Infinite Dimensional ODEs with Discrete Convolution of Higher Dimension and Combinatorial Analysis

Fei Xu
(Jilin University)

Joint work with David Damanik (Rice University)
and Yong Li (Jilin University)

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As the title suggests, what are Infinite Dimensional ODEs with Discrete Convolution of Higher Dimension and how do they arise?

To put it simply, they can be generated by **a nonlinear evolution PDE** (KdV, NLS, KG, etc) and **spatially quasi-periodic solutions** (represented by spatially quasi-periodic Fourier series).

In what follows, we will give a more detailed introduction in the generalized KdV framework.

Let's start from **the quasi-periodic Cauchy problem** for gKdV.

The generalized KdV

On the real line $\mathbb{R}$, consider the generalized KdV (gKdV for short) equation

$$\partial_t u + \partial_x^3 u + u^{\mathfrak{p}-1}\partial_x u = 0, \qquad (1)$$

where ∂_t and ∂_x stand for the derivative w.r.t. to time and space respectively, $2 \leq \mathfrak{p} < \infty$ is a natural number.

[1] D. J. Korteweg and G. de Vries. "On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves". In: *Philos. Mag. (5)* 39.240 (1895), pp. 422–443. ISSN: 1941-5982. DOI: 10.1080/14786449508620739. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3363408.

[2] George L. Lamb Jr. *Elements of soliton theory*. Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1980, pp. xiii+289. ISBN: 0-471-04559-4. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=591458.

[3] Masayoshi Tsutsumi, Toshio Mukasa, and Riichi Iino. "On the generalized Korteweg-deVries equation". In: *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 46.9 (1970). DOI: 10.3792/pja/1195520159. URL: https://doi.org/10.3792%2Fpja%2F1195520159.

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- gKdV is a generalization of KdV (i.e., $\mathfrak{p} = 2$). It appears in the study of waves on shallow water[1], as well as other areas of physics[2]. In addition, it is closely to the study of anharmonic lattices[3].

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[1] D. J. Korteweg and G. de Vries. "On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves". In: *Philos. Mag. (5)* 39.240 (1895), pp. 422–443. ISSN: 1941-5982. DOI: 10.1080/14786449508620739. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3363408.

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Quasi-periodic Initial Data

- Let $1 \leq \nu < \infty$ be a given finite natural number, $(\omega_1, \cdots, \omega_\nu) = \omega \in \mathbb{R}^\nu$ be a non-resonant frequency vector, that is, $n \cdot \omega = 0$ implies that $n = 0 \in \mathbb{Z}^\nu$, here $n \cdot \omega$ is the Euclidean inner product defined by letting $n \cdot \omega = \sum_{j=1}^{\nu} n_j \omega_j$.

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- Consider the quasi-periodic initial data defined by the spatially quasi-periodic Fourier series

$$u(0, x) = \sum_{n \in \mathbb{Z}^\nu} c(n) e^{\mathrm{i}(n \cdot \omega) x}, \tag{2}$$

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◇ For the sake of simplicity, we call (1)-(2) the quasi-periodic Cauchy problem for gKdV.

Spatially Quasi-periodic Solutions

What we are interested in is to study the existence and uniqueness of spatially quasi-periodic solutions with the same frequency vector as the initial data to the quasi-periodic Cauchy problem (1)-(2). Such solutions are defined by the following spatially quasi-periodic Fourier series

$$u(t,x)=\sum_{n\in\mathbb{Z}^\nu} c(t,n)e^{\mathrm{i}(n\cdot\omega)x},\quad x\in\mathbb{R}, \tag{3}$$

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- It should be emphasized that such functions are non-periodic w.r.t. the space ($\nu \geq 2$ and ω is rationally independent), and
- they don't decay to zero w.r.t. the space (oscillation at infinity).

Regarding the quasi-periodicity in space, Deift P. said:"With the discovery of **quasi-crystals**, one can anticipate, in particular, that **interest in PDE problems on quasi-periodic backgrounds will surely grow**."[4]

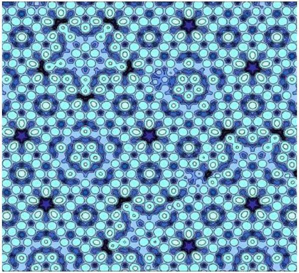

Figure: quasi-crystal (from wikipedia)

[4] Percy Deift. "Some open problems in random matrix theory and the theory of integrable systems. II". In: *SIGMA Symmetry Integrability Geom. Methods Appl.* 13 (2017), Paper No. 016, 23. DOI: 10.3842/SIGMA.2017.016. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3622647.

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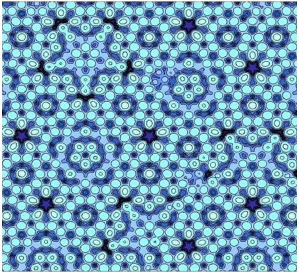

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◇ Regarding PDE in the quasi-periodic setting, there are some significant works, such as

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♡ Iooss G.[5] studied the steady Swift-Hohenberg $(1+\Delta)^2 u - \mu u + u^3 = 0$, where μ is a bifurcation parameter, u is a real function of $(x, y) \in \mathbb{R}^2$ and quasi-periodic in all directions, i.e., its Fourier expansion with wave vectors belonging to a quasi-lattice Γ (see the left Figure 2 below), spanned by two concentric hexagonal with a rotation angle, in the real plane. This implies that u has the Fourier expansion as follows

$$u(x, y) = \sum_{(v_1, v_2) \in \Gamma} \hat{u}_{v_1, v_2} e^{\mathrm{i}(v_1 x + v_2 y)}.$$

[5] Gérard Iooss. "Existence of quasipatterns in the superposition of two hexagonal patterns". In: *Nonlinearity* 32.9 (2019), pp. 3163–3187. ISSN: 0951-7715. DOI: 10.1088/1361-6544/ab230a. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3987802.

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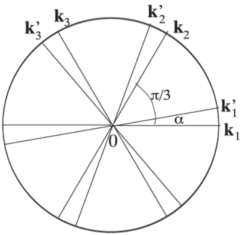

Figure: quasi-lattice Γ (from Iooss)

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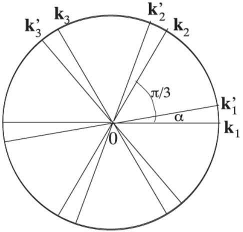

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- This is the so-called quasi-pattern, which was discovered in nonlinear pattern-forming systems in the Faraday wave experiment.

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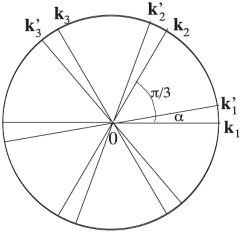

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- Mathematical existence of quasi-patterns is one of the outstanding problems in pattern formation theory.

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♡ Wang W.-M.[6] considered the NLS

$$i\partial_t U = -\Delta U - |U|^{2p} U \qquad (4)$$

where $\mathbb{N} \ni p \geq 1$ is arbitrary, $U = U(t, x)$ is a complex valued function on $\mathbb{R} \times \mathbb{R}^d$ (d is an arbitrary dimension), and she studied the space quasi-periodic standing wave solutions

$$U(t, x) = e^{-iEt} u(x),$$

where $E \in \mathbb{R}$, u is even and quasi-periodic in each x_k, given by a quasi-periodic cosine series

$$u(x) = u(x_1, \cdots, x_d) = \sum_{j_1, \cdots, j_d} \hat{u}(j_1, \cdots, j_d) \prod_{k=1}^{d} \cos(j_k \cdot \lambda_k) x_k.$$

NLS (4) is used to study Bose-Einstein condensation, and is usually called the Gross-Pitaevskii equation, when seeking non-decaying solutions.

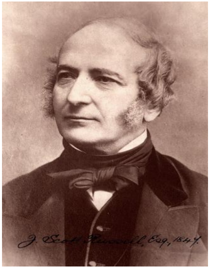
[6] W.-M. Wang. "Space quasi-periodic standing waves for nonlinear Schrödinger equations". In: *Comm. Math. Phys.* 378.2 (2020), pp. 783–806. ISSN: 0010-3616. DOI: 10.1007/s00220-020-03798-x. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=4134934.

KdV

In the case $\mathfrak{p} = 2$, gKdV (1) is the classical KdV

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0.$$

It's a mathematical model of waves on shallow water surfaces, which can be traced back to the experiments by John S-cott Russell.



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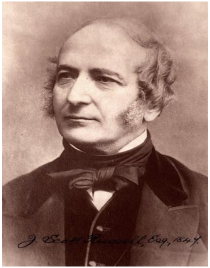

Figure: John Scott Russell

- Regarding the Cauchy problem, especially for those with some periodicity structure, to the KdV equation, there are some significant works on the periodic, quasi-periodic and almost periodic initial data, which are associated with the so-called **Deift conjecture**.

Deift Conjecture

The Deift conjecture[7] states that the solution to KdV with almost periodic initial data has a unique global solution $u(t, x)$ that is almost periodic in x for any t and almost periodic in t for any x.

[7] Percy Deift. "Some open problems in random matrix theory and the theory of integrable systems". In: *Integrable systems and random matrices*. Vol. 458. Contemp. Math. Amer. Math. Soc., Providence, RI, 2008, pp. 419–430. DOI: 10.1090/conm/458/08951. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=2411922.

Periodic Setting

- Lax P.D.[8]
- McKean H.P., Trubowitz E.[9]
- This is also a motivation of Deift conjecture.

[8] Peter D. Lax. "Periodic solutions of the KdV equation". In: *Comm. Pure Appl. Math.* 28 (1975), pp. 141–188. ISSN: 0010-3640. DOI: 10.1002/cpa.3160280105. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=369963.

[9] H. P. McKean and E. Trubowitz. "Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points". In: *Comm. Pure Appl. Math.* 29.2 (1976), pp. 143–226. ISSN: 0010-3640. DOI: 10.1002/cpa.3160290203. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=427731.

Quasi-periodic Setting

- Tsugawa K.[10]: local result; poly. decay condition w.r.t. $|n|$; Bourgain's Fourier restriction method.
- Damanik D., Goldstein M.[11]-[12]: global result; exp. decay condition w.r.t. $|n|$, Diophantine condition on the frequency vector ω; combinatorial analysis method (local) + iso-spectral argument (global; Lax pair). (This means that they can iterate the local result with a uniform length of time interval infinitely so that the global result is achieved)

[10] Kotaro Tsugawa. "Local well-posedness of the KdV equation with quasi-periodic initial data". In: *SIAM J. Math. Anal.* 44.5 (2012), pp. 3412–3428. ISSN: 0036-1410. DOI: 10.1137/110849973. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3023416.

[11] David Damanik and Michael Goldstein. "On the inverse spectral problem for the quasi-periodic Schrödinger equation". In: *Publ. Math. Inst. Hautes Études Sci.* 119 (2014), pp. 217–401. ISSN: 0073-8301. DOI: 10.1007/s10240-013-0058-x. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3210179.

[12] David Damanik and Michael Goldstein. "On the existence and uniqueness of global solutions for the KdV equation with quasi-periodic initial data". In: *J. Amer. Math. Soc.* 29.3 (2016), pp. 825–856. ISSN: 0894-0347. DOI: 10.1090/jams/837. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3486173.

Almost Periodic Setting

- Binder I., Damanik D., Goldstein M., Lukic, M.[13]: partially solved this conjecture.
- Eichinger B., VandenBoom T. and Yuditskii P.[14]: a more general existence result (in the study of KdV hierarchy).

[13] Ilia Binder et al. "Almost periodicity in time of solutions of the KdV equation". In: *Duke Math. J.* 167.14 (2018), pp. 2633–2678. ISSN: 0012-7094. DOI: 10.1215/00127094-2018-0015. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3859361.

[14] B. Eichinger, T. VandenBoom, and P. Yuditskii. "KdV hierarchy via abelian coverings and operator identities". In: *Trans. Amer. Math. Soc. Ser. B* 6 (2019), pp. 1–44. DOI: 10.1090/btran/30. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3894927.

KdV Hierarchy

Set

$$L = -\partial_x^2 + u \text{ and } P_{2n+1} = \sum_{\ell=0}^{n} \left(f_{n-\ell}\partial_x - (1/2)f_{n-\ell,x}\right) L^\ell,$$

where $n \in \mathbb{N}_0$, $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ is recursively defined by letting $f_0 = 1$ and

$$f_{\ell,x} = -(1/4)f_{\ell-1,xxx} + uf_{\ell-1,x} + (1/2)u_x f_{\ell-1}, \quad \ell \in \mathbb{N}.$$

Here (L, P_{2n+1}) is called **Lax pair**[15]. The KdV hierarchy can be represented by Lax pair, or rather,

$$\partial_{t_n} u = [L, P_{2n+1}](u) = -2f_{n+1,x}u, \quad n \in \mathbb{N}. \tag{5}$$

[15] Fritz Gesztesy and Helge Holden. *Soliton equations and their algebro-geometric solutions. Vol. I.* Vol. 79. Cambridge Studies in Advanced Mathematics. (1 + 1)-dimensional continuous models. Cambridge University Press, Cambridge, 2003, pp. xii+505. ISBN: 0-521-75307-4. DOI: 10.1017/CBO9780511546723. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=1992536.

KdV Hierarchy

Eichinger B., VandenBoom T. and Yuditskii P.[16] studied KdV hierarchy (5) with almost periodic initial data

$$u(0, \cdot) = V(\cdot). \tag{6}$$

Under some conditions on V in terms of the spectral properties of L, they proved that quasi-periodic Cauchy problem (5)-(6) admits a global solutions $u(t_n, x)$ in the classical sense which is uniformly almost periodic in time and space coordinates.

[16] 14, Eichinger-VandenBoom-Yuditskii-2019-TAMS.

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◇ The case of $n = 1$ is the existence result for KdV.

[16] 14, Eichinger-VandenBoom-Yuditskii-2019-TAMS.

A Possible Negative Answer to Deift Conjecture

- Damanik D., Lukic M., Volberg A., Yuditskii P.[17]: **a program** which **intends to** show that this conjecture is not true in general (by constructing smooth almost periodic initial data for which the solution is not almost periodic in time).

[17] David Damanik et al. "The Deift Conjecture: A Program to Construct a Counterexample". In: (Nov. 2021). eprint: 2111.09345. URL: https://arxiv.org/pdf/2111.09345.pdf.

mKdV

The case $\mathfrak{p} = 3$ of gKdV is the modified KdV equation

$$\partial_t u + \partial_x^3 u + u^2 \partial_x u = 0.$$

There are many studies on mKdV in the usual Sobolev space $H^s(\mathbb{R})$ and periodic Sobolev space $H^s(\mathbb{T})$, and the sharp results are $H^{1/4}(\mathbb{R})$ and $H^{1/2}(\mathbb{T})$ respectively[18].

[18] J. Colliander et al. "Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$". In: *J. Amer. Math. Soc.* 16.3 (2003), pp. 705–749. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-03-00421-1. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=1969209.

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- However, to the best of our knowledge, there is no research on the quasi-periodicity in space for mKdV.

[18] J. Colliander et al. "Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$". In: *J. Amer. Math. Soc.* 16.3 (2003), pp. 705–749. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-03-00421-1. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=1969209.

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- However, to the best of our knowledge, there is no research on the quasi-periodicity in space for mKdV.
- These motivate us to study gKdV (1) with quasi-periodic initial data (2), and our main result is the following theorem.

[18] J. Colliander et al. "Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$". In: *J. Amer. Math. Soc.* 16.3 (2003), pp. 705–749. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-03-00421-1. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=1969209.

Theorem (arXiV: 2110.11263[19] by Damanik-Li-X) If the initial Fourier data is exponentially decaying, that is, there exist $\mathcal{A} > 0$ and $0 < \kappa \leq 1$ such that

$$|c(n)| \leq \mathcal{A}^{\frac{1}{\mathfrak{p}-1}} e^{-\kappa|n|}, \quad \forall n \in \mathbb{Z}^{\nu},$$

then the quasi-periodic Cauchy problem (1)-(2) has a unique spatially quasi-periodic solution (3) on $[0, t_0) \times \mathbb{R}$, where t_0 is a positive number and it depends on $\mathfrak{p}, \mathcal{A}, \kappa, \nu$ and $|\omega|$. What's more, the Fourier coefficient $c(t, n)$ of solution retains the exponential decay with a slightly worse constant, that is,

$$|c(t, n)| \leq \Box e^{-\frac{\kappa}{2}|n|}, \quad \forall n \in \mathbb{Z}^{\nu} \text{ and } \forall 0 \leq t < t_0,$$

where $\Box = 2(6\kappa^{-1})^{\nu}\mathcal{A}^{\frac{1}{\mathfrak{p}-1}}$.

[19] David Damanik, Yong Li, and Fei Xu. "Local Existence and Uniqueness of Spatially Quasi-periodic Solutions to the Generalized KdV Equation". In: (Oct. 2021). eprint: 2110.11263. URL: https://arxiv.org/pdf/2110.11263.pdf.

- The case $\mathfrak{p} = 2$ coincides with Damanik and Goldstein's local result[20].

[20] 12, Damanik-Goldstein-2016-JAMS.

[21] M. Christ. "Power series solution of a nonlinear Schrödinger equation". In: *Mathematical aspects of nonlinear dispersive equations*. Vol. 163. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 2007, pp. 131–155. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=2333210.

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- The case $\mathfrak{p} = 2$ coincides with Damanik and Goldstein's local result[20].
- Regarding the proof, we first reduce the quasi-periodic Cauchy problem to a nonlinear system of infinite coupled ODEs in the Fourier space. Due to the difficulty of discrete convolution of higher dimension, we apply a **combinatorial analysis method**[21]-[22] to obtain the local existence and uniqueness under the exp. decay condition w.r.t $|n|$.

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- Next we will introduce the skeleton of proof with some details.

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Step 1: Infinite Dimensional ODEs with Discrete Convolution of Higher Dimension

To reduce the nonlinear PDE problem with Fourier expansion to the following nonlinear system of infinite coupled ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t,n) - \mathrm{i}(n\cdot\omega)^3 c(t,n) + \frac{\mathrm{i}n\cdot\omega}{\mathfrak{p}} c^{*\mathfrak{p}}(t,n) = 0, \quad \forall n \in \mathbb{Z}^\nu,$$

with the discrete convolution of higher dimension $c^{*\mathfrak{p}}(\cdot,n)$, defined by the following formula

$$c^{*\mathfrak{p}}(t,n) \triangleq \underbrace{c * c \cdots * c}_{\mathfrak{p}}(t,n) := \sum_{n_1,\cdots,n_{\mathfrak{p}}\in\mathbb{Z}^\nu : \sum_{j=1}^{\mathfrak{p}} n_j = n} \prod_{j=1}^{\mathfrak{p}} c(t,n_j).$$

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- This equation is a type of infinite dimensional ODEs with discrete convolution of higher dimension, and it is impossible to solve it directly. Here we use its integral form and define the Picard sequence to approximate the solution.

Step 2: Picard Iteration

Choosing the solution $e^{\mathrm{i}(n\cdot\omega)^3 t}c(n)$ to the linear equation as the initial guess $c_0(t,n)$, and defining $c_k(t,n)$ successively by letting

$$c_k(t,n) = c_0(t,n) - \frac{\mathrm{i}n\cdot\omega}{\mathfrak{p}}\int_0^t e^{\mathrm{i}(n\cdot\omega)^3(t-\tau)}c_{k-1}^{*\mathfrak{p}}(\tau,n)\mathrm{d}\tau, \quad k\geq 1.$$

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- Notice that $c_k(t,n) - c_0(t,n)$ can be viewed as a multi-linear form of $c_{k-1}(t,n)$ for all $k \geq 1$. Hence $c_k(t,n) - c_0(t,n)$ is indeed a multi-linear form of c, that is, $c_k(t,n) - c_0(t,n)$ has the continued product form $c \cdot c \cdots c \cdot c$.

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- Due to the difficulty of calculating the discrete convolution of higher dimension, we cannot control the iteration process effectively. By applying a combinatorial analysis method, we obtain the local result.

Step 3: Combinatorial Analysis

For all $k \geq 1$, the Picard sequence $c_k(t,n)$ has the following combinatorial tree form

$$c_k(t,n) = \sum_{\gamma \in \mathfrak{T}^{(k)}} \sum_{\substack{n^{(k)} \in \mathfrak{N}^{(k,\gamma)} \\ \mu(n^{(k)})=n}} \mathfrak{F}^{(k,\gamma)}(n^{(k)}) \mathfrak{I}^{(k,\gamma)}(t, n^{(k)}) \mathfrak{C}^{(k,\gamma)}(n^{(k)}), \qquad (7)$$

where these abstract symbols $\mathfrak{T}^{(k)}, \mathfrak{N}^{(k,\gamma)}, \mathfrak{F}^{(k,\gamma)}, \mathfrak{I}^{(k,\gamma)}$ and $\mathfrak{C}^{(k,\gamma)}$ are defined inductively.

With the help of the combinatorial form, along with the exponential decay condition, and by induction, we can prove that the Picard sequence is exponentially decaying, with the decay rate $\kappa/2$, uniformly in time.

Lemma

If $0 < t \le \frac{\kappa^{(\mathfrak{p}-1)\nu+1}}{2^{\mathfrak{p}+1}6^{(\mathfrak{p}-1)\nu+1}\mathcal{A}|\omega|}$, *then*

$$|c_k(t,n)| \le \Box e^{-\frac{\kappa}{2}|n|}, \quad \forall k \ge 1,$$

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[23] 18, Damanik-Li-Xu-2021-arXiv 2110.11263.

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where $\square \triangleq 2(6\kappa^{-1})^{\nu}\mathcal{A}^{\frac{1}{\mathfrak{p}-1}}$.

- Regarding the proof of this Lemma, many complicated estimates should be made (due to the complicated combinatorial indices generated by the nonlinearity), and we omit them here. If you are interested in them, you can find them in **arXiv: 2110.11263**[23].

[23] 18, Damanik-Li-Xu-2021-arXiv 2110.11263.

Step 4: Cauchy Sequence, Existence and Uniqueness

By the exponential decay property of the Picard sequence, we can prove that it is a Cauchy sequence with the following estimates.

Lemma

For $0 < t < \min\left\{ \frac{\kappa^{(\mathfrak{p}-1)\nu+1}}{2^{\mathfrak{p}+1}6^{(\mathfrak{p}-1)\nu+1}\mathcal{A}|\omega|}, \frac{\kappa^{(\mathfrak{p}-1)\nu+1}}{2(\mathfrak{p}-1)e\Box^{\mathfrak{p}-1}12^{(\mathfrak{p}-1)\nu+1}|\omega|} \right\} \triangleq t_0 > 0$*, and all* $k \geq 1$*, we have*

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\begin{aligned}
&|c_{k+\bigstar}(t,n) - c_k(t,n)| \\
\leq &\frac{\Theta e^{-\frac{\kappa}{4}|n| + \frac{1}{\mathfrak{p}-1}} \left\{2(\mathfrak{p}-1)e\Box^{\mathfrak{p}-1}(12\kappa^{-1})^{(\mathfrak{p}-1)\nu+1}|\omega| t\right\}^{k+1}}{1 - 2(\mathfrak{p}-1)e\Box^{\mathfrak{p}-1}(12\kappa^{-1})^{(\mathfrak{p}-1)\nu+1}|\omega| t}
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uniformly for $\bigstar$.

- Furthermore, by some standard argument, we can obtain the local existence result.
- In addition, the solution we construct is unique and is in the classical sense (thanks to the exponential decay property).

Summary

[24] 12, Damanik-Goldstein-2016-JAMS.

[25] 6, Wang-2020-CMP.

Summary

- We obtain the local existence and uniqueness of spatially quasi-periodic solutions to the generalized KdV equation, under the exponential decay condition. This result is a generalization of Damanik and Goldstein's work[24] in terms of the local result on KdV to gKdV.

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- The solutions we consider are non-periodic w.r.t. space (compare with the spatially periodic solutions) and don't decay to zero w.r.t. space (compare with fast decay at infinity case).

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- The solutions we consider are non-periodic w.r.t. space (compare with the spatially periodic solutions) and don't decay to zero w.r.t. space (compare with fast decay at infinity case).
- The setting we are in is non-compact real line $\mathbb{R}$. "Generally speaking, due to the non-compact setting, there are very few known results on space quasi-periodic solutions to nonlinear PDEs." This is from Wang's paper[25].

[24] 12, Damanik-Goldstein-2016-JAMS.

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Summary

- The main difficulty we encounter is the operation of higher dimensional discrete convolution (the complicated combinatorial indices and estimates). The remaining global problem is much more difficult and it may need the integrability of gKdV and deep spectral analysis; compare Damanik and Goldstein's paper[26].

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- The main difficulty we encounter is the operation of higher dimensional discrete convolution (the complicated combinatorial indices and estimates). The remaining global problem is much more difficult and it may need the integrability of gKdV and deep spectral analysis; compare Damanik and Goldstein's paper[26].
- Our method works for arbitrary natural number $\mathfrak{p}$.

[26] 12, Damanik-Goldstein-2016-JAMS.

Thanks for your attention!

Fei Xu
(Jilin University)