A Tale of Three Coauthors: Comparison of Ising Models

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$$\langle f\rangle_{\mu,\Lambda} = Z^{-1}\langle fe^{-H}\rangle_{\mu,0}; \qquad Z = \langle e^{-H}\rangle_{\mu,0}$$

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$$\langle \sigma^B \rangle_{\mu,\Lambda} \leq \langle \sigma^B \rangle_{\nu,\Lambda}$$

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$$T_c(\mu) \geq T_-(\mu)^2 T_c(\textit{classical Ising})$$

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The left hand side is an Ising expectation and the right with the apriori measure of the $2D$ rotor with only couplings of the 1 components. So this was part of what seems to be an Ising domination result (the 2 indicates the Ising measure should really be $b_{1/\sqrt{2}}$).

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A *Ginibre system* is a triple $\langle X, \mu, \mathcal{F} \rangle$ of a compact Hausdorff space, X, a probability measure, μ, on X (with expectations $\langle \cdot \rangle_\mu$) and a class of continuous real valued functions $\mathcal{F} \subset C(X)$ that obeys:

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We will see shortly that $(G2) \Rightarrow (G1)$

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Ginibre Theorem 1 *If a triple* $\langle X, \mu, \mathcal{F} \rangle$ *obeys* $(G2)$*, so does* $\langle X, \mu, \mathcal{C}(\mathcal{F}) \rangle$.

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What makes the notion so powerful is that there are three theorems for getting new Ginibre systems from old ones.

Given a family of functions, $\mathcal{F} \subset C(X)$, we define the *Ginibre cone*, $\mathcal{C}(\mathcal{F})$, as the set of linear combinations with non-negative coefficients of products of functions from $\mathcal{F}$.

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The proof is easy. The normalization is irrelevant and we expand the exponential $\exp(-H(x) - H(y))$.

Classical Ising System

Ginibre Theorem 4 *Let X be $\mathbb{R}$ or a compact subset of the form $[-A, A]$ and let $d\mu$ be a probability measure which is invariant under $x \mapsto -x$ and so that (only non-trivial in case X is not compact) $\int x^{2n}\, d\mu(x) < \infty$ for all n. Let $\mathcal{F}$ contain the single function, $f(x) = x$. Then $\langle X, \mu, \mathcal{F} \rangle$ is a Ginibre system.*

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Interchanging x and y implies the integral is zero if m is odd and $x \mapsto -x$ symmetry implies the integral is zero if $m+k$ is odd. Thus the only possible non-zero integrals are when m and k are even in which case the integrand is positive!

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$$-H = \sum_{A \subset \Lambda} J(A)\sigma^A \qquad \sigma^A = \prod_{j \in A} \sigma_j$$

with ANY (!!!) even apriori measure, one has positive expectations and positive correlations of the σ^A.

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Basic Definition

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for all non-negative integers, n and m in which case we use the symbol $\triangleleft$ without being explicit about $\mathcal{F}$. Since the measures are even, one need only check this when $n+m$ is even. It is trivial if both are even, so we only need worry about the case that both are odd. Since the measures are different, we don't have the exchange symmetry that makes the integral vanish if both are odd but symmetry under $y \mapsto -y$ implies invariance under interchange of m and n, so we need only check for $m \geq n$. We'll see examples later.

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Corollary *If for $j = 1, \dots, n$, $\mu_j \lhd \nu_j$ for probability measures on spaces X_j with respect to sets of functions $\mathcal{F}_j$ on X_j, then if $-H \in \mathcal{C}(\cup_{j=1}^n \mathcal{F}_j)$ and if μ_H, ν_H are formed from the underlying product measures $\otimes_{j=1}^n \mu_j$ and $\otimes_{j=1}^n \nu_j$, then for all $F \in \mathcal{C}(\cup_{j=1}^n \mathcal{F}_j)$, one has that $\int f(x)\, d\mu_H(x) \leq \int f(x)\, d\nu_H(x)$. In particular, if each $X_j = \mathbb{R}$, (so implicitly F_j is the single function σ_j) and if H has the general Ising form, then for all $A \subset 2^{\{1,\dots,n\}}$ one has that*

$$\langle \sigma^A \rangle_{\mu_H} \leq \langle \sigma^A \rangle_{\nu_H}$$

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Since Ising domination is trivially transitive, for applications, this lack isn't so important.

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Big Theorem *Let $d\mu$ be an even probability measure on $\mathbb{R}$ with compact support that is not a point mass at 0.*

Statement of the Theorem

We say an even probability measure is non-trivial if and only if it is not a unit mass at 0. The following theorem says that any non-trivial measure of compact support is Ising dominated by a scaling of any other such measure and gives quantitative optimal bounds when one of the measures is the Bernoulli measure.

Big Theorem *Let $d\mu$ be an even probability measure on $\mathbb{R}$ with compact support that is not a point mass at 0. Then there are two strictly positive numbers $T_-(\mu)$ and $T_+(\mu)$ so that $\mu \lhd b_S$ if and only if $S \geq T_+$ and $b_S \lhd \mu$ if and only if $S \leq T_-$. Moreover*

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$$S \leq T_- \iff \forall_{n \in \mathbb{N}} \int_{\mathbb{R}} (x^2 - S^2)^n \, d\mu(x) \geq 0$$

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It is an interesting question when one has equality. Before leaving this theorem, I should mention I happened to look at a 1981 paper of Bricmont, Lebowitz and Pfister that includes in an appendix a proof (with attribution to Wells) of Wells result about the existence of $T_- > 0$.

Three Spin Values

For $0 \leq \lambda \leq 1$, consider the probability measure supported by the three points $\{0, \pm 1\}$ given by

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$$T_{-}(\lambda) = \begin{cases} \sqrt{\lambda}, & \text{if } \lambda \leq \frac{1}{2} \\ \sqrt{\frac{1}{2}}, & \text{if } \lambda \geq \frac{1}{2} \end{cases}$$

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So we see there are cases where $T_{-} = \langle x^2 \rangle^{1/2} = \sqrt{\lambda}$ and other cases where the inequality is strict. Note also that at $\lambda = \frac{1}{2}$, the integral $\langle (x^2 - T_{-}^2)^{2m+1} \rangle_\lambda$ vanishes for all n, a sign that the distribution of $x^2 - T_{-}^2$ is symmetric about 0.

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Conjecture For $S = \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$ one has that

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As explained earlier, because Wells domination implies Ising domination, one has that for pair interactions

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Not only is this an improvement of Griffiths by more than $\frac{4}{3}$ but in the result for $S \neq 1$, the improved constant is optimal!! For one has equality if T_c is replaced by its mean field values and as noted by Dyson, Lieb and Simon, mean field theory is exact in the nearest neighbor infinite dimension limit.

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For S an integer, this is the usual kind of sum.

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It seemed a shame not to make one more push to prove the conjecture so I did the obvious thing.

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His note had one wonderful idea (using Karamata's inequality) and 5 dense pages of calculation to implement it.

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José also suggested it would be good to try again to locate Daniel Wells.

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Majorization

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If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{+,\geq}$ (the set with $x_1 \geq x_2 \geq \ldots x_n \geq 0$), we say that $\mathbf{x}$ *majorizes* $\mathbf{y}$, written $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{y} \prec \mathbf{x}$ if an only if

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j; \quad S_k(\mathbf{x}) \equiv \sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j,\ k = 1, \ldots, n{-}1$$

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Karamata's Inequality

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Theorem (Karamata's Inequality) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{+,\geq}$ with $\mathbf{x} \succ \mathbf{y}$ and let φ be an arbitrary continuous convex function on $[0, x_1]$. Then

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Even though this is widely referred to as Karamata's inequality after Karamata's 1932 paper, it or theorems that imply it appear in a 1923 paper of Schur and a 1929 paper of Hardy-Littlewood-Pólya.

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That said, we note that HLP doesn't have a proof which may not have appeared until their 1934 book and that Karamata proved a converse, namely, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{+,\geq}$ and the inequality holds for all convex φ, then $\mathbf{x} \succ \mathbf{y}$.

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The proof of Karamata's theorem is simple. One proves the convex hull result and then one notes the function $\mathbf{w} \mapsto \sum_{j=1}^{n} \varphi(w_j)$ is convex and permutation symmetric.

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$$m < p \Rightarrow \psi\,(m+1) - \psi\,(m) \leq \psi\,(p+1) - \psi\,(p)$$

which is true by convexity of ψ.

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