Differential equations $w'' + Aw = 0$, where A is entire of finite order, having a basis of solutions with only real zeros

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$$w'' + Aw = 0, \quad A \quad \text{is entire,}$$

of finite order:

$$\rho(A) = \limsup_{r \to \infty} \frac{\max_{|z|=r} \log \log |A(z)|}{\log r} < \infty.$$

When there is a pair of linearly independent solutions with all zeros real?

If A is a polynomial, this is only possible when $A = \mathrm{const}$ (Hellerstein, Chen and Williamson, 1984).

Our first result is an analog of this theorem without any a priori restriction on A:

Theorem 1. *If $w'' + Aw = 0$ with entire A has three pairwise linearly independent solutions with all zeros real then A is constant*

This can be derived from a result of Bergweiler and Eremenko (2009) saying that if for a meromorphic function F the preimage of three points is contained in the real line, then F maps the real line into a circle, unless the Schwarzian derivative of F is constant. Howewer the main topic of this talk is equations $w'' + Aw = 0$ having two linearly independent solutuons with all zeros real, and this leads to new interesting classes of entire and meromorphic functions.

We show that such A are quite exceptional, in particular, $\rho(A)$ is either an odd integer, or half of an odd integer. Moreover, A has completely regular growth in the sense of Levin–Pfluger.

If w_1, w_2 are linearly independent solutions, then $F = w_1/w_2$ is a *locally univalent* meromorphic function, and every locally univalent meromorphic function arises in this way: w_1, w_2 can be recovered by the formulas

$$w_1 = 1/\sqrt{F'}, \quad w_2 = F/\sqrt{F'},$$

and

$$2A = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2.$$

The expression in the RHS is called the *Schwarzian*. Let

$$E = w_1 w_2 = \frac{F}{F'}.$$

$E = w_1 w_2$ is an entire function with the property

$$E(z) = 0 \quad \to \quad E'(z) \in \{\pm 1\}.$$

Entire functions with this property are called Bank-Laine functions, and each Bank–Laine function is a product of a normalized pair of solutions of $w'' + Aw = 0$ with some entire A. Notice also the differential equation

$$-2\frac{E''}{E} + \left(\frac{E'}{E}\right)^2 = \frac{1}{E^2} = 4A.$$

If A is transcendental, all non-trivial solutions of $w'' + Aw = 0$ are of infinite order. However their product E can be of finite order, for example, when $A = p'' - (p')^2 - e^{4p}$, with a polynomial p, then there are two solutions

$$w_{1,2}(z) = \exp\left(-p(z) \pm \int_0^z e^{2p(\zeta)} d\zeta\right),$$

whose product is $w_1 w_2 = e^{-2p}$.

Theorem 2 *If zeros of E lie on finitely many rays, then E is of finite order if and only if A is of finite order.*

Theorem 2 is a relatively simple consequance of the deep result of Joe Miles saying that if zeros of an entire function of infinite order belong to finitely many rays then 0 is almost deficient:

$$\lim_{r\to\infty, r\notin L} \frac{N(r,1/f)}{T(r,f)} = 0,$$

where L is a set of zero logarithmic density.

Langley's Theorem (2020). *Suppose that* E *is of finite order and has only real zeros, and* A *is non-constant. Then* F^{-1} *has infinitely many singularities over* $0, \infty$*, but the number* m *of singularities over points in* $\mathbf{C}^*$ *is finite. Moreover,*
If the sequence of zeros of E *is infinite and one-sided then* $m \geq 2$ *and* $\rho(E) \geq m - 1/2$.
If the sequence of zeros of E *is two-sided, then* $m \geq 4$*, and* $\rho(E) \geq m - 1$.

This suggests that one has to study locally univalent meromorphic functions whose inverses have infinitely many singularities over $0, \infty$ and finitely many singularities over points in $\mathbf{C}^*$.

Let $F : \mathbf{C} \to \overline{\mathbf{C}}$ be a local homeomorphism. An *asymptotic curve* is a curve $\gamma : [0, 1) \to \mathbf{C}$ such that

$$\gamma(t) \to \infty \quad \text{and} \quad f(\gamma(t)) \to a \in \overline{\mathbf{C}}.$$

This a is called an asymptotic value. If there are only finitely many asymptotic values a_j, then there are disjoint disks $D_j \subset \overline{\mathbf{C}}$ containing them. An unbounded component of $F^{-1}(D_j)$ is called a *tract over* a_j.

Tracts are in bijective correspondence with *singularities* of F^{-1}.

A local homeomorphism which commutes with complex conjugation is called *symmetric*.

For every local homeomorphism $F : \mathbf{C} \to \overline{\mathbf{C}}$ there is a homeomorphism ϕ such that $F \circ \phi$ is a meromorphic function, *either in the unit disk or in* $\mathbf{C}$. This is a corollary of the Uniformization Theorem.

Theorem 3 *Let* $F : \mathrm{C} \to \overline{\mathrm{C}}$ *be a symmetric local homeomorphism with all zeros and poles real, and such that* F^{-1} *has infinitely many singularities over* $0, \infty$ *and* $m < \infty$ *singularities over points in* C^*. *Then there is a symmetric homeomorphism* $\phi : \mathrm{C} \to \mathrm{C}$ *such that* $F_0 = F \circ \phi$ *is a meromorphic function and its Schwarzian* $2A$ *and* $E = F/F'$ *have the following properties:*
a) If the sequence of zeros and poles of F *is finite, then* $m \geq 1$ *and* $\rho(A) = \rho(E) = m$,
b) If the sequence of zeros and poles of F *is infinite and one-sided, then* $m \geq 2$ *and* $\rho(A) = \rho(E) = m - 1/2$,
c) If the sequence of zeros and poles of F *is two-sided, then* $m \geq 4$ *and* $\rho(A) = \rho(E) = m - 1$,
d) A *and* E *are of completely regular growth in the sense of Levin–Pfluger.*
e) All indicated values of m *can actually occur.*

An entire function f is of completely regular growth in the sense of Levin–Pfluger, if

$$r^{-\rho} \log |f(rz)|, \quad \rho = \rho(f)$$

has a non-zero limit when $r \to \infty$ in the space of Schwartz distributions $D'(\mathrm{C})$, or equivalently in L^1_{loc} with respect to the Lebesgue measure.
The limit function necessarily has the form $r^\rho h(\theta)$ and h is called the *indicator* of f. Theorem 3 permits to find easily the indicators of A and E. They are determined up to a positive multiple by the number m and by the presence of infinitely many of positive or negative zeros.

In particular, Theorem 3 implies that A and $E = F_0/F_0'$ are of *normal type*.

One can prove a version of this theorem using methods of classical analysis (estimates for subharmonic functions etc.) In this version one assumes that E is a real entire function of finite order, with all zeros real, and satisfying the Bank–Laine property

$$E(z) = 0 \quad \rightarrow \quad E(z) \in \{\pm 1\}.$$

However this proof also gives a weaker conclusion: it does not give the normal type, but gives a completely regular growth in the generalized sense:

$$\lim_{r\to\infty} \frac{\log |E(rz)|}{\log M(r, E)} \quad \text{exists.}$$

Our main result, Theorem 3, can be compared with the classical

Theorem of R. Nevanlinna (1932) *Let* $F : \mathbf{C} \to \overline{\mathbf{C}}$ *be a local homeomorphism whose inverse has* $m < \infty$ *singularities. Then there is a homeomorphism* $\phi : \mathbf{C} \to \mathbf{C}$ *such that* $F_0 = F \circ \phi$ *is a meromorphic function whose Schwarzian is a polynomial of degree* $m - 2$.

In comparison with this, we allow infinitely many singularities over two points, but have an important additional condition that all zeros and poles are real.

A modern way to prove Nevanlinna's theorem is to show that

a) To show that the Riemann surface spread over the sphere corresponding to the function F^{-1} is a result of gluing of finitely many *logarithmic ends*, and then writing an explicit quasiregular uniformization of it.

b) If the quasiconformal dilatation of this explicit uniformization is small, the quasiconformal uniformization is close to a conformal one in view of Teichmüller-Wittich–Belinskii distortion theorem.

We recall that a *logarithmic end* is a Riemann surface spread over the sphere uniformized by the restriction of function $f = L \circ \exp$ onto the upper half-plane, where L is a linear-fractional function.

Our method follows the same general idea, but in addition to the logarithmic end we need one more type of end, which we call a *B-end*. It is uniformized by the restriction of a function of the form

$$f(z) = a \exp\left(\int_0^z R(\zeta) e^{-\zeta^2} d\zeta\right) + b$$

onto the upper half-plane. Here R is a rational function such that f is meromorphic in C.

So our proof consists of the same two parts as the proof of Nevanlinna's theorem:

a) Topological part, which shows that the Riemann surface spread over the sphere corresponding to F^{-1} in the theorem can be split into at most two logarithmic ends, finitely many B-ends and a compact part.

b) Analytic part, where we use the explicit uniformization of the parts, and construct a quasiregular uniformization of the whole surface, which has controlled dilatation, allowing to use the Teichmüller–Wittich–Belinski theorem. This permits to make conclusions about the asymptotic behavior of F.

Let D be a bordered surface, and $f : D \to \overline{\mathbb{C}}$ a *topologically holomorphic* map, which means that in local coordinates it looks like $z \mapsto z^n$ for some integer $n \geq 1$. For example, local homeomorphisms are topologically holomorphic, with $n = 1$ at all points.

A *Riemann surface spread over the sphere* is a pair (D, f), where D is a bordered surface, and f a topologically holomorphic map, modulo the following equivalence relation: $f_1 = f_2 \circ \phi$, where $\phi : D_1 \to D_2$ is a homeomorphism. We call f a uniformization function of this Riemann surface.

By the Uniformization theorem, when D is open and simply connected, there is always a holomorphic uniformization function defined in a plane region.

Let (D_1, f_1), (D_2, f_2) be two Riemann surfaces spread over the sphere. Let $I_1 \subset \partial D_1$ and $I_2 \subset \partial D_2$ be two boundary arcs, and $\phi : I_1 \to I_2$ is an orientation reversing homeomorphism (we assume that ∂D_j are equipped with their standard orientations). Assume that

$$f_2(\phi(z)) = f_1(z), \quad z \in I_1.$$

Then one can *glue* (D_1, f_1) with (D_2, f_2) and obtain a new Riemann surface spread over the sphere (D, f) such that D is divided by an arc into two subregions, and the restrictions of f on these subregions are equivalent to the initial elements.

Topological part of the proof of the Theorem 3.
Notice that singularities of the Riemann surface spread over $\overline{\mathbf{C}}$ corresponding to F^{-1} lie over finitely many points. We cut it by making simple cuts which project into some intervals $(a_j, a_j + \epsilon]$, where a_j are all asymptotic values in $\mathbf{C}^*$, and $\epsilon > 0$ is small enough. Preimages of these cuts γ_j are disjoint curves in $\mathbf{C}$ tending to ∞. Connecting their starting points by some Jordan curve, we break the plane into unbounded regions D_j and a bounded region G. This picture can be made symmetric with respect to the real line. So at most two regions G_j are symmetric and contain rays of the real line, and all other G_j can be made disjoint from the real line and they are paired by the symmetry.

Suppose that F has infinitely many positive zeros and poles. Then the region G_0 (containing a positive ray) is symmetric and contains these zeros and poles. We show that restriction of F onto G_0 has no asymptotic curves with asymptotic values 0 or ∞, and maps G_0 onto a logarithmic end which can be uniformized by

$$a \tan z + b.$$

Similar logarithmic end we obtain if F has infinitely many negative zeros and poles.

Restrictions of F onto all other regions G_j have no zeros, no poles, and the only asymptotic curves with asymptotic values over C^* are those along the boundary curves of G_j. We show that this implies that this restrictions are equivalent to B-ends.
This completes the first (topological) part of the proof.

Analytic part.

Notice that our explicit uniformizing functions of pieces have exponential asymptotics on the boundary curves of their regions G_j. So one can transplant them to appropriate angular sectors so that their behaviors on the boundary rays almost match.

To make them match completely, one performs quasiconformal deformations in these sectors, which are close to identity maps.

Gluing these pieces together, we obtain a quasiregular uniformization of the Riemann surface corresponding to F^{-1} (minus a compact piece) by a quasiregular map F_1 which is actually conformal except the set of finite logarithmic area:

$$\int\int_E \frac{dxdy}{1+x^2+y^2} < \infty.$$

This last condition is the condition of Teichmüller–Wittich–Belinskii theorem which ensures the existence of a quasiconformal homeomorphism ϕ such that $F = F_0 \circ \psi$, where $F_0 : \mathrm{C} \to \overline{\mathrm{C}}$ is a meromorphic function. This homeomorphism has the property

$$\phi(z) \sim z,$$

which allows us to derive all asymptotic properties of F_0 claimed in the Theorem 3.