Long time asymptotics of Toda shock waves

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Complex Analysis, Spectral Theory and Approximation meet in Linz,
July 2022

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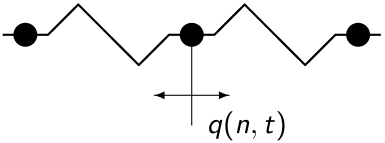
FWF Der Wissenschaftsfonds.

Based on

- Long-time asymptotics for the Toda shock problem: Non-overlapping spectra, with I. Egorova and J. Michor, Zh. Mat. Fiz. Anal. Geom. **14**, 406–451 (2018)
- Long-time asymptotics for Toda shock waves in the modulation region, with I. Egorova, J. Michor, and A. Pryimak, arXiv:2001.05184

Motion of a chain of particles coupled via nonlinear springs



with potential

$$V(r) = \mathrm{e}^{-r} + r - 1 = \frac{r^2}{2} - \frac{r^3}{6} + O(r^4).$$

Applications: Used to model Langmuir oscillations in plasma physics, to investigate conducting polymers, in quantum cohomology, etc. (several monographs about the Toda equation).

In Flaschka's variables

$$a(n,t) = \frac{1}{2}\mathrm{e}^{-(q(n+1,t) - q(n,t))/2}, \quad b(n,t) = -\frac{1}{2}\dot{q}(n,t)$$

the Toda equation explicitly reads:

$$\dot{a}(n,t) = a(t)\Big(b(n+1,t) - b(n,t)\Big),$$
$$\dot{b}(n,t) = 2\Big(a(n,t)^2 - a(n-1,t)^2\Big).$$

Here $\dot{} = \frac{d}{dt}$.

More specific, we consider the Cauchy problem for the Toda lattice equation with initial data which is asymptotically constant

$$a(n,0) \to \frac{1}{2}, \quad b(n,0) \to 0, \quad \text{as } |n| \to -\infty.$$

Hence the corresponding operator L is a *small* perturbation of the background operator

$$(L_0 y)(n) := \frac{1}{2} y(n-1) + \frac{1}{2} y(n+1).$$

One can show that this asymptotic behavior is preserved by the time evolution.

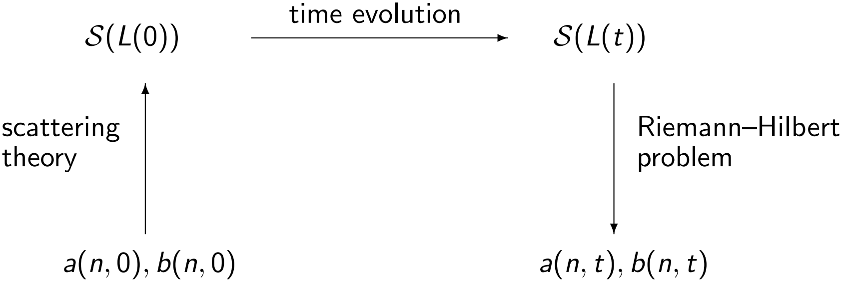
- The Toda lattice admits the Lax representation: $\frac{d}{dt}L = [L, P]$,

$$(L(t)y)(n) = a(n-1,t)y(n-1) + b(n,t)y(n) + a(n,t)y(n+1),$$
$$(P(t)y)(n) = -a(n-1,t)y(n-1) + a(n,t)y(n+1).$$

- Spectrum is preserved: $L(t) = U(t)L(0)U(-t)$. Here U is the solution of $\dot{U}(t) = P(t)U(t)$, $U(0) = \mathbb{I}$ and is unitary since P is skew-adjoint.
- Infinitely many preserved quantities $\mathrm{tr}(L(t)^n - L_0^n)$, $n \in \mathbb{N}$.

The inverse scattering transform

The initial value problem for the Toda lattice can be solved via the inverse scattering transform:



The long-time asymptotics can then be found via a nonlinear steepest descent analysis (Manakov, Its, Deift & Zhao).

Here we consider the Cauchy problem for the Toda lattice with steplike initial data

$$a(n,0) \to a, \quad b(n,0) \to b, \quad \text{as } n \to -\infty,$$
$$a(n,0) \to \frac{1}{2}, \quad b(n,0) \to 0, \quad \text{as } n \to +\infty.$$

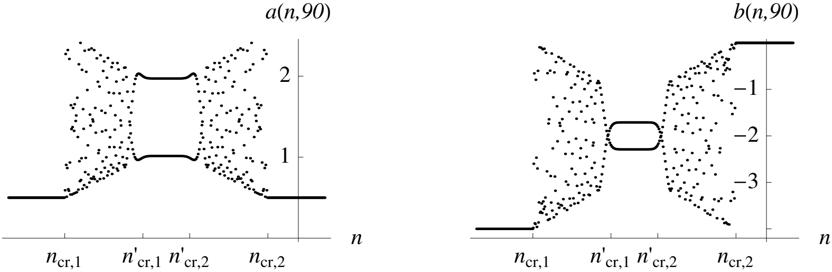
Note that now there are two different background operators.
The Toda shock problem is the case satisfying

$$2a + b < -1.$$

Note that the spectra of the two background operators, $[b - 2a, b + 2a]$ and $[-1, 1]$ are nonoverlapping in this case. Hence this condition should be thought of a a condition on the mutual location of the background spectra.

Classical shock problem: $a(n,0) = \frac{1}{2}$, $b(n,0) = \mathrm{sign}(n)b$, $b > 1$.
(Note that $b < -1$ would be the rarefaction problem.)

Numerically the situation looks as follows:



Problem: Explain/prove this picture.

Numerical investigations

- B. L. Holian and G. K. Straub (1978).
- B. L. Holian, H. Flaschka, and D. W. McLaughlin (1981).

Theoretical

- S. Venakides, P. Deift, and R. Oba (1991)
- A.M. Bloch, Y. Kodama (1991, 1992)
- S. Kamvissis (1993)
- A. Boutet de Monvel, I. Egorova, and E. Khruslov (1997)
- I. Egorova, J. Michor, and G.T. (2018)
- I. Egorova and J. Michor (2021)

Also mentioned in the list of open problems by P. Deift in SIGMA (2017).

The operator $L(t)$ has a continuous spectrum $\mathfrak{S}$,
$\mathfrak{S} = [b-2a, b+2a] \cup [-1,1]$ and a finite discrete spectrum. The Jacobi equation

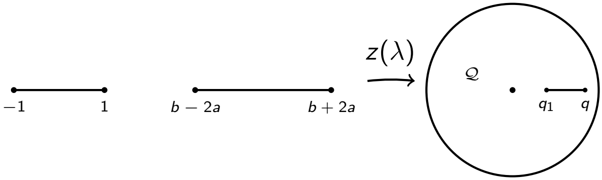
$$a(n-1,t)y(n-1) + b(n,t)y(n) + a(n,t)y(n+1) = \lambda y(n)$$

has two **Jost solutions**

$$\phi(z,n,t) \sim z^n, \quad n \to +\infty, \qquad \psi(z,n,t) \sim \zeta^{-n}, \quad n \to -\infty.$$

Two associated Joukowsky transforms of the spectral parameter:

$$\lambda = \frac{1}{2}\left(z + z^{-1}\right) = b + a\left(\zeta + \zeta^{-1}\right), \ |z| \le 1, \ |\zeta| \le 1.$$



- Wronskian of the Jost solutions ($z \in \mathcal{Q}$):

  $$W(z) = a(n-1,0)(\phi(z,n-1,0)\psi(z,n,0) - \phi(z,n,0)\psi(z,n-1,0)).$$

  A boundary point $\tilde{q} \in \{q, q_1\}$ of the spectrum is called **resonant** if $W(\tilde{q}) = 0$. Here

  $$z([b-2a, b+2a]) = [q_1, q].$$

- Right scattering data (for the initial conditions $t = 0$):

  $$\{R(z),\ z \in \mathbb{T};\ \ \chi(z),\ z \in [q_1, q];\quad z_j, \gamma_j > 0\},$$

  where

  $$\chi(z) = 2a\frac{(z - z^{-1})(\zeta(z) - \zeta^{-1}(z))}{|W(z)|^2} = -\overline{T(z)}\, T_{left}(z),$$

  $T(z) = T(z,0)$ the right transmission coefficient, $\lambda_j = \frac{z_j + z_j^{-1}}{2}$ an eigenvalue and γ_j the corresponding norming constant.

Initial RHP – Setup

In $\mathcal{Q}$ introduce the vector-function $m(z) = m(z, n, t)$

$$m(z) = (m_1(z), m_2(z)) = \left(T(z,t)\psi(n,z,t)z^n,\ \phi(z,n,t)z^{-n} \right).$$

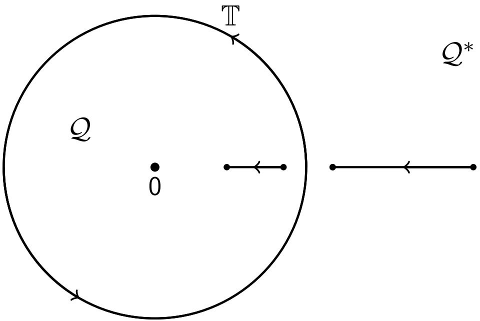
The solution $\{a(n,t), b(n,t)\}$ can be obtained from m via:

$$\frac{m_1(0,n,t)}{m_1(0,n+1,t)} = 2a(n,t),$$
$$\lim_{z\to 0} \frac{1}{2z} \left(m_1(z,n,t) m_2(z,n,t) - 1 \right) = b(n,t).$$

Set $\mathcal{Q}^* = \{z : \ z^{-1} \in \mathcal{Q}\}$ and extend $m(z)$ to $\mathcal{Q}^*$ by

$$m(z) = m(z^{-1})\sigma_1, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Below is a visualization of the jump contour Σ consisting of the unit circle $\mathbb{T}$ and two intervals:



The vector $m(z)$ is the **unique** solution of the following RHP: Find a vector-valued function m which is meromorphic in $\mathcal{Q} \cup \mathcal{Q}^*$ and continuous up to Σ except at possibly the points $q^{\pm 1}, q_1^{\pm 1}$. It has simple poles at $z_j^{\pm 1}$, $j = 1, \dots, N$, and satisfies:

- the jump condition: $m_+(z) = m_-(z) v(z)$, where

$$v(z) = \begin{cases} \begin{pmatrix} 0 & -\overline{R(z)}\mathrm{e}^{-2t\Phi(z)} \\ R(z)\mathrm{e}^{2t\Phi(z)} & 1 \end{pmatrix}, & z \in \mathbb{T}, \\ \begin{pmatrix} 1 & 0 \\ \chi(z)\mathrm{e}^{2t\Phi(z)} & 1 \end{pmatrix}, & z \in [q, q_1], \\ \sigma_1 (v(z^{-1}))^{-1} \sigma_1, & z \in [q_1^{-1}, q^{-1}]; \end{cases}$$

where

$$\Phi(z) := \Phi(z, \xi) = \frac{1}{2}\left(z - z^{-1}\right) + \xi \log z, \quad \xi := \frac{n}{t},$$

is the right phase function.

- the residue conditions:

$$\mathrm{Res}_{z=z_j} m(z) = \lim_{z\to z_j} m(z) \begin{pmatrix} 0 & 0 \\ -z_j\gamma_j \mathrm{e}^{2t\Phi(z_j)} & 0 \end{pmatrix},$$

$$\mathrm{Res}_{z=z_j^{-1}} m(z) = \lim_{z\to z_j^{-1}} m(z) \begin{pmatrix} 0 & z_j^{-1}\gamma_j \mathrm{e}^{2t\Phi(z_j)} \\ 0 & 0 \end{pmatrix};$$

- the symmetry condition: $m(z^{-1}) = m(z)\sigma_1$.
- the normalization condition: $m_1(0)\cdot m_2(0) = 1$ and $m_1(0) > 0$.
- the resonant/non-resonant condition:
  - If $\chi(z) = C(z-\tilde{q})^{1/2}(1+o(1))$ at $\tilde{q}$ then $m(z)$ has finite limits $m(\tilde{q})$ as $z\to\tilde{q}$, $\tilde{q}\in\{q, q_1\}$.
  - If $\chi(z) = \frac{C}{(z-\tilde{q})^{1/2}}(1+o(1))$ then

$$m(z) = \left(\frac{C_1}{(z-\tilde{q})^{1/2}},\ C_2\right)(1+o(1)), \quad C_1C_2\neq 0, \text{ or}$$

$$m(z) = (C_1, C_2(z-\tilde{q}))(1+o(1)), \quad z\to\tilde{q}, \quad C_1C_2\neq 0.$$

At $\tilde{q}^{-1}$ an analog of the above condition holds by symmetry.

Vector RHP vs matrix RHP

Symmetry and normalization conditions:

We use conjugation/deformation techniques preserving the vector form of the RH problem. To ensure uniqueness we impose the following requirements: I. All contours should be symmetric with respect to the map $z \mapsto z^{-1}$.

II. For transformations of the form $\tilde{m}(z) = m(z)[d(z)]^{-\sigma_3}$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $d(z)$ is a sectionally analytical function, we require: (1) the jump contour $\hat{\Sigma}$ to be symmetric; (2) $d(z^{-1}) = d^{-1}(z)$ for $z \in \mathbb{C} \setminus \hat{\Sigma}$; (3) $d(\infty) > 0$.

Advantage of the vector RHP

- Easy to prove the uniqueness for both, the initial and the model RHPs;
- The matrix statement of the model RHP for the shock wave case does not have invertible solutions in the class of matrices with L^2-singularities for certain sufficiently large n, t.

Remarks

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- We work with a vector RHP in comparison to a more common matrix RHP.
- The symmetry condition is important for uniqueness!
- The matrix problem fails to have a nonsingular solution at certain critical parameters (n, t).
- We investigate the problem on an appropriate Riemann surface.

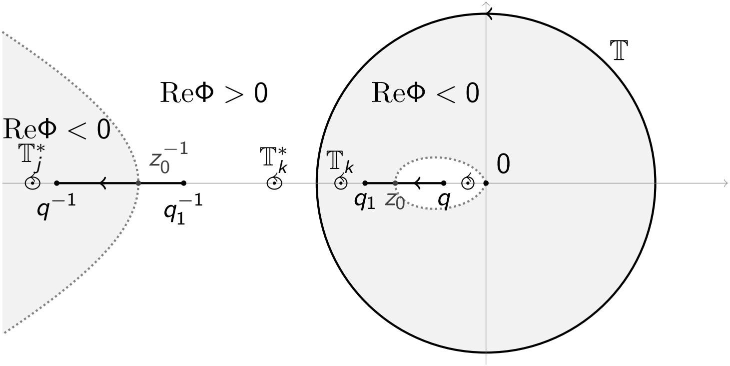
Nonlinear steepest decent

Basic tools:

- Contour deformation (to move the pieces of the jump into regions of the complex domain, where they decay)
- Factorization of the jump matrix (Schur complements) to separate decaying/growing pieces (non-commutativity of matrix multiplication causes problems)
- Conjugation to *replace* the phase function in case the matrices cannot be properly factorized
- Scalar problems can be solved (Sokhotski–Plemelj formulas)
- Problems with constant jumps can be explicitly solved on the Riemann surface

The original phase function

The signature table for the original phase function does not allow for a proper deformation of the RHP.



The key transformation:

$$m(z) \mapsto m(z)\mathrm{e}^{t(g(z)-\Phi(z))\sigma_3}.$$

Consider the Riemann surface associated with the function

$$\mathcal{R}(\lambda) = \sqrt{(\lambda^2 - 1)((\lambda - b)^2 - 4a^2)}.$$

Let Ω_0 be the Abel differential of the second kind with second order poles at $\infty_\pm$ and ω be the Abel differential of the third kind with logarithmic poles at $\infty_\pm$, both normalized as $\int_{\mathfrak{a}} \Omega_0 = \int_{\mathfrak{a}} \omega = 0$. The function

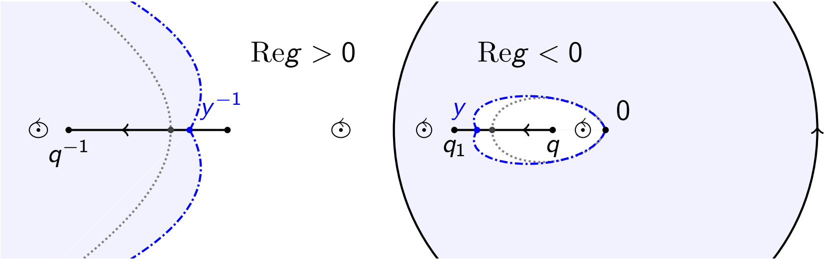
$$\tilde{g}(\lambda, \xi) = \int_1^\lambda (\Omega_0 + \xi\omega) = \int_1^\lambda \frac{(\lambda - \mu_1(\xi))(\lambda - \mu_2(\xi))}{\mathcal{R}(\lambda)} d\lambda,$$

approximates $\Phi(z, \xi)$ as $\lambda \to \infty$ up to a constant.

In modulation region with each ξ we associate the Riemann surface for $\mathcal{R}(\lambda,\xi) = \sqrt{(\lambda^2-1)(\lambda-b+2a)(\lambda-\alpha(\xi))}$ and

$$\tilde{g}(\lambda,\xi) = \int_1^\lambda (\Omega_0(\xi) + \xi\omega(\xi)) = \int_1^\lambda \frac{(\lambda-\mu(\xi))(\lambda-\alpha(\xi))d\lambda}{\mathcal{R}(\lambda,\xi)}.$$

The second zero $\alpha(\xi) = \frac{y+y^{-1}}{2} \in (b-2a, b+2a)$ is chosen such that it lies at the branch point!



Remaining steps

- Conjugate to replace Φ by g
- Contour deformation and further conjugations to remove solvable parts (keep singularities under control!)
- Solve the resulting model problem (using theta functions on the elliptic surface)
- Solve the paramtrix problem (to control the difference between the original and the model problem)

The end

Dear Peter, thanks for being such a great colleague

and

many more happy years!