Extremal polynomials on a Jordan arc and on compact subsets of the real line

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Widom factors

Let $K \subset \mathbb{C}$ be a compact set with $\mathrm{Cap}(K) > 0$. Let w be a weight function (non-negative and upper semicontinuous on K and positive on a non-polar subset of K) on K, and let $\|\cdot\|_K$ denote the sup-norm on K.

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Then the n-th (weighted) Chebyshev polynomial with respect to w is the minimizer of $\|wP_n\|_K$ over all monic polynomials P_n of degree n and we denote it by $T_{n,w}^{(K)}$. Let

$$t_n(K,w) := \|wT_{n,w}^{(K)}\|_K. \tag{1.1}$$

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$$t_n(K,w) := \|wT_{n,w}^{(K)}\|_K. \tag{1.1}$$

We define the n-th Widom factor for the sup norm with respect to weight w on K by

$$W_{\infty,n}(K,w) := \frac{t_n(K,w)}{\mathrm{Cap}(K)^n}. \tag{1.2}$$

Widom factors

For a finite (positive) Borel measure μ with $\mathrm{supp}(\mu) = K$, the n-th monic orthogonal polynomial for μ is the minimizer of $\|P_n\|_{L_2(\mu)}$ over all monic polynomials of degree n and we denote it by $P_n(\cdot;\mu)$. We define the n-th Widom factor for μ by

$$W_{2,n}(\mu) := \frac{\|P_n(\cdot;\mu)\|_{L_2(\mu)}}{\mathrm{Cap}(K)^n}. \tag{1.3}$$

Universal lower bound

Let μ be a finite (positive) Borel measure with $\mathrm{supp}(\mu) = K$ where K is a compact non-polar subset of $\mathbb{C}$. Let us consider the Lebesgue decomposition of μ with respect to equilibrium measure μ_K of K:

$$d\mu = f\, d\mu_K + d\mu_s.$$

We define exponential relative entropy for μ by

$$S(\mu) := \exp\left[\int \log f(x)\, d\,\mu_K(x)\right]. \tag{1.4}$$

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It was shown in [A. 19] that, for all $n \in \mathbb{N}$, the universal lower bound

$$[W_{2,n}(\mu)]^2 \geq S(\mu) \tag{1.5}$$

holds.

Widom's results

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Theorem (Widom, 1969)

Let $\Gamma = \cup_{k=1}^{p} E_k$ be a system of C^{2+} Jordan curves. Let w be a weight function such that $S(w) > 0$. Then the limit points of $W_{\infty,n}(\Gamma, w)$ is a finite union of closed subintervals of $[S(w), S(w)\exp\{\sum_{j=1}^{p-1} g_\Gamma(z_j)\}]$ where z_j's are critical points of g_Γ counting multiplicity.

Widom's results

Theorem (Widom, 1969)

Let $\Gamma = \cup_{k=1}^{p} E_k$ *be a system of* C^{2+} *Jordan curves and arcs. Let* w *be a weight function such that* $S(w) > 0$*. Then*
$\limsup_{n\to\infty} W_{\infty,n}(\Gamma, w) \leq 2S(w)\exp\{\sum_{j=1}^{p-1} g_\Gamma(z_j)\}$.

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Widom also conjectured that once Γ includes an arc, this is the correct upper bound and the corresponding lower bound for $\liminf$ should be $2S(w)$. He even verified this conjecture when Γ is union of finitely many intervals.

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Nonetheless, his conjecture was proven to be wrong.

On this Conjecture

Theorem (Totik-Yuditskii, 2015)

Let $\Gamma = \cup_{k=1}^{p} E_k$ be a system of C^{2+} Jordan curves and arcs which includes at least one curve. Let w be a weight function such that $S(w) > 0$. Then there is a $C(\Gamma) < 2$ such that
$\limsup_{n\to\infty} W_{\infty,n}(\Gamma, w) \leq C(\Gamma) S(w) \exp\{\sum_{j=1}^{p-1} g_\Gamma(z_j)\}$.

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-Totik and Yuditskii also mentioned the following result. Let $w \equiv 1$, Γ be a subarc on the unit circle of central angle 2α. Then $S(w) = 1$ and $\lim_{n\to\infty} W_{\infty,n}(\Gamma, w) = 2\cos^2(\alpha/4)$. (Thiran-Detaille (1991)). Thus the lower bound conjecture was also wrong since the limit is smaller than 2.

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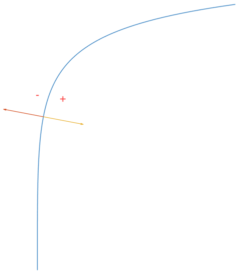
The proof of the above Theorem by Totik and Yuditskii improves the upper bound given in Widom's paper if the system includes a Jordan curve.
We discuss an improved upper bound if Γ is an arbitrary C^{2+} Jordan arc.

Widom factors on a Jordan arc

Let Γ be a C^{2+} Jordan arc.
We call two sides of Γ positive and negative sides.



Orthogonal polynomials on a Jordan arc

If $n_\pm$ are the two normals at $z \in \Gamma$ then

$$g'_\pm(z) := \frac{\partial g_\Gamma(z)}{\partial n_\pm}. \tag{2.1}$$

Both $g'_\pm$ are continuous and positive except for the endpoints. Let

$$\omega_\Gamma(z) := \frac{1}{2\pi}(g'_+(z) + g'_-(z)). \tag{2.2}$$

Then

$$d\mu_\Gamma = \omega_\Gamma\, ds \tag{2.3}$$

where ds is the arc-length measure on Γ.

Orthogonal polynomials on a Jordan arc

A finite (positive) Borel measure μ of the form $d\mu = fd\mu_\Gamma$ is in the Szegő class $\mathrm{Sz}(\Gamma)$ if

$$S(\mu) = S(f) = \exp\left[\int \log fd\mu_\Gamma\right] > 0. \tag{2.4}$$

Note that $d\mu_\Gamma, ds \in \mathrm{Sz}(\Gamma)$.

Orthogonal polynomials on a Jordan arc

Widom proved that if $\mu \in \mathrm{Sz}(\Gamma)$ then $[W_{2,n}(\mu)]^2$ has a limit and we have

$$\nu(\mu) := \lim_{n\to\infty} [W_{2,n}(\mu)]^2 = 2\pi R_\mu(\infty)\mathrm{Cap}(\Gamma)$$

where R_μ is an analytic function determined by μ in $\overline{\mathbb{C}} \setminus \Gamma$.

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Theorem

[A. 2022] The quantity $\nu(\mu_\Gamma)$ satisfies

$$1 < \nu(\mu_\Gamma) \leq 2. \tag{2.5}$$

The equality

$$\nu(\mu_\Gamma) = 2 \tag{2.6}$$

holds if and only if $g'_+(z) = g'_-(z)$ for all $z \in \Gamma_0$ where Γ_0 is the interior of Γ.

Orthogonal polynomials on a Jordan arc

As a corollary of the above result, we obtain a result which can be considered a generalization of the Szegő theorem on an interval:

Corollary

[A. 2022] Let $\mu \in \mathrm{Sz}(\Gamma)$*. Then*

$$1 < \lim_{n\to\infty} [W_{2,n}(\mu_\Gamma)]^2 = \lim_{n\to\infty} \frac{[W_{2,n}(\mu)]^2}{S(\mu)} \leq 2. \tag{2.7}$$

The equality

$$\lim_{n\to\infty} \frac{[W_{2,n}(\mu)]^2}{S(\mu)} = 2 \tag{2.8}$$

holds if and only if

$$g'_+(z) = g'_-(z) \ \text{ for all } z \in \Gamma_0. \tag{2.9}$$

Weighted Chebyshev polynomials on a Jordan arc

Let ρ be a weight function on Γ. Widom's result concerning the upper bound of Widom factors for the sup-norm is as follows:
$\limsup_{n\to\infty} W_{\infty,n}(\Gamma,\rho) \leq 2S(\rho)$.

The following result gives an improved upper bound:

Theorem

[A. 2022] Let ρ be a weight on Γ and $S(\rho) > 0$. Then

$$\limsup_{n\to\infty} W_{\infty,n}(\Gamma,\rho) \le \sqrt{2\nu(\mu_\Gamma)}S(\rho). \tag{2.10}$$

If there is a $z \in \Gamma_0$ such that $g'_+(z) \neq g'_-(z)$ then $\limsup_{n\to\infty} W_{\infty,n}(\Gamma,\rho) < 2S(\rho)$.

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We obtain also a weaker result which looks more useful to construct sets with improved upper bounds. We still assume Γ is C^{2+} below.

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Theorem

[A. 2022] Let $\mu \in \mathrm{Sz}(\Gamma)$ and ρ be a weight with $S(\rho) > 0$. If the interior Γ_0 of Γ is not analytic then

1. $\lim_{n\to\infty} \frac{[W_{2,n}(\mu)]^2}{S(\mu)} < 2.$
2. $\limsup_{n\to\infty} W_{\infty,n}(\Gamma,\rho) < 2S(\rho).$

Classical Chebyshev polynomials

Let T_n, U_n, V_n, W_n be the classical monic Chebyshev polynomials of the first, second, third and fourth kinds respectively on $K = [-1,1]$. Then they are the weighted Chebyshev polynomials for the weights $w(x) = 1, w(x) = \sqrt{1-x^2}, w(x) = \sqrt{1+x}, w(x) = \sqrt{1-x}$, respectively.

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It is also well known that they are the orthogonal polynomials associated with measures $\frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}$, $\frac{1}{\pi}\sqrt{1-x^2}\,dx$,$\frac{1}{\pi}\frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx$, $\frac{1}{\pi}\frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx$ on $[-1,1]$.

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Since $d\mu_{[-1,1]}(x)=\frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}$, these four measures can be written as $d\mu_K$, $(1-x^2)d\mu_K(x)$, $(1+x)d\mu_K(x)$, $(1-x)d\mu_K(x)$.

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Hence in all four cases, they are of the form $w^2 d\mu_K$.

Parreau-Widom sets

Let $K \subset \mathbb{R}$ be a compact set that is regular with respect to the Dirichlet problem and let $\{c_j\}_j$ denote the set of critical points of g_K. Then K is called a *Parreau-Widom* set if $\mathrm{PW}(K) := \sum_j g_K(c_j) < \infty$. The set of critical points of a regular set is countable and a Parreau-Widom set has positive Lebesgue measure.

Widom factors for the sup-norm

Theorem (Schiefermayr 08, Totik 11, Christiansen-Simon-Zinchenko 17,20)

Let $K \subset \mathbb{R}$ *be a compact set with* $\mathrm{Cap}(K) > 0$ *and* $w \equiv 1$. *Then*

(i) *For all* $n \in \mathbb{N}$, *we have*

$$W_{\infty,n}(K, w) \geq 2. \tag{3.1}$$

Equality is satisfied in (3.1) *if and only if there is a polynomial* P_n *of degree* n *such that* $K = \{z \in \mathbb{C} : P_n(z) \in [-1, 1]\}$.

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   Equality is satisfied in (3.1) if and only if there is a polynomial P_n of degree n such that $K = \{z \in \mathbb{C} : P_n(z) \in [-1,1]\}$.

2. *If, in addition K is a Parreau-Widom set then for all n,*

$$W_{\infty,n}(K,w) \leq 2e^{\mathrm{PW}(K)}. \tag{3.2}$$

   Equality is satisfied in (3.2) if and only if K is an interval.

Widom factors for the sup-norm

Theorem (Schiefermayr-Zinchenko 21)

Let K be a compact non-polar subset of $[-1,1]$ and $w(x)=\sqrt{1-x^2}$. Then

1. *For each $n\in\mathbb{N}$*

$$W_{\infty,n}(K,w)\geq 2\mathrm{Cap}(K). \tag{3.3}$$

Equality is attained in (3.3) if and only if there exists a polynomial S_n of degree n such that $K=\{z\in\mathbb{C}:(1-z^2)S_n^2(z)\in[0,1]\}$.

2. *In addition, let us assume that K is a Parreau-Widom set. Then*

$$W_{\infty,n}(K,w)\leq 2\,\mathrm{Cap}(K)e^{(1/2)g_K(1)+(1/2)g_K(-1)+\mathrm{PW}(K)}. \tag{3.4}$$

Equality is attained in (3.4) if and only if $K=[-1,1]$.

Widom factors for the sup-norm

Theorem (A. 22)

Let K be a compact non-polar subset of $[-1,1]$ and $w(x)=\sqrt{1+x}$. Then

1. *For each $n\in\mathbb{N}$*

$$W_{\infty,n}(K,w)\geq 2\sqrt{\mathrm{Cap}(K)}. \tag{3.5}$$

   Equality is attained in (3.5) if and only if there exists a polynomial S_n of degree n such that $K=\{z\in\mathbb{C}:(1+z)S_n^2(z)\in[0,1]\}$.

2. *In addition, let us assume that K is a Parreau-Widom set. Then*

$$W_{\infty,n}(K,w)\leq 2\sqrt{\mathrm{Cap}(K)}e^{(1/2)g_K(-1)+\mathrm{PW}(K)}. \tag{3.6}$$

   Equality is attained in (3.6) if and only if $K=[-1,b]$ for some $b\in(-1,1]$.

Widom factors for the sup-norm

Theorem (A. 22)

Let K be a compact non-polar subset of $[-1,1]$ and $w(x) = \sqrt{1-x}$. Then

1. *For each $n \in \mathbb{N}$*

$$W_{\infty,n}(K,w) \geq 2\sqrt{\mathrm{Cap}(K)}. \tag{3.7}$$

Equality is attained in (3.7) if and only if there exists a polynomial S_n of degree n such that $K = \{z \in \mathbb{C} : (1-z)S_n^2(z) \in [0,1]\}$.

2. *In addition, let us assume that K is a Parreau-Widom set. Then*

$$W_{\infty,n}(K,w) \leq 2\sqrt{\mathrm{Cap}(K)}e^{(1/2)g_K(1)+\mathrm{PW}(K)}. \tag{3.8}$$

Equality is attained in (3.8) if and only if $K = [a,1]$ for some $a \in [-1,1)$.

Thank you for your attention.