Spectra of Periodic and Limit-Periodic Dirac Operators

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Complex Analysis, Spectral Theory and Approximation

Motivation

This talk centers around the spectral theory of periodic and almost-periodic operators , specifically the size of the spectrum.

There is a scheme due to A. Avila that one can use to prove very fine measure estimates on spectra of periodic and limit-periodic discrete Schrödinger operators , which in turn leads to the construction of almost-periodic operators with very thin spectra.

Many steps of the scheme are essentially model-independent.

One step that is model-dependent is the careful construction of suitable spectral gaps.

In some recent joint work on Dirac operators, we had to find a new way to implement this aspect of the scheme, using some notions from group theory, complex analysis, and inverse spectral theory.

A General Problem

Given:

- A class of linear self-adjoint operators $\mathscr{L}$.
- An operator $L \in \mathscr{L}$.
- A spectral parameter $\lambda \in \mathbb{R}$.

How to perturb L within $\mathscr{L}$ and open a spectral gap around λ?

If $\mathscr{L}$ is some collection of periodic differential/difference operators in dimension one, one wants to know how to produce hyperbolicity of specific matrices.

This can be (and has been) achieved in suitable settings by fine/delicate/hard/technical analysis.

Later on, we'll discuss a soft approach via some helpful facts from group theory, complex analysis, and inverse spectral theory.

Philosophy

We consider a family $\{A(t)\}_{t\in\mathfrak{T}}$ of $\mathbb{SU}(1,1)$ matrices that depend on analytically on a parameter $t \in \mathfrak{T}$, where $\mathfrak{T}$ is a Banach space.

- Hyperbolicity can be achieved via noncommutation (to be described later).
- Noncommutation at a single point t leads to noncommutation everywhere outside a set with empty interior (identity principle).
- Commutation everywhere is a very strict statement. Depending on the context, it may directly imply conclusions via direct calculations or inverse spectral theory.

Dirac Operators

Dirac operator

Given $\varphi : \mathbb{R} \to \mathbb{C}$, the associated Dirac operator Λ_φ (in the Zakharov–Shabat gauge) is given by

$$\Lambda_\varphi = -\mathbb{i} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{=:\mathbf{j}} \partial_x + \underbrace{\begin{bmatrix} 0 & \varphi(x) \\ \overline{\varphi(x)} & 0 \end{bmatrix}}_{=:\mathbf{\Phi}(x)}.$$

Goal. Study the spectrum:

$$\sigma(\Lambda) = \{z \in \mathbb{C} : \Lambda - zI \text{ not invertible}\}.$$

Dirac operators arise in relativistic quantum mechanics.

The Zakharov–Shabat operators come from a Lax pair representation of a nonlinear Schrödinger equation.

Transfer Matrices

The transfer matrices are given by

$$\mathbf{A}_z(x, x_0, \varphi) = \begin{bmatrix} U_1(x) & V_1(x) \\ U_2(x) & V_2(x) \end{bmatrix},$$

where U and V solve

$$\Lambda_\varphi U = zU, \quad \Lambda_\varphi V = zV, \quad U(x_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad V(x_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Facts.

- $\det \mathbf{A}_z(x, x_0, \varphi) \equiv 1$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathbf{A}_\lambda(x, x_0, \varphi)^* \mathbf{j} \mathbf{A}_\lambda(x, x_0, \varphi)\right] = 0$ for $\lambda \in \mathbb{R}$. ($\mathbf{j} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$)
- Thus, $\mathbf{A}_\lambda(x, x_0, \varphi)) \in \mathbb{SU}(1, 1)$ for $\lambda \in \mathbb{R}$.
- $\mathbf{A}_z(x, x_0, \varphi)$ analytic as a function of z and of φ.

Periodic Dirac Operators

If $\varphi : \mathbb{R} \to \mathbb{C}$ is periodic of period T:

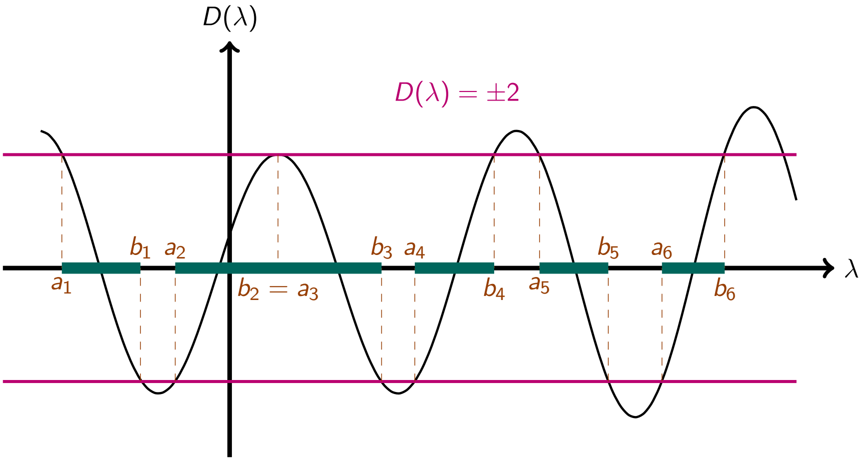
Monodromy matrices: $\mathbf{M}_z(x, \varphi) = \mathbf{A}_z(x + T, x, \varphi)$.

Discriminant: $D(z, \varphi) = \mathrm{Tr}(\mathbf{M}_z(x, \varphi))$.

The spectrum is purely a.c. and is given by a union of closed intervals (bands):

$$
\begin{aligned}
\sigma(\Lambda_\varphi) &= \overline{\{\lambda \in \mathbb{R} : \Lambda_\varphi U = \lambda U \text{ has a polynomially bounded solution}\}} \\
&= \{\lambda \in \mathbb{R} : \Lambda_\varphi U = \lambda U \text{ has a bounded solution}\} \\
&= \{\lambda \in \mathbb{R} : \mathrm{spr}(\mathbf{M}_\lambda(0, \varphi)) = 1\} \\
&= \{\lambda \in \mathbb{R} : D(\lambda, \varphi) \in [-2, 2]\} \\
&= \overline{\{\lambda \in \mathbb{R} : \mathbf{M}_\lambda(0, \varphi) \text{ is conjugate to a rotation}\}} \\
&=: \bigcup_{n \in \mathbb{Z}} [a_n, b_n].
\end{aligned}
$$

Floquet Discriminant



Limit-Periodic Functions

Recall, φ is (uniformly) limit-periodic if it is a uniform limit of continuous periodic functions , e.g.

$$\varphi(x) = \sum_{m=1}^{\infty} \mathrm{e}^{\frac{2\pi \mathrm{i} x}{m!} - m^{m!}}$$

Denote the set of limit-periodic functions by

$$\mathrm{LP}(\mathbb{R}) = \overline{\{\varphi \in C(\mathbb{R}) : \varphi \text{ is periodic}\}}^{\|\cdot\|_\infty}$$

$\mathrm{LP}(\mathbb{R})$ is a complete metric space.

Not, however, a Banach space. E.g. $\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{\mathrm{i}\pi x}$.

Spectral Theory of limit-periodic operators studied by many including Moser, Avron, Simon, Chulaevskii, Molchanov, Pöschel, Pastur, Tkachenko, Egorova, Peherstorfer, Volberg, **Yuditskii**, Bellissard, Geronimo, Avila, Damanik, Gan, Krüger, Lukic, Yessen, Ong, VandenBoom, G. Young, C. Wang, Gwaltney, Eichinger,...

Theorem

Theorem (Eichinger–F.–Gwaltney–Lukic, (2022))

For generic $\varphi \in \mathrm{LP}(\mathbb{R})$, $\sigma(\Lambda_\varphi)$ is an extended Cantor set of zero Lebesgue measure.

For a dense subset $\varphi \in \mathrm{LP}(\mathbb{R})$, $\sigma(\Lambda_\varphi)$ has zero Hausdorff dimension as well.

Benjamin Eichinger (Johannes Kepler University)

Ethan Gwaltney (Rice)

Milivoje Lukic (Rice)

The proof follows Avila's scheme in the bulk, with some new approaches to certain model-dependent steps.

Proof Ideas: Bird's-Eye View

- The mapping $\varphi \mapsto \sigma(\Lambda_\varphi)$ is 1-Lipschitz if the domain has the L^∞ metric and the target has the Hausdorff metric.

That is: $\mathrm{dist}(\sigma(\Lambda_\varphi), \sigma(\Lambda_\psi)) \leq \|\varphi - \psi\|_\infty$, where

$$\mathrm{dist}(F, K) = \inf\{\varepsilon > 0 : F \subseteq B_\varepsilon(K) \text{ and } K \subseteq B_\varepsilon(F)\}$$

- So, the set of φ for which $\sigma(\Lambda_\varphi)$ has zero measure is always a G_δ.

Namely, the set of φ such that $|(\sigma(\Lambda_\varphi) \cap [-M, M]| < \delta$ is open for every M, δ.

Lemma

Given $\varphi \in C(\mathbb{R})$ T-periodic , $M > 0$, and $\varepsilon > 0$.
There exist $c_0 = c_0(\varphi, M, \varepsilon) > 0$ and $N_0 = N_0(\varphi, M, \varepsilon) \in \mathbb{N}$ such that:
for every integer $N \geq N_0$ there exists $\widetilde{\varphi}$ of period $\widetilde{T} = NT$ such that

$$\|\varphi - \widetilde{\varphi}\|_\infty < \varepsilon \quad \text{and} \quad |\sigma(\Lambda_{\widetilde{\varphi}}) \cap [-M, M]| < e^{-c_0 \widetilde{T}}$$

Lemma Implies Theorems

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- The zero-measure spectrum result is immediate (Baire Category)
- The zero-dimensional result follows from noting that one can produce T_n-periodic $\varphi_n \to \varphi_\infty$ in such a way that

$$[-M_n, M_n] \cap \sigma(\varphi_{\Lambda_n})$$

  can be covered efficiently by small intervals.
- Singularity of spectral measures is immediate.
- Continuity of spectral measures follows from Gordon's lemma.

Avila's Scheme

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$$\|\varphi - \widetilde{\varphi}\|_\infty < \varepsilon \quad \text{and} \quad |\sigma(\Lambda_{\widetilde{\varphi}}) \cap [-M, M]| < e^{-c_0 \widetilde{T}}$$

Let us describe the overall structure.

- Begin with φ periodic, $M > 0$, $\varepsilon > 0$.
- Produce a finite family of perturbations $\varphi_1, \varphi_2, \ldots, \varphi_\ell$ that are close to φ and whose resolvent sets cover $[-M, M]$.
- Form $\widetilde{\varphi}$ by concatenating each φ_j many times.
- For each $\lambda \in [-M, M]$, transfer matrices grow on long intervals.
- This can be used to get lower bounds on the derivative of the rotation number.
- Hence, upper bounds on measure of spectrum (in $[-M, M]$).

Avila's Scheme

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Opening Spectral Gaps

There are two steps that depend on the particular structure of the model in question: opening spectral gaps, and transferring bounds on solutions to bounds on the rotation number/integrated density of states.

In this talk, I will focus on the first of these.

Recall that the spectrum of a T-periodic operator Λ_φ was characterized as those energies λ for which the monodromy matrix $\mathbf{A}_\lambda(T, 0, \varphi)$ is not hyperbolic.

So, one wants to know how to perturb φ so as to make $A_\lambda(T, 0, \varphi)$ hyperbolic.

As phrased, this is problematic, because... ellipticity is open. Need to pass to higher periods.

Hyperbolicity via Noncommutation

Recall that $\mathbf{A} \in \mathbb{SU}(1,1) \setminus \{\pm \mathbf{I}\}$ is

- elliptic if $|\mathrm{Tr}\,\mathbf{A}| < 2$
- parabolic if $|\mathrm{Tr}\,\mathbf{A}| = 2$
- hyperbolic if $|\mathrm{Tr}\,\mathbf{A}| > 2$

Let us write $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ for the commutator of $\mathbf{A}$ and $\mathbf{B}$.

Lemma

If $\mathbf{A}, \mathbf{B} \in \mathbb{SU}(1,1)$ are elliptic and $[\mathbf{A}, \mathbf{B}] \neq 0$, then the semigroup they generate contains a hyperbolic matrix.

Proof Sketch.

It is well known that the closed subgroup generated by A and B contains a hyperbolic element.

Approximate $\mathbf{A}^{-1}$ (resp. $\mathbf{B}^{-1}$) by positive powers of $\mathbf{A}$ (resp. $\mathbf{B}$) to see that the closed semigroup generated by $\mathbf{A}$ and $\mathbf{B}$ is the same as the closed subgroup.

Hyperbolicity is an open condition. □

Opening Spectral Gaps

Recall: The goal is to begin with φ periodic, $\lambda \in \sigma(\Lambda_\varphi)$ and push λ into the resolvent set of a perturbed operator.

- **Case 1.** $D(\lambda) \in (-2, 2)$.
- By the Lemma from previous slide, it suffices to find $\widetilde{\varphi}$ near φ of the same period for which

$$[\mathbf{A}_\lambda(T, 0, \varphi), \mathbf{A}_\lambda(T, 0, \widetilde{\varphi})] \neq 0$$

- Well, if you cannot do that , the commutator vanishes everywhere by analyticity.
- Calculation: The centralizer of the set of Dirac monodromies of a given fixed period T is $\{\pm \mathbf{I}\}$. Contradiction.
- Having found the nearby $\widetilde{\varphi}$ for which the monodromies don't commute , use the lemma to concatenate $\{\varphi, \widetilde{\varphi}\}$ so as to make the resulting monodromy hyperbolic.
- **Case 2.** $D(\lambda) \in \{-2, 2\}$. Perturb φ a bit. You either push λ into the resolvent set or you push yourself into Case 1.

Thank you!