An approach to universality using Weyl m-functions

Milivoje Lukić

joint work with
Benjamin Eichinger
and
Brian Simanek

arXiv:2108.01629

Christoffel–Darboux kernel

- Let μ be a probability measure on $\mathbb{R}$ with all finite moments,

$$\int |\xi|^n \, d\mu(\xi) < \infty, \qquad \forall n \in \mathbb{N}.$$

  Assume that μ has infinite support (in sense of cardinality).
- From the sequence of monomials $\{z^j\}_{j=0}^{\infty}$ in $L^2(\mathbb{R}, d\mu)$, the Gram–Schmidt process gives orthonormal polynomials $\{p_j(z)\}_{j=0}^{\infty}$
- The Christoffel–Darboux (CD) kernel is

$$K_n(z,w) = \sum_{j=0}^{n-1} p_j(z)\overline{p_j(w)}.$$

  Reproducing kernel for subspace $\operatorname{span}\{1, z, \dots, z^{n-1}\} \subset L^2(\mathbb{R}, d\mu)$

Universality limits

- Universality limits of CD kernels are double scaling limits

$$\lim_{n\to\infty} \frac{1}{\tau_n} K_n \left(\xi + \frac{z}{\tau_n}, \xi + \frac{w}{\tau_n} \right)$$

  for an appropriate sequence $\tau_n \to \infty$ and $z, w \in \mathbb{C}, \xi \in \mathbb{R}$.

- They are called universality limits because the limit is often found to be a standard kernel and not depend on exact measure we started with: the most common phenomenon is bulk universality, associated with sine kernel

$$\frac{\sin(\overline{w} - z)}{\overline{w} - z}$$

- Wigner 1955: local eigenvalue statistics of random matrices as a model for local statistical behavior of resonances in scattering theory

Bulk universality

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{z}{f_\mu(\xi)K_n(\xi,\xi)}, \xi + \frac{w}{f_\mu(\xi)K_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin(\pi(\overline{w}-z))}{\pi(\overline{w}-z)}$$

Bulk universality was proved in several settings:

- For Gaussian measure $d\mu = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2}\, d\xi$ follows from properties of Hermite polynomials
- Deift–Kriecherbauer–McLaughlin–Venakides–Zhou: Riemann–Hilbert techniques for measures

  $$d\mu = e^{-Q(\xi)} d\xi$$

  Q a polynomial of even degree
- Lubinsky, with extensions by Totik, Findley, Simon, Mitkovski: Stahl–Totik regular measures $d\mu$ with local Lebesgue point/local Szegő conditions at ξ

A local criterion for bulk universality

Theorem (Eichinger–Lukić–Simanek)

Let μ be a probability measure on $\mathbb{R}$ with infinite support and finite moments, corresponding to a determinate moment problem. Let

$$m(z) = \int \frac{1}{x-z}\, d\mu(x), \qquad z \in \mathbb{C}_+.$$

Let $\xi \in \mathbb{R}$ and assume that for some $0 < \alpha < \pi/2$,

$$f_\mu(\xi) := \frac{1}{\pi} \lim_{\substack{z \to \xi \\ \alpha \le \arg(z-\xi) \le \pi - \alpha}} \operatorname{Im} m(z) \in (0, \infty)$$

Then uniformly on compact regions of $(z, w) \in \mathbb{C} \times \mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{z}{f_\mu(\xi) K_n(\xi,\xi)}, \xi + \frac{w}{f_\mu(\xi) K_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin(\pi(\overline{w} - z))}{\pi(\overline{w} - z)}$$

Nontangential limits of $m(z)$

- The nontangential limit

$$f_\mu(\xi) := \frac{1}{\pi} \lim_{\substack{z\to\xi \\ \alpha\le\arg(z-\xi)\le\pi-\alpha}} \operatorname{Im} m(z)$$

exists for Lebesgue-a.e. $\xi \in \mathbb{R}$

- Pointwise, it exists at every Lebesgue point of the measure μ
- This limit recovers the a.c. part of the measure:

$$d\mu(\xi) = f_\mu(\xi)d\xi + d\mu_{\mathrm{s}}(\xi)$$

- The essential support for a.c. spectrum is the set

$$\Sigma_{\mathrm{ac}}(\mu) = \{\xi \in \mathbb{R} \mid f_\mu(\xi) \in (0,\infty)\}$$

In particular, this solves a conjecture of Avila–Last–Simon:

Corollary

Bulk universality holds almost everywhere on $\Sigma_{\mathrm{ac}}(\mu)$.

Local zero spacing

- Denote by $\xi_j^{(n)}(\xi)$ for $j \in \mathbb{Z}$ the zeros of p_n counted from ξ, i.e.,

$$\dots < \xi_{-2}^{(n)}(\xi) < \xi_{-1}^{(n)}(\xi) < \xi \le \xi_0^{(n)}(\xi) < \xi_1^{(n)}(\xi) < \dots$$

- Freud–Levin theorem: The bulk universality limit

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{z}{f_\mu(\xi)K_n(\xi,\xi)}, \xi + \frac{w}{f_\mu(\xi)K_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin(\pi(\overline{w}-z))}{\pi(\overline{w}-z)}$$

implies

$$\lim_{n\to\infty} f_\mu(\xi)K_n(\xi,\xi)(\xi_{j+1}^{(n)}(\xi) - \xi_j^{(n)}(\xi)) = 1 \qquad \forall j \in \mathbb{Z}.$$

Statements of this type are commonly described as “clock behavior”.

Rescaling

- Some of the cited prior literature is actually formulated at some explicit polynomial scale $\tau_n = n^\alpha$
- If
$$\lim_{n\to\infty} \frac{1}{\tau_n} K_n\left(\xi + \frac{z}{\tau_n}, \xi + \frac{w}{\tau_n}\right)$$
converges to a sine kernel, evaluating at $z = w = 0$ gives
$$\lim_{n\to\infty} \frac{K_n(\xi,\xi)}{\tau_n} \in (0,\infty)$$
- Conversely, if
$$\lim_{n\to\infty} \frac{K_n(\xi,\xi)}{\tau_n} \in (0,\infty)$$
one scale can be replaced by the other

Growth rate of $K_n(\xi,\xi)$

- For measures $d\mu = e^{-Q(\xi)}\,d\xi$, $K_n(\xi,\xi) \sim n^\alpha$, $\alpha \in (0,1)$
- For compactly supported measures:

$$\lim_{n\to\infty} \frac{K_n(\xi,\xi)}{n} = \frac{f_{\mathsf{E}}(\xi)}{f_\mu(\xi)}$$

if μ is Stahl–Totik regular, f_{E} denotes the density of the equilibrium measure of the essential spectrum $\mathsf{E} = \operatorname{ess\,supp}\mu$, $f_\mu(\xi) > 0$, $\log f_\mu$ is integrable in a neighborhood of ξ, and ξ is a Lebesgue point of both the measure μ and the function $\log f_\mu$ (Máté–Nevai–Totik for $\mathsf{E} = [-2,2]$, generalized by Totik)

Second kind polynomials

- Jacobi recursion: for some sequence of $a_n > 0$, $b_n \in \mathbb{R}$,

$$zp_n(z) = a_n p_{n-1}(z) + b_{n+1} p_n(z) + a_{n+1} p_{n+1}(z)$$

  with convention $p_{-1}(z) = 0$
- Second kind polynomials for μ are defined by

$$q_n(z) = \int \frac{p_n(z) - p_n(\xi)}{z - \xi} d\mu(\xi)$$

  for $n = 0, 1, 2, \ldots$ and $q_{-1}(z) = -1$.
- Matrix version of Christoffel–Darboux kernel defined by

$$\mathcal{K}_n(z,w) = \begin{pmatrix} \sum_{j=0}^{n-1} p_j(z)\overline{p_j(w)} & \sum_{j=0}^{n-1} q_j(z)\overline{p_j(w)} \\ \sum_{j=0}^{n-1} p_j(z)\overline{q_j(w)} & \sum_{j=0}^{n-1} q_j(z)\overline{q_j(w)} \end{pmatrix}$$

Limits of m-function

- If $m : \mathbb{C}_+ \to \mathbb{C}_+$ has a normal limit at ξ, then

$$\eta = \lim_{y \downarrow 0} m(\xi + iy) \in \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}.$$

- For $\eta \in \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$, define

$$\mathring{H}_\eta := \frac{1}{1+|\eta|^2} \begin{pmatrix} 1 & -\operatorname{Re}\eta \\ -\operatorname{Re}\eta & |\eta|^2 \end{pmatrix} \qquad \eta \in \mathbb{C}_+ \cup \mathbb{R}$$

$$\mathring{H}_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Denote also

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and define

$$\mathring{\mathcal{K}}_\eta(z,w) = \int_0^1 e^{-t\overline{w}\mathring{H}_\eta j} \mathring{H}_\eta e^{tzj\mathring{H}_\eta}\, dt$$

Bulk universality for matrix CD kernel

Theorem (Eichinger–Lukić–Simanek)

Denote $\tau(n) = \operatorname{tr} \mathcal{K}_n(\xi, \xi)$. *The following are equivalent:*

1. *m has a normal limit at* ξ,

$$\lim_{y \downarrow 0} m(\xi + iy) = \eta \in \overline{\mathbb{C}_+}$$

2. *For some* 2×2 *matrix* H,

$$\lim_{n \to \infty} \frac{1}{\tau(n)} \mathcal{K}_n(\xi, \xi) = H$$

3. *For some function* $\mathcal{K}(z, w)$, *uniformly on compact subsets of* $(z, w) \in \mathbb{C} \times \mathbb{C}$,

$$\lim_{n \to \infty} \frac{1}{\tau(n)} \mathcal{K}_n \left(\xi + \frac{z}{\tau(n)}, \xi + \frac{w}{\tau(n)} \right) = \mathcal{K}(z, w).$$

Moreover, in this case, $H = \mathring{H}_\eta$ *and* $\mathcal{K}(z, w) = \mathring{\mathcal{K}}_\eta(z, w)$.

Remark: connection to subordinacy theory

- Recall

$$\mathcal{K}_n(\xi,\xi) = \begin{pmatrix} \sum_{j=0}^{n-1} |p_j(\xi)|^2 & \sum_{j=0}^{n-1} q_j(\xi)\overline{p_j(\xi)} \\ \sum_{j=0}^{n-1} p_j(\xi)\overline{q_j(\xi)} & \sum_{j=0}^{n-1} |q_j(\xi)|^2 \end{pmatrix}$$

- By Cauchy–Schwarz,

$$\lim_{n\to\infty} \frac{1}{\operatorname{tr} \mathcal{K}_n(\xi,\xi)} \mathcal{K}_n(\xi,\xi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \iff \qquad \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} p_j(\xi)^2}{\sum_{j=0}^{n-1} q_j(\xi)^2} = 0$$

- This recovers subordinacy result of Gilbert–Pearson, Kahn–Pearson:

$$\lim_{y\downarrow 0} m(\xi + iy) = \infty \qquad \iff \qquad \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} p_j(\xi)^2}{\sum_{j=0}^{n-1} q_j(\xi)^2} = 0$$

j-monotonic families of transfer matrices

- Let $A, B : [0,\infty) \to \mathbb{C}^{2\times 2}$ be locally integrable and

$$A(x) \geq 0, \qquad B(x)^* = B(x), \qquad \mathrm{tr}(A(x)j) = \mathrm{tr}(B(x)j) = 0$$

  for Lebesgue-a.e. x.
- Let $T : [0,\infty) \times \mathbb{C} \to \mathbb{C}^{2\times 2}$ be the solution of the initial value problem

$$j\partial_x T(x,z) = (-zA(x) + B(x))T(x,z), \quad T(0,z) = I$$

- Applications: transfer matrices of Schrödinger operators $L_V = -\frac{d^2}{dx^2} + V$, Dirac operators, orthogonal polynomials on the real line
- With some effort: orthogonal polynomials on the unit circle
- Assume the limit point condition

$$\mathrm{tr} \int_0^\infty T(x,0)^* A(x) T(x,0)\, dx = \infty$$

Weyl disks and m-functions

- CD formula implies j-monotonic property

$$\frac{T(x_2,z)^* j T(x_2,z) - T(x_1,z)^* j T(x_1,z)}{i} \le 0, \qquad z \in \mathbb{C}_+, \quad 0 \le x_1 \le x_2$$

- For any $z \in \mathbb{C}_+$, the Weyl disks are defined by

$$D(x,z) = \{w \in \hat{\mathbb{C}} \mid T(x,z)w \in \overline{\mathbb{C}_+}\}$$

The Weyl disks are nested,

$$D(x_2,z) \subset D(x_1,z), \qquad z \in \mathbb{C}_+, \quad 0 \le x_1 \le x_2$$

- In the limit point case, Weyl function $m : \mathbb{C}_+ \to \mathbb{C}_+$ is defined by

$$\{m(z)\} = \bigcap_{x \ge 0} D(x,z).$$

Kernels and variation of parameters

- Matrix CD kernel and CD formula given by

$$\mathcal{K}_L(z,w) = \int_0^L T(x,w)^* A(x) T(x,z)\, dx = \frac{T(L,w)^* j T(L,z) - j}{\overline{w} - z}$$

- Scalar CD kernel defined by

$$K_L(z,w) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \mathcal{K}_L(z,w) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- For $U(L) \in \mathrm{SL}(2,\mathbb{R})$, gauge transformation

$$\{T(L,z)\} \mapsto \{U(L) T(L,z)\}$$

  doesn't affect the kernels because $U(L)^* j U(L) - j = 0$
- $M(L,z) = T(L,0)^{-1} T(L,z)$ solution of a canonical system with

$$H(x) = T(x,0)^* A(x) T(x,0)$$

A local criterion for bulk universality

Theorem (Eichinger–Lukić–Simanek)

Let $\xi \in \mathbb{R}$ *and assume that for some* $0 < \alpha < \pi/2$*,*

$$f_\mu(\xi) := \frac{1}{\pi} \lim_{\substack{z \to \xi \\ \alpha \le \arg(z-\xi) \le \pi - \alpha}} \operatorname{Im} m(z) \in (0, \infty)$$

Then uniformly on compact regions of $(z, w) \in \mathbb{C} \times \mathbb{C}$*,*

$$\lim_{L \to \infty} \frac{K_L\left(\xi + \frac{z}{f_\mu(\xi) K_L(\xi,\xi)}, \xi + \frac{w}{f_\mu(\xi) K_L(\xi,\xi)}\right)}{K_L(\xi, \xi)} = \frac{\sin(\pi(\overline{w} - z))}{\pi(\overline{w} - z)}$$

Bulk universality for matrix kernel

Theorem (Eichinger–Lukić–Simanek)

Denote $\tau(L) = \operatorname{tr} \mathcal{K}_L(\xi,\xi)$. The following are equivalent:

1. *m has a normal limit at ξ,*
$$\lim_{y \downarrow 0} m(\xi + iy) = \eta \in \overline{\mathbb{C}_+}$$
2. *For some 2×2 matrix H,*
$$\lim_{L \to \infty} \frac{1}{\tau(L)} \mathcal{K}_L(\xi,\xi) = H$$
3. *For some function $\mathcal{K}(z,w)$, uniformly on compact subsets of $(z,w) \in \mathbb{C} \times \mathbb{C}$,*
$$\lim_{L \to \infty} \frac{1}{\tau(L)} \mathcal{K}_L\left(\xi + \frac{z}{\tau(L)}, \xi + \frac{w}{\tau(L)}\right) = \mathcal{K}(z,w).$$

Moreover, in this case, $H = \mathring{H}_\eta$ and $\mathcal{K}(z,w) = \mathring{\mathcal{K}}_\eta(z,w)$.

de Branges homeomorphism

- Canonical systems are initial value problems of the form

$$j\partial_x M(x,z) = -zH(x)M(x,z), \qquad M(0,z) = I$$

- Reparametrize x to impose $\operatorname{tr} H = 1$ a.e..
- de Branges: map $H \mapsto m$ is a bijection
- The correspondences between $H, m, M, \mathcal{K}$ are homeomorphisms (homeomorphisms between first three previously known, see Eckhart–Kostenko–Teschl or Remling)

Scaling operation

- Consider a trace-parametrized canonical system

$$j\partial_t M(t,z) = -zH(t)M(t,z), \qquad M(0,z) = I$$

with Weyl function $m(z)$ and kernel $\mathcal{K}_t(z,w)$

- For $r > 0$, a scaling operation

$$\begin{aligned} m_r(z) &= m(z/r) \\ H_r(t) &= H(rt) \\ M_r(t,z) &= M(rt, z/r) \\ (\mathcal{K}_r)_t(z,w) &= \frac{1}{r}\mathcal{K}_{rt}(z/r, w/r) \end{aligned}$$

found by Kasahara for Krein strings; used by Eckhardt–Kostenko–Teschl and Langer–Pruckner–Woracek for canonical systems to investigate large energy asymptotics of m-function

- We use the scaling operation to "zoom in" towards $\xi \in \mathbb{R}$

Proofs of Theorems

Proof of Theorem 2:

- Start from transfer matrices $T(L,z)$ with Weyl function $m(z)$
- Apply gauge transformation and trace-parametrize

$$M(t,z) = T(L,\xi)^{-1} T(L,\xi+z)$$

- Consider family of canonical systems corresponding to Weyl functions

$$m_r(z) = \begin{cases} m(\xi + z/r) & r \in [1,\infty) \\ \eta & r = \infty \end{cases}$$

- Characterize continuity of this family in terms of $H, m, M, \mathcal{K}$

Proof of Theorem 1:

- In addition, use a translation trick and consider the family

$$\tilde{m}_r(z) = \begin{cases} m(\xi + z/r) - \operatorname{Re} m(\xi + i/r) & r \in [1,\infty) \\ i f_\mu(\xi) & r = \infty \end{cases}$$

Translation trick

- Action by a matrix $A \in \mathrm{SL}(2,\mathbb{R})$ on a canonical system,

$$m_A(z) = A^{-1} m(z), \quad H_A = A^* H A, \quad M_A = A^{-1} M A, \quad \mathcal{K}_A = A^* \mathcal{K} A$$

- Translations in the spectral parameter correspond to

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$A\binom{1}{0} = \binom{1}{0}$ implies that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* (\mathcal{K}_A)_L(0,0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \mathcal{K}_L(0,0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so the scalar CD kernel is unaffected by A, except for a change in parametrization

- $K_L(0,0)$ is not a parameter (often not injective), but it is injective for a constant coefficient canonical system with $\eta \in \mathbb{C}_+ \cup \mathbb{R}$

Jacobi recursion (OPRL)

- Modify Jacobi transfer matrices by a conjugation,

$$T(n,z) = j_1 \prod_{k=1}^{n} \begin{pmatrix} \frac{z-b_k}{a_k} & -\frac{1}{a_k} \\ a_k & 0 \end{pmatrix} j_1, \qquad j_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Define

$$M(n,z) = T(n,0)^{-1} T(n,z)$$

and interpolate linearly,

$$M(x,z) = M(\lfloor x \rfloor, z) + (x - \lfloor x \rfloor)(M(\lfloor x \rfloor + 1, z) - M(\lfloor x \rfloor, z))$$

- This is the solution of a canonical system

$$j\partial_x M(x,z) = -zH(x)M(x,z), \qquad M(0,z) = I$$

with piecewise constant data

$$H(x) = \begin{pmatrix} p_n(0)^2 & q_n(0)p_n(0) \\ p_n(0)q_n(0) & q_n(0)^2 \end{pmatrix}, \qquad x \in [n, n+1)$$

- Linear interpolation works because jH is nilpotent

Szegő recursion (OPUC)

- Correspondence between measure μ on unit circle, Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$, Caratheodory function F
- Szegő recursion can be written in matrix form as

$$S(n,z) = \prod_{k=0}^{n-1} A(\alpha_k, z), \quad A(\alpha_n, z) = \frac{1}{\sqrt{1-|\alpha_n|^2}} \begin{pmatrix} z & -\overline{\alpha_n} \\ -\alpha_n z & 1 \end{pmatrix}$$

- Denote

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

- Lemma: the functions

$$T(n,z) = e^{-inz/2} \mathcal{C}^{-1} \mathcal{J} S(n, e^{iz}) \mathcal{J} \mathcal{C}$$

are a j-monotonic family of entire j-inner functions and obey $T(0,z) = I$, $\det T(n,z) = 1$, limit point case with

$$m(z) = iF(e^{iz}).$$

- Lemma is inspired by a substitution used by Damanik–Yuditskii to relate comb domains

Thank you!