(k))$, $k \in \mathbb{C}$, of the jump problem

$$m_+(k) = m_-(k)v(k), \quad k \in \Sigma.$$

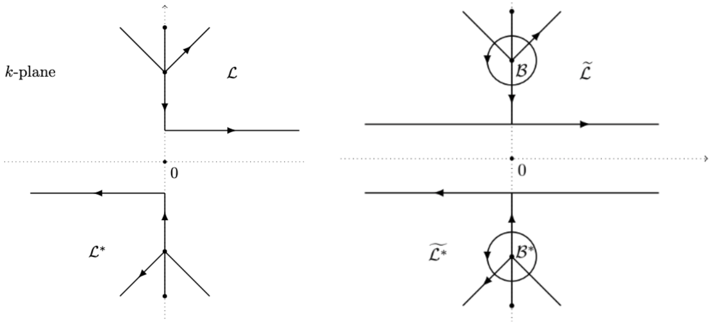
- It satisfies the symmetry condition $m_1(k) = m_2(-k)$, i.e.

$$m(-k) = m(k)\sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

- and the normalization condition: $m(k) \to (1, \ \ 1)$, $k \to \infty$.
- $q(x,t)$ is connected with $m(k) = m(k,x,t)$ by formula

$$q(x,t) = \lim_{k\to\infty} 2k^2 \left(m_1(k,x,t) m_2(k,x,t) - 1 \right).$$

Vector RHP, symmetry requirements and uniqueness



- The jump contour Σ is symmetric with respect to the map $k \mapsto -k$;
- The jump matrix is bounded on Σ and satisfies: $\det v(k) = 1$ and

$$v(-k) = \sigma_1 v(k) \sigma_1, \quad k \in \Sigma.$$

- Function $m(k)$ is holomorphic in $\mathbb{C} \setminus \Sigma$, continuous up to the boundary except at the node points of Σ (ends, self-intersections and points of discontinuity of the jump matrix), where the fourth root singularities are admissible.

From initial to pre-limit RHP

- Any transformations (conjugations, deformations) should respect the properties above.

- If $\tilde{m}(k) = m(k)D(k)$, for $|k| \gg 1$, then $D(k) = d(k)^{-\sigma_3}$, where $d(k) \to 1$ and $d(-k) = d(k)^{-1}$. In particular,

$$q(x,t) = \lim_{k\to\infty} 2k^2 \left(\tilde{m}_1(k,x,t)\tilde{m}_2(k,x,t) - 1\right).$$

- The aim of transformations is to get $\tilde{m}(k)$ satisfying a jump problem $\tilde{m}_+(k) = \tilde{m}_-(k)\tilde{v}(k)$ and all properties above, and such that

$$\|\tilde{v}(k) - v^{mod}(k)\|_{L^p(\tilde{\Sigma}\setminus(\mathcal{B}\cup\mathcal{B}^*))} = O(t^{-\alpha}), \quad \alpha > 0,\ p \geq 1,$$

  where the model vector RHP $m_+^{mod}(k) = m_-^{mod}(k)v^{mod}(k)$ with piecewise constant jump matrix $v^{mod}(k)$ is exactly solvable.

- Uniqueness of the model problem solution?

- Can we compare $\tilde{m}_1(k)\tilde{m}_2(k)$ with $m_1^{mod}(k)m_2^{mod}(k)$ as $k \to \infty$?

Standard asymptotic analysis for vector RHP

Let $M^{mod}(k)$ be an invertible matrix solution of model RHP,

$$M_+^{mod}(k) = M_-^{mod}(k) v^{mod}(k), \quad M^{mod}(k) \to \mathbb{I}, \quad k \to \infty.$$

Then

$$M^{mod}(-k) = \sigma_1 M^{mod}(k) \sigma_1, \quad k \in \mathbb{C},$$

and

$$m^{mod}(k) = (1, \;\; 1) M^{mod}(k).$$

In vicinities of the parametrix points we solve the local matrix RHP problems. Solutions are symmetric, invertible, and satisfy

$$M^{par}(k)[M^{mod}(k)]^{-1} = \mathbb{I} + O(t^{-\alpha}), \quad k \in \partial B \cup \partial B^*, \;\; \alpha = 1/2, 1.$$

Introduce "the error vector"

$$m^{err}(k) = \tilde{m}(k)(M^{\mathsf{as}}(k))^{-1}, \quad M^{\mathsf{as}}(k) := \begin{cases} M^{\mathrm{par}}(k), & k \in (\mathcal{B} \cup \mathcal{B}^*), \\ M^{\mathrm{mod}}(k), & k \in \mathbb{C} \setminus (\mathcal{B} \cup \mathcal{B}^*). \end{cases}$$

"Small norm" arguments

$m^{err}(k)$ possess symmetry and normalization conditions,

$$m_+^{err}(k) = m_-^{err}(k)(\mathbb{I} + W(k)), \quad \|k^j W(k)\|_{L^p(\hat{\Sigma})} \le C t^{-\alpha}, \quad p \in [1, \infty].$$

$$m^{err}(k) = (1,\ 1) + \frac{1}{2\pi \mathrm{i}} \int_{\hat{\Sigma}} \frac{(1,\ 1)W(s)}{s-k} ds + (1,\ -1) O(t^{-2\alpha}) O(k^{-1}).$$

$$\tilde{m}(k) = m^{err}(k) M^{\mathrm{mod}}(k) = m^{\mathrm{mod}}(k) + \frac{f_0(\xi, t)}{2\mathrm{i}kt}(1,\ -1) M^{\mathrm{mod}}(k)(1 + O(k^{-1}))(1 + O(t^{-\alpha})).$$

This allows us to conclude that

$$q(x,t) = q^{mod}(x,t) + O(t^{-\alpha}), \quad t \to \infty$$

uniformly with respect to ξ in a given region.

This scheme works only for some regions in rarefaction problem.

Matrix model solution, existence, uniqueness

Proposition

Assume that $m_\pm(0) = (0,0)$. Then there is no invertible matrix solution with admissible singularities.

For the KdV shock problem in the soliton and in the middle regions there are arbitrary large pairs (x,t) such that $m_\pm^{mod}(0,x,t) = (0,0)$.
Thus, one has to admit a pole for the associated matrix solution. Such a solution is not unique.
A suitable solution $M^{mod}(k)$ should be such that the error vector does not have pole at point 0!

The initial RHP does not have invertible matrix solutions iff $\phi(0,x,t) = 0$, where $\phi(0,x,t)$ is the right Jost solution.

RHP for the KdV shock problem via right scattering data

Assume that $q(\cdot,t)\in\mathcal{L}_1^3$. Set $\Sigma=\mathbb{R}_+\cup[\mathrm{i}c,0]$, $\Sigma^*=\{k:-k\in\Sigma\}$.

In $\overline{\mathbb{C}^+\setminus(\Sigma\cup\Sigma^*)}$ introduce the vector function $m(k)=(m_1(k),m_2(k))$ (x and t - parameters):

$$m(k,x,t)=\begin{cases}\begin{pmatrix}T(k,t)\phi_1(k,x,t)\mathrm{e}^{\mathrm{i}kx}, & \phi(k,x,t)\mathrm{e}^{-\mathrm{i}kx}\end{pmatrix}, & k\in\mathbb{C}^+\setminus(0,\mathrm{i}c],\\ m(-k,x,t)\sigma_1, & k\in\mathbb{C}^-\setminus[-\mathrm{i}c,0),\end{cases}$$

$T(k,t)=\frac{2\mathrm{i}k}{W(k,t)}$ is the right transmission coefficient.

$$m(k)=\begin{pmatrix}1 & 1\end{pmatrix}-\frac{1}{2\mathrm{i}k}\int_x^{+\infty}q(y,t)dy\begin{pmatrix}-1 & 1\end{pmatrix}+O\left(\frac{1}{k^2}\right).$$

$$m_1(k)m_2(k)=2\mathrm{i}kG(x,x,k,t)=1+\frac{q(x,t)}{2k^2}(1+o(1)).$$

Statement of RHP

$m(k)$ is the unique solution of the following RHP: to find a meromorphic away from $\Sigma \cup \Sigma^*$ function $m(k)$ satisfying:

1. the jump condition $m_+(k) = m_-(k) v(k)$, where

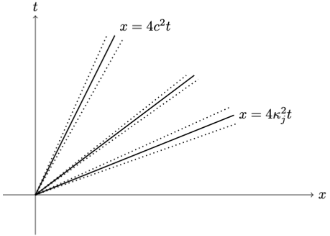
$$v(k) = \begin{cases} \begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)} \mathrm{e}^{-2t\Phi(k)} \\ R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, & k \in \mathbb{R}_+, \\ \begin{pmatrix} 1 & 0 \\ \chi(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, & k \in [\mathrm{i}c, 0], \\ \sigma_1(v(-k))\sigma_1, & k \in \Sigma^*; \end{cases}$$

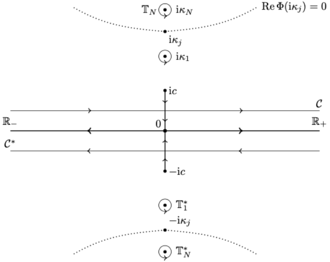
2. the symmetry and the normalizing conditions;
3. the pole conditions: $\mathrm{Res}_{\mathrm{i}\kappa_j} m_1(k) = \lim_{k \to \mathrm{i}\kappa_j} \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} m_2(\mathrm{i}\kappa_j)$,
4. For $\ell = 1$ vector $m(k)$ is bounded as $k \to \mathrm{i}c$. For $\ell = -1$:

$$m(k) = \Big(C_1(x,t)(k - \mathrm{i}c)^{-1/2}, C_2(x,t) \Big) (1 + o(1)) \quad C_1 C_2 \neq 0;$$

Soliton region, $q \in \mathcal{L}_4^7$, $\xi = \frac{x}{12t} \geq \frac{c^2}{3}$

Phase function: $\Phi(k) = \Phi(k,x,t) = 4\mathrm{i}k^3 + \mathrm{i}k\frac{x}{t} = 4\mathrm{i}k^3 + 12\mathrm{i}k\xi$.





Factorization:

$$v(k) = \begin{pmatrix} 1 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, \quad k \in \mathbb{R}.$$

Important identity for analytical continuation:

$$(\mathrm{i})^{l+1} R^{(l)}(+0) + (-\mathrm{i})^{l+1} R^{(l)}(-0) = \lim_{h\to+0} \frac{d^l}{dh^l}\chi(\mathrm{i}h).$$

[J. Lenells, *The nonlinear steepest descent method for Riemann-Hilbert problems of low regularity*, Indiana Univ. Math. J. **66:4** (2017), 1287–1332.]

Matrix model RHP for $(4\kappa_j^2 - \varepsilon)t \le x \le (4\kappa_j^2 + \varepsilon)t$

$$\mathrm{Res}_{\mathrm{i}\kappa_j} M(k,j) = \lim_{k\to \mathrm{i}\kappa_j} M(k,j) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma_j^2(x,t) & 0 \end{pmatrix},$$

$$\mathrm{Res}_{-\mathrm{i}\kappa_j} M(k,j) = \lim_{k\to -\mathrm{i}\kappa_j} M(k,j) \begin{pmatrix} 0 & -\mathrm{i}\gamma_j^2(x,t) \\ 0 & 0 \end{pmatrix},$$

$$\gamma_j^2(x,t) = \gamma_j^2 \mathrm{e}^{8\kappa_j^3 t - 2\kappa_j x} \prod_{l=j+1}^{N} \left(\frac{\kappa_l - \kappa_j}{\kappa_l + \kappa_j} \right)^2.$$

$$M(\infty, j) = \mathbb{I}, \quad M(-k,j) = \sigma_1 M(k,j) \sigma_1.$$

$$M(k,j) = \begin{pmatrix} 1 + \frac{\mu_j(x,t)}{2k} & -\frac{\mu_j(x,t)}{2k}\frac{k-\mathrm{i}\kappa_j}{k+\mathrm{i}\kappa_j} \\ \frac{\mu_j(x,t)}{2k}\frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j} & 1 - \frac{\mu_j(x,t)}{2k} \end{pmatrix},$$

$$\mu_j(x,t) = \frac{\mathrm{i}\gamma_j^2(x,t)}{1 + (2\kappa_j)^{-1}\gamma_j^2(x,t)}.$$

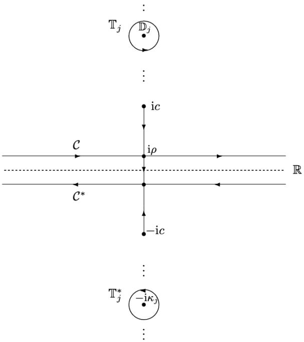
[K. Grunert and G. Teschl, *Long-time asymptotics for the Korteweg–de Vries equation via nonlinear steepest descent*, Math. Phys. Anal. Geom. **12** (2009), 287–324.]

Region $-c^2/2+\varepsilon \le \xi \le c^2/3-\varepsilon$

Assumption:

$$\int_0^\infty \mathrm{e}^{\eta x}\left(|q(-x)+c^2|+|q(x)|\right)dx<\infty,\ \eta>0.$$

Previously $\eta>c$. From meromorphic to holomorphic statement of RHP:



$$m^{\mathrm{ini}}(k)=\begin{cases} m(k)A_j(k)\,(P(k)Q(k))^{-\sigma_3}, & k\in\mathbb{D}_j,\\ m(k)A_0(k)\,(P(k)Q(k))^{-\sigma_3}, & k\in\Omega;\\ m(k)\,(P(k)Q(k))^{-\sigma_3}, & k\in\mathbb{C}^+\setminus(\Omega\cup_j\mathbb{D}_j),\\ m^{\mathrm{ini}}(-k)\sigma_1, & k\in\mathbb{C}^-,\end{cases}$$

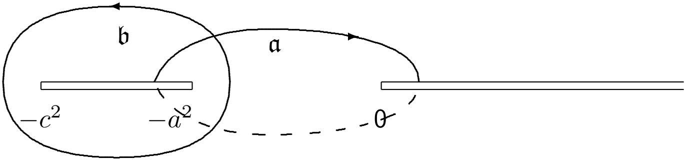
where

$$P(k):=\prod_{j=1}^N\frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j},\quad Q(k):=\left(\frac{k-\mathrm{i}c}{k+\mathrm{i}c}\right)^{\frac{\ell}{4}},$$

$$A_j(k)=\begin{pmatrix}1 & -\frac{k-\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2\mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}\\ 0 & 1\end{pmatrix},\quad A_0(k)=\begin{pmatrix}1 & 0\\ -R(k)\mathrm{e}^{t\Phi(k)} & 1\end{pmatrix},$$

$$\Omega=\{k:0<\mathrm{im}k<\rho\},\ \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

"g" - function as Abel integral



RS $\mathbb{M}$ is associated with $\mathcal{R}(\lambda) = \sqrt{-\lambda(\lambda + c^2)(\lambda + a^2)}$.

- $\Omega_j(p)$ - Abel integrals of the 2nd kind: $\int_{\hat{\mathfrak{a}}} d\Omega_{1,3} = 0$.

$$d\Omega_1 = \frac{\mathrm{i}}{2\sqrt{\lambda}}(1 + O(\lambda^{-1}))d\lambda, \quad d\Omega_3 = -\frac{3\mathrm{i}}{2}\sqrt{\lambda}(1 + O(\lambda^{-1}))d\lambda.$$

$$\Omega_1(p)x + 4\Omega_3(p)t = -6t \int_{-c^2}^{p} \frac{(\lambda - \mu_1)(\lambda - \mu_2)}{\mathcal{R}(\lambda)} d\lambda$$
$$= \mathrm{i}kx + 4\mathrm{i}k^3 t + O(k^{-1}) = t\Phi(k) + O(k^{-1}).$$

- Normalization implies $\mu_2(\xi) \in (-a^2, 0)$. For any $\xi \in (-c^2/2, c^2/3)$ there exist unique $a(\xi) \in (0, c)$ such that $\mu_1(\xi) = a(\xi)$. Function $a(\xi)$ is monotonous with $a(-c^2/2) = 0$ and $a(c^2/3) = c$.

g - function on k plane

Conjugation: $m^{(1)}(k) = m^{\mathrm{ini}}(k)\mathrm{e}^{(\mathrm{i}tg(k) - t\Phi(k))\sigma_3}$,

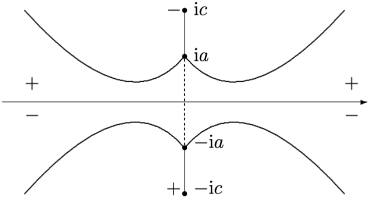
$$g(k) = 12\int_{\mathrm{i}c}^{k}\left(k^2 + \xi + \frac{c^2 - a^2}{2}\right)\sqrt{\frac{k^2+a^2}{k^2+c^2}}\,dk.$$

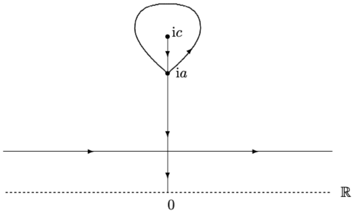
Properties:

- $g_-(k) + g_+(k) = 0$ as $k \in [\mathrm{i}c, \mathrm{i}a] \cup [-\mathrm{i}a, -\mathrm{i}c]$;
- $t(g_-(k) - g_+(k)) = Vx + 4Wt$ as $k \in [\mathrm{i}a, -\mathrm{i}a]$,

where $\mathrm{i}V = \int_{\mathfrak{b}} d\Omega_1$; $\mathrm{i}W = \int_{\mathfrak{b}} d\Omega_3$.

Signature table for $\operatorname{Im} g$:





Model vector RHP

The following RHP has a unique solution: find a vector-valued function $m^{\mathrm{mod}}(k) = (m_1^{\mathrm{mod}}(k)\ m_2^{\mathrm{mod}}(k))$ holomorphic in the domain $\mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$, which is continuous up to the boundary except at points of the set $\mathcal{G}^{\mathrm{mod}} := \{\mathrm{i}c, \mathrm{i}a, -\mathrm{i}a, -\mathrm{i}c\}$, and satisfies the jump condition $m_+^{\mathrm{mod}}(k) = m_-^{\mathrm{mod}}(k) v^{\mathrm{mod}}(k)$:

$$v^{\mathrm{mod}}(k) = v^{\mathrm{mod}}(k,x,t,\xi) = \begin{cases} \mathrm{i}\sigma_1, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \mathrm{e}^{(-4\mathrm{i}Wt - \mathrm{i}Vx - \mathrm{i}\Delta)\sigma_3}, & k \in [\mathrm{i}a, 0], \\ \sigma_1 v^{\mathrm{mod}}(-k)\sigma_1, & k \in [-\mathrm{i}c, 0], \end{cases}$$

where

$$\Delta = \frac{\int_{\mathrm{i}a}^{\mathrm{i}c} \frac{2\log\left|T(s)\prod_{j=1}^{N}\frac{s-\mathrm{i}\kappa_j}{s+\mathrm{i}\kappa_j}\right| + \log\left|\frac{s+\mathrm{i}c\ell}{s}\right|}{|(s^2+c^2)(s^2+a^2)|^{1/2}} ds}{\int_0^{\mathrm{i}a}\left((s^2+c^2)(s^2+a^2)\right)^{-1/2} ds} - \frac{\pi\ell}{2};$$

the symmetry $m^{\mathrm{mod}}(-k) = m^{\mathrm{mod}}(k)\sigma_1$, and normalization $m^{\mathrm{mod}}(\infty) = (1\ \ 1)$ conditions.

At any point $\kappa \in \mathcal{G}^{\mathrm{mod}}$ the vector function $m^{\mathrm{mod}}(k)$ can have at most a fourth root singularity: $m^{\mathrm{mod}}(k) = O((k-\kappa)^{-1/4})$, $k \to \kappa$.

Solution of model RHP

- RS $\mathbb{X} = \mathbb{X}(\xi)$ is associated with $w(k) = \sqrt{(k^2+c^2)(k^2+a^2)}$.
- $d\omega$ - Abel holomorphic differential, $\int_{\mathfrak{a}} d\omega = 1$, $\tau = \int_{\mathfrak{b}} d\omega$, $A(k) = \int_{\mathrm{i}c}^{k} d\omega$ - Abel map on the upper sheet of $\mathbb{X}$.
- $\theta_3(z \,|\, \tau)$ - the Jacobi theta-function.
- Put $\gamma(k) = \sqrt[4]{\frac{k^2+a^2}{k^2+c^2}}$, where $\arg\gamma(0) = 0$, and

$$\alpha(k) = \frac{\theta_3\left(2A(k) - \frac{1}{2} - \Lambda \,\middle|\, 2\tau\right)}{\theta_3\left(2A(k) - \frac{1}{2} \,\middle|\, 2\tau\right)}, \quad \Lambda = \frac{Vx + 4tW + \Delta}{2\pi}.$$

The solution of model vector RHP is given by:

$$m^{\mathrm{mod}}(k) = \left(\gamma(k)\frac{\alpha(k)}{\alpha(\infty)},\ \gamma(k)\frac{\alpha(-k)}{\alpha(\infty)}\right).$$

Computing asymptotics:

- We observe that:

$$\alpha_{\pm}(0) = \frac{\theta_3\left(\mp\tau - 1 - \Lambda \,\middle|\, 2\tau\right)}{\theta_3\left(\pm\tau + 1 \,\middle|\, 2\tau\right)},$$

therefore for $\Lambda = \frac{1}{2}(\mathrm{mod}\, n)$ we have $m_{\pm}^{\mathrm{mod}}(0) = (0, 0)$. Thus, for x, t such that

$$xV + 4tW + \Delta = \pi(2n + 1)$$

there is no nonsingular matrix solution of model RHP.

- Function

$$S(k) = \frac{\alpha(k)\alpha(-k)}{\alpha^2(\infty)},$$

does not have jumps, is even, and $S(\infty) = 1$. It is a meromorphic (rational) function of λ. As a function of λ it has one simple zero at point $\lambda(x, t) \in [-a^2, 0]$ and simple pole at point $\lambda = -a^2$. Thus

$$S(k) = \frac{k^2 - \lambda(x,t)}{k^2 + a^2}, \quad m_1^{mod}(k) m_2^{mod}(k) = \frac{k^2 - \lambda(x,t)}{\sqrt{(k^2 + a^2)(k^2 + c^2)}}.$$

Computing asymptotics:

Point $p(x,t) = (\lambda(x,t), \pm) \in \mathbb{M}$ solves the Jacobi inversion problem:

$$\int_{p_0}^{p(x,t)} d\hat{\omega} = \mathrm{i}(Vx + 4Wt)(\text{mod } 2\pi\mathrm{i}), \quad \int_{-a^2}^{p_0} d\hat{\omega} = \mathrm{i}\Delta,$$

where $d\hat{\omega}$ is a holomorphic differential on $\mathbb{M}$ normalized as $\int_{\mathfrak{a}} d\hat{\omega} = 2\pi\mathrm{i}$. Thus, for any $\xi \in (-c^2/2, c^2/3)$ function

$$q^{\mathsf{mod}}(x,t,\xi) = -c^2 - a^2(\xi) - 2\lambda(x,t,\xi),$$

is a periodic one gap solution to the KdV equation on the spectrum $[-c^2, -a^2] \cup \mathbb{R}_+$ associated with the initial Dirichlet divisor $p_0 = (\lambda(0,0,\xi), \pm)$.

Asymptotics in the modulated elliptic wave region

I.E., M. Piorkowski, G. Teschl, 22'

Assume that the initial datum satisfies:

$$\int_0^{+\infty} \mathrm{e}^{\eta x}(|q(x)| + |q(-x) + c^2|)dx < \infty, \qquad q \in \mathcal{L}_4^7.$$

Then for $x \to \infty$, $t \to \infty$ such that $\frac{x}{t} \in I_\varepsilon = [-6c^2 + \varepsilon, 4c^2 - \varepsilon]$ the following asymptotics is valid:

$$q(x,t) = q^{mod}(x, t, \frac{x}{12t}) + O(t^{-1}),$$

uniformly with respect to $\frac{x}{t} \in I_\varepsilon$.

Generalization: a finite gap background

- Let $p(x)$ be a one gap periodic potential. Initial data $q(x)$:

$$\int_0^\infty (1+|x|^{m_0})(|q(x)|+|q(-x)-p(-x)|)dx<\infty.$$

- Right scattering data $\{\chi(k),\ k\in[\mathrm{i}b,\mathrm{i}c];\quad R(k),\ k\in\mathbb{R}\ \}$.
- Reflectionless case, RHP: $m_+(k)=m_-(k)v(k)$,

$$v(k)=\begin{pmatrix}1 & 0\\ \chi(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix}, k\in[\mathrm{i}c,\mathrm{i}b];\ v(-k)=\sigma_1 v(k)\sigma_1.$$

- This RHP coincides the RHP for the KdV solitonic gas.

Thank you!