Weighted estimates of Hardy operator and Poincaré inequality on multi-trees

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This is about multi-parameter paraproducts, and their two weight estimates. This also about embedding theorems of certain spaces of holomorphic functions in the polydisc. It turns out those are equivalent problem. It is about why Carleson quilt counterexample does NOT hold for embeddings of Dirichlet spaces in the polydisc while works for Hardy spaces in the polydisc.

What are (multi)-parameter paraproducts and why they are needed. Coifman–Meyer bilinear (bi)-parameter multipliers. Leibniz rules. They are ubiquitous in PDE: local well-posedness of NS, KdV, optimal smoothing in Schródinger semi-group. Bi-parameter Coifman–Meyer multipliers estimates were used by Kenig for Kadamtsev–Petviashvili well-posedness.

Geometric problems

We are given a collection of non-negative numbers $\{\alpha_I\}_{I\in\mathcal{D}(I_0)}$ enumerated by the family $\mathcal{D}$ of dyadic subintervals of unit interval $I_0=[0,1]$. We wish to find an assignment $I\to E_I$, $I\in\mathcal{D}$, of measurable sets in such a way that

1. sets E_I are pairwise disjoint;
2. $m(E_I)=\alpha_I$.

There is an obvious necessary condition:

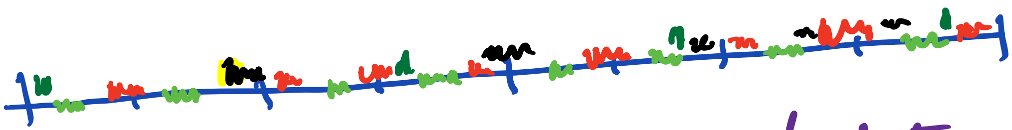
$$\forall J\in\mathcal{D}(I_0)\quad \sum_{I\in\mathcal{D}(J)}\alpha_I\le m(J)\,. \tag{1}$$

A simple construction shows that (1) is not only necessary but also sufficient.

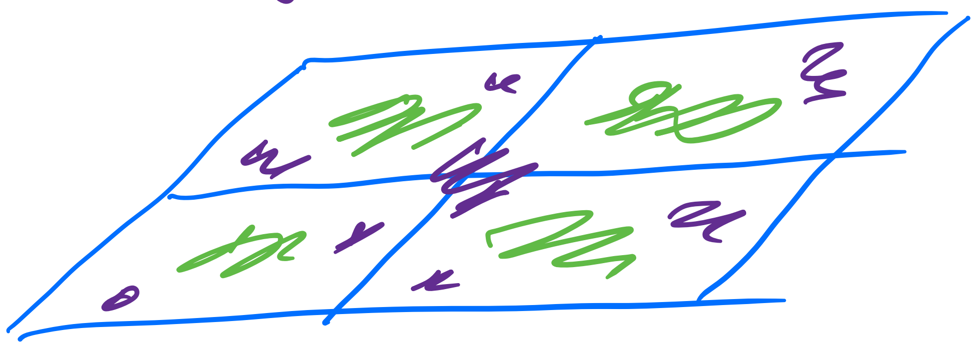
Geometric problem for dyadic intervals and dyadic cubes

Assignment $I \mapsto E_I$

$Q \mapsto E_Q$



disjoint, and $|E_I| = \alpha_I$.



disjoint, and $|E_Q| = \alpha_Q$

Now let us make the problem harder. We augment the collection of sets in $\mathbb{R}^d$. It is very natural and useful to consider the collection of dyadic rectangles $\mathcal{D}^k = \mathcal{D} \times \cdots \times \mathcal{D}$ k times, $k \geq 2$. It is much harder to prove that the condition

$$\forall \mathscr{S} \subset \mathcal{D}^2 \quad \sum_{I \times J \in \mathscr{S}} \alpha_{I\times J} \leq \mu(\cup_{I\times J \in \mathscr{S}} I \times J) \tag{2}$$

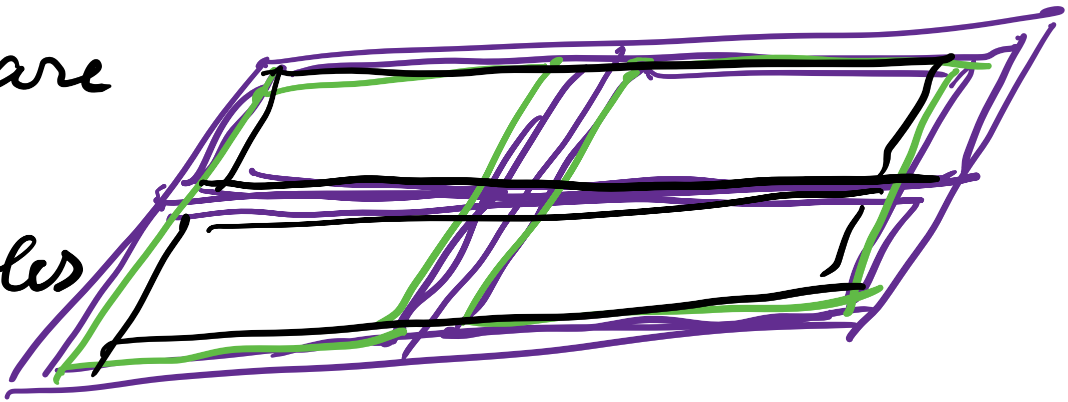
for μ without point masses is sufficient for the existence of the assignment $I \times J \to E_{I\times J}$, of measurable sets in such a way that

1. sets $E_{I\times J}$ are pairwise disjoint;
2. $\mu(E_{I\times J}) = \alpha_{I\times J}$.

Assignment for $I \times J \to E_{I \times J}$ is more difficult.

However, several proofs exist, they are quite non-trivial, and methods range from geometric ones, Barron–Pipher, to convex analysis/functional analysis, Hänninen. Moreover, Hänninen proved that dyadic rectangles can be replaced by arbitrary collection of Borel sets.

There are 9 rectangles here.



Hänninen

Moreover, Hänninen proved that dyadic rectangles can be replaced by arbitrary collection of Borel sets.

Definition

(Carleson coefficients in the generality of a collection of Borel sets). Let μ be a locally finite Borel measure on $\mathbb{R}^d$. Let $\mathscr{S}$ be a countable collection of Borel sets. A family $\{\alpha_S\}_{S\in\mathscr{S}}$ of non-negative reals is Carleson (with the constant $C = 1$) if we have

$$\sum_{S\in\mathscr{S}, S\subset\Omega} \alpha_S \le \mu(\Omega) \tag{3}$$

for every union Ω of sets of the collection $\mathscr{S}$.

Hänninen proved that the **disjoint measurable assignment** $S \to E_S$ exists iff $\{\alpha_S\}_{s\in\mathscr{S}}$ satisfies Carleson packing condition,

$$\mathscr{S}' \subset \mathscr{S} \quad \sum_{S\in\mathscr{S}'} \alpha_S \le \mu(\cup_{S\in\mathscr{S}'} S) \tag{4}$$

From geometry to weighted embedding

We indicate connections of the above "combinatorial" problems to *two-weight embedding theorems* = **two weight multi-parameter paraproduct estimates**.

1) 1 **dimensional dyadic case**: Let T be dyadic tree. We fix bijection $\mathcal{D}(I_0) \to T$, whose vertices we will still call I, and I_0 is the root of T.
Fix μ on $[0,1]$. It is one of our two weights. The second weight lives on T and it is just a sequence of non-negative numbers enumerated by vertices (=dyadic intervals): $w := \{w_I\}_{I\in T}$.

The two-weighted problem is to find **necessary and sufficient conditions on (w,μ) to have**

$$\sum_{I\in T} w_I \cdot \Big(\int_I f d\mu\Big)^2 \le C \int_0^1 f^2 d\mu \tag{5}$$

Simply solved by Carleson 60's and Sawyer, 80's

There is an obvious necessary condition for (5) to hold: just plug $f = \mathbf{1}_J$, $J \in \mathcal{D}$, to obtain in terms of (μ, w):

$$\forall J \in \mathcal{D} \quad \sum_{I \in \mathcal{D}(J)} w_I \cdot \mu(I)^2 \leq C\mu(J)\,. \tag{6}$$

We can now use the assignment mentioned above for $\alpha_I := \frac{w_I \mu(I)^2}{C}$. We will get disjoint $\{E_I\}_{I \in \mathcal{D}}$.
Next step: One use that the dyadic maximal function with respect to any μ is bounded in $L^2(I_0, \mu)$. This will finish the proof. The fact that (6) is necessary and sufficient for the embedding (5) is called Carleson–Sawyer theorem.
Carleson proved it in the 60's and used in his interpolation and corona famous results. Sawyer's generalization appeared in the 80's. **Both results are fundamental in the dyadic approach to the theory of Calderón–Zygmund operators.**

Make life harder

Two (or multi) parameter paraproducts require a solution of a much more involved two-weight problem. We fix a measure μ on $[0,1]^2$, it is the first of two weights.
The second weight lives on T^2 and it is just a sequence of non-negative numbers enumerated by vertices (=dyadic rectangles): $w := \{w_{I\times J}\}_{I,J\in T}$.
Find necessary and sufficient conditions on (w,μ) to have

$$\sum_{I,J\in T} w_{I\times J}\cdot\Big(\int_{I\times J} f d\mu\Big)^2 \le C\int_{[0,1]^2} f^2 d\mu \qquad (7)$$

Necessary condition. Carleson's counter-example

Bi-tree T^2 is a rooted graph with vertices being dyadic rectangles, and the root being $I_0 \times I_0 = [0,1]^2$. It is a much more complicated graph than simple T, in particular, it has cycles. However, again there are simple necessary condition for (12). We get one by plugging $f = \mathbf{1}_{I_1 \times J_1}$, $I_1, J_1 \in \mathcal{D}$.

But Carleson gave an example of weight w on T^2 such that even with $\mu = m_2$, Lebesgue measure on the plane, this necessary condition **is not sufficient**. But there is a stronger necessary condition.

It belongs to S.-Y. A. Chang.

Carleson–Chang packing condition

Choose now $f = \mathbf{1}_{\cup_{k=1}^\infty I_k \times J_k}$. In other words choose a subset $S' \subset \mathcal{D}(I_0) \times \mathcal{D}(I_0)$, consider $\Omega = \cup_{R' \in S'} R'$, and choose $f = \mathbf{1}_\Omega$ to plug into (12). Then we immediately and trivially get the following necessary for embedding (12) condition: $\forall S' \subset \mathcal{D}(I_0) \times \mathcal{D}(I_0)$ put $\Omega := \cup_{R' \in S'}$, and then

$$\forall \text{ such } \Omega, \quad \sum_{R \subset \Omega} w_R \cdot (\mu(R))^2 \leq C\mu(\Omega)\,. \tag{8}$$

Again, the assignment of disjoint $E_R, R \in \mathcal{D}(I_0) \times \mathcal{D}(I_0)$, is the first step. And we know that assignment always exists.

But the second step breaks down: strong maximal (even dyadic strong maximal) operator with respect to μ is rarely bounded in $L^2(\mu)$.

Question: But who said that one needs maximal operators to prove embedding as above?

This is what we know about embedding w.r.t to dyadic rectangles, I

- A. S.-Y. Chang proved that if $\mu = m_2$ (or $\mu = m_d$) then necessary condition (14) is sufficient and this holds for any w on T^2 (and correspondingly T^d).
- For any μ such that strong dyadic maximal function is bounded in $L^2(\mu)$ (14) is sufficient and this holds for any w on T^2 (and correspondingly T^d if we consider measure μ on $[0,1]^d$).
- Moreover, if (14) is sufficient for the embedding (12) with arbitrary w, then μ is such that strong dyadic maximal function is bounded in $L^2(\mu)$. This holds in any dimension d.
- There exists w such that (14) does *not* hold, but the following simplified version does hold:

$$\forall I_1 \times J_1 \in \mathcal{D}(I_0) \times \mathcal{D}(I_0) \qquad \sum_{R \subset I_1 \times J_1} w_R \cdot (\mu(R))^2 \le C\mu(I_1 \times J_1)\,. \tag{9}$$

This is what we know about embedding w.r.t to dyadic rectangles, II

- Such an example exists even with $\mu = m_2$ (Carleson, Tao).
- There exists (w, μ) such that (14) does hold, but the following more complicated (but obviously necessary, plug $f = \mathbf{1}_F$ into (12)) condition does *not* hold: $\forall F \subset [0,1]^2$

$$\forall \Omega \quad \sum_{R \subset \Omega} w_R \cdot (\mu(R \cap F))^2 \leq C\mu(F)\,. \tag{10}$$

- The latter example has w having only values 1 and 0, and moreover the support of w is a connected subgraph of T^2.
- In general the necessary and sufficient condition for embedding (12) are unknown, and hardly can be found at all.
- The case $w \equiv 1$ is interesting and has interesting applications to complex analysis.

This is what we know about embedding w.r.t to dyadic rectangles, $w \equiv 1$, III

- For arbitrary μ, given that $w \equiv 1$ (on T^2 and/or T^3), we can give simple necessary and sufficient condition for the embedding (12) to hold
- We conjecture that the same answer holds for T^d, $d \geq 4$, but we cannot prove this.
- Our answer for the case $w \equiv 1$ for T^2 and T^3 is counterintuitive. **Our answer seems to contradict Carleson's example (but it it does not contradict it)**.
- Embedding (12) holds iff (for $d = 2$, the same answer holds for $d = 3$, and **this is the main result of the current talk**):

$$\forall I_1 \times J_1 \in \mathcal{D}(I_0) \times \mathcal{D}(I_0) \qquad \sum_{R \subset I_1 \times J_1} (\mu(R))^2 \leq C_0 \mu(I_1 \times J_1)\,. \tag{11}$$

Of course constant C in (12) can be calculated by C_0 in (13), **but it is a non-linear relationship**.

Weighted Poincaré inequality on Multi-trees

Consider any rooted directed graph Γ without directed cycles (but possibly with cycles, like T^d). It induces partial order $\leq$ on vertices, the root o is the maximal vertex.
Let Γ satisfy the following. For every $v \in \Gamma$

$$\sharp\{u : v \leq u \leq o\} = F(\text{dist}(v, o))\,,$$

where F is any finite function on $\mathbb{Z}_+$. Let $\mathbb{I}f(v) := \sum_{u:v\leq u\leq o} f(u)$.

Theorem

Let μ be any probability measure on graph. Inequality $\int_\Gamma |\mathbb{I}f - \int_\Gamma \mathbb{I}f\, d\mu|^2\, d\mu \leq C \int_\Gamma f^2$ holds with universal C for all f on Γ iff another inequality holds with a universal constant: $\int_\Gamma |\mathbb{I}f|^2\, d\mu \leq \tilde{C} \int_\Gamma f^2$.

So Poincaré inequality holds iff embedding holds.

Statement of result

Theorem

Embedding with $w \equiv 1$ *(**not only**)*

$$\sum_{I,J \in T} \cdot \left(\int_{I \times J} f d\mu\right)^2 \le C \int_{[0,1]^2} f^2 d\mu \tag{12}$$

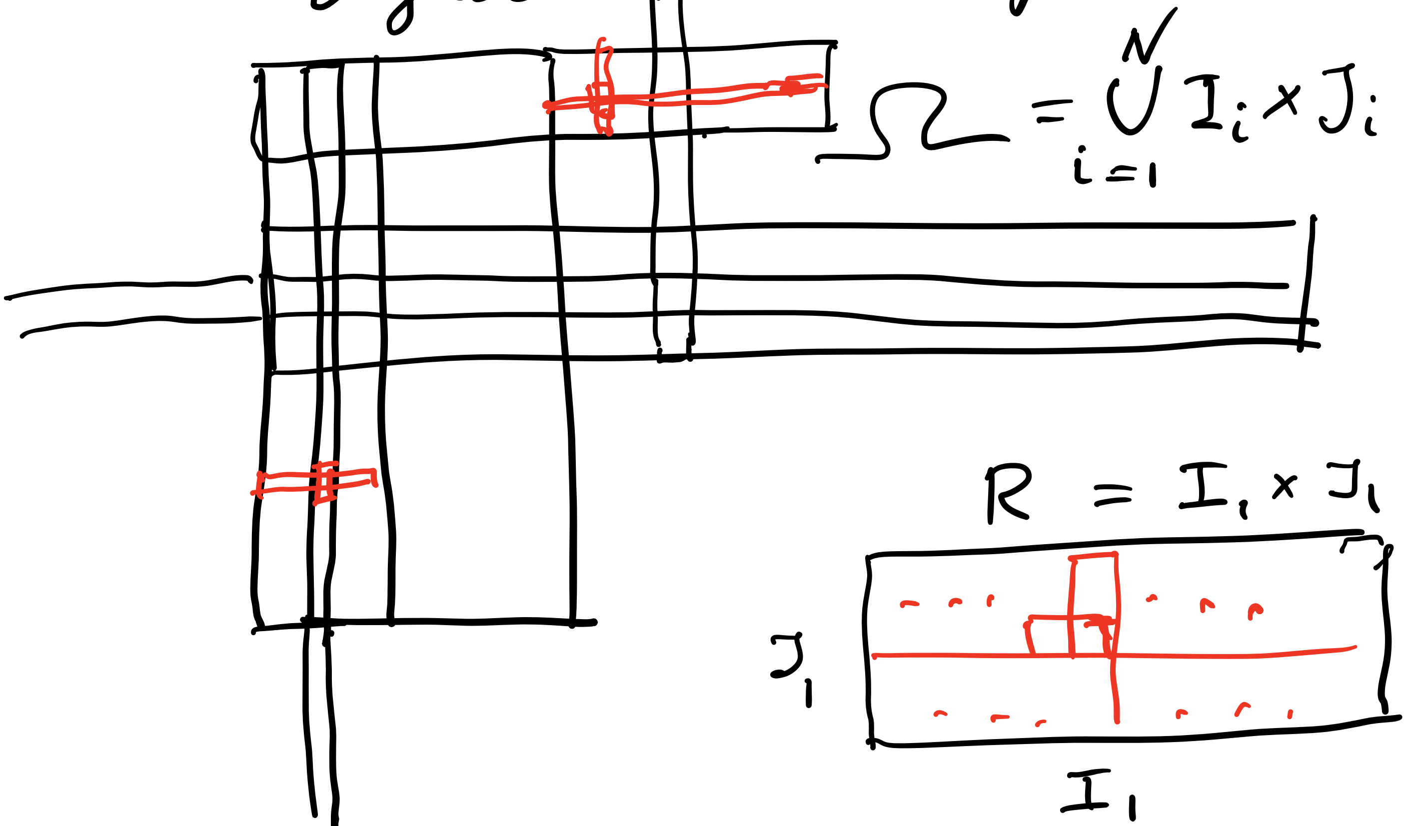
holds if and only if

$$\forall I_1 \times J_1 \in \mathcal{D}(I_0) \times \mathcal{D}(I_0) \qquad \sum_{R \subset I_1 \times J_1} (\mu(R))^2 \le C_0 \mu(I_1 \times J_1)\,. \tag{13}$$

In particular, (13) implies (counter-intuitively) a much "stronger" property: $\forall S' \subset \mathcal{D}(I_0) \times \mathcal{D}(I_0)$ put $\Omega := \cup_{R' \in S'}$, and then

$$\forall \text{ such } \Omega, \quad \sum_{R \subset \Omega} (\mu(R))^2 \le C \mu(\Omega)\,. \tag{14}$$

Huge overlapping of dyadic rectangles.



Two technical lemmas

First **simple weighted estimate**:

Lemma

Let (S, ν) *be a measure space and* J *be an operator with positive kernel. Then for two positive functions* f, g*, we have*

$$\int (Jf)^2 g \leq \sup_{\operatorname{supp} g} JJ^* g \int f^2 \,.$$

Lemma

Let T *be dyadic tree,* $g, h \geq 0$ *on* T*. Let Let* $If(v) := \sum_{u: v \leq u \leq o} f(u)$*, integration on* T*. Let* g *be superadditive. Let* $Ih \leq \lambda$ *on* $\operatorname{supp} g$*. Then*

$$I^*(gh)(v) \leq \lambda g(v) \quad \forall v \in T \,.$$

Let μ be measure on T^d. Put

$$\mathbb{V}^\mu(v) = \mathbb{I}(\mathbb{I}^*\mu)(v),\ \mathcal{E}[\mu] = \int_{T^d} \mathbb{V}^\mu \, d\mu = \int_{T^d} (\mathbb{I}^*\mu)^2 .$$

$$E_\delta := \{u \in T^d : \mathbb{V}^\mu(u) \le \delta\} .$$

$$\mathbb{V}^\mu_\delta(v) = \mathbb{I}(\mathbf{1}_{E_\delta}\mathbb{I}^*\mu)(v),\ \mathcal{E}_\delta[\mu] = \int_{T^d} \mathbb{V}^\mu_\delta \, d\mu = \int_{E_\delta} (\mathbb{I}^*\mu)^2 .$$

For $d = 1$ trivially $\mathbb{V}^\mu_\delta \le \delta$. For $d \ge 2$ this is **false in general**. For $d = 1$, $\sup \mathbb{I}f \le \sup_{\mathrm{supp}\, f} \mathbb{I}f$. For $d \ge 2$ this is **false in general**. So

$$d \ge 2 \text{ in general, } \sup_{T^d} \mathbb{V}^\mu >> \sup_{\mathrm{supp}\, \mathbb{I}^*\mu} \mathbb{V}^\mu \ge \sup_{\mathrm{supp}\, \mu} \mathbb{V}^\mu .$$

NO MAXIMUM PRINCIPLE IF $d \ge 2$.

$$d = 1 \Rightarrow \sup_T \mathbb{V}^\mu = \sup_{\mathrm{supp}\, \mu} \mathbb{V}^\mu \Rightarrow \mathbb{E}_\delta[\mu] \le \delta \|\mu\| .$$

See Figures.

Figures

d = 1

● = supp f

● = where we measure $\mathbb{I}f$

$$\mathbb{I}f(\bullet) \le \mathbb{I}f(\blacksquare)$$

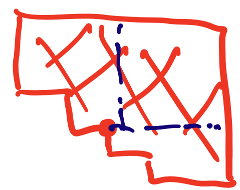
$$\Longrightarrow \sup \mathbb{I}f \le \sup_{\text{supp } f} \mathbb{I}f$$

for d = 1.

d = 2

● = supp f

$\mathbb{I}f(\bullet) = \sum$  ≤ 1

But $\mathbb{I}f(v) = \sum$  $\gg 1.$

Main tool: surrogate maximum principle

Theorem

Let μ *be a measure on* T^2*, then for any* $\varepsilon \in (0,1)$,

$$1)\mathbb{E}_\delta[\mu] \lesssim_\varepsilon \delta^{1-\varepsilon} \|\mu\|^{1-\varepsilon}\, \mathbb{E}[\mu]^\varepsilon\,.$$

$$2)\ \textit{Moreover},\ \ \mathbb{E}_\delta[\mu] \le C\delta e^{\sqrt{\log\frac{1}{\delta}}}\, \mathbb{E}[\mu]$$

for any μ *such that* $\|\mu\| \le \mathbb{E}[\mu]$.

Theorem

Let μ *be a measure on* T^3*, then for any* $\tau \in (0,1)$,

$$\mathbb{E}_\delta[\mu] \le C\delta^{1/2} \|\mu\|^{1/2}\, \mathbb{E}[\mu]^{1/2}\,.$$

Main Tool for main tool

Theorem (Majorization with small energy on bi-tree)

Let $f \geq 0$ *on* T^2. *Let* $\operatorname{supp} f \subset \{\mathbb{I}f \leq \delta\}$ *(e.g.* $f = \mathbf{1}_{E_\delta}\mathbb{I}^*\mu$*). Let* $\lambda \geq 10\delta$. *Then there exists* $\varphi \geq 0$ *on* T^2 *such that*

1. $\mathbb{I}\varphi \geq \mathbb{I}f$ *on* $\{\mathbb{I}f \geq 40\lambda\}$, **domain of majorization**;
2. $\operatorname{supp}\varphi \subset \{\delta < \mathbb{I}f \leq 3\lambda\}$, **support of majorant**;
3. $\int_{T^2}\varphi^2 \lesssim \frac{\delta}{\lambda}\int_{T^2} f^2$, **energy drops a lot**.

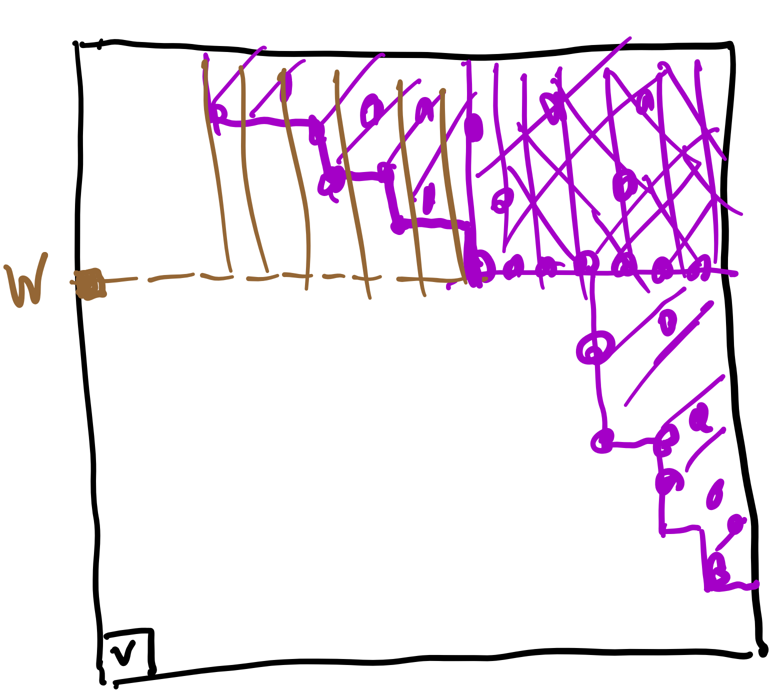
Theorem (Majorization with small energy on 3-tree)

Let $f \geq 0$ *on* T^3. *Let* $\operatorname{supp} f \subset \{\mathbb{I}f \leq \delta\}$ *(e.g.* $f = \mathbf{1}_{E_\delta}\mathbb{I}^*\mu$*). Let* $\lambda \geq 10\delta$. *Then there exists* $\varphi \geq 0$ *on* T^3 *such that*

1. $\mathbb{I}\varphi \geq \mathbb{I}f$ *on* $\{\mathbb{I}f \geq 40\lambda\}$, **domain of majorization**;
2. $\int_{T^3}\varphi^2 \lesssim \frac{\delta}{\lambda}\int_{T^3} f^2$, **energy drops a lot**.

If to drop support requirement in the first theorem, then $\lesssim \frac{\delta^2}{\lambda^2}$.

Wrong but instructive proof by picture



$$\mathbb{I}f(v) \ge 40\lambda$$

$$\frac{1}{K} + \dots + \frac{1}{K} = 1$$

$$\frac{1}{K^2} + \dots + \frac{1}{K^2} = \frac{1}{K}$$

$(\mathbb{I}f)(\bullet) \le \delta$

$\bullet$ = supp f

$(\mathbb{I}f)(w) \ge \lambda \Rightarrow$

$\# ||| \gtrsim \frac{\lambda}{\delta} \# |||$

$\Rightarrow$ distribute ••••• over ••••• to have the same Σ

but $\Sigma(\ )^2 \lesssim \frac{\delta}{\lambda} \Sigma(\ )^2$.

DONE ?!?

The mistake is: The pictures above will be built $\forall v$: $\mathbb{I}f(v) \geq 40\lambda$.

The problem is they are not independent pictures.

The difficulty is to glue, to reconcile, the above construction on all those pictures simultaneously.

Application to Carleson embedding on poly-discs.

$\mathcal{H}$ a space of holomorphic functions on $\mathbb{D}^2, \mathbb{D}^3, \dots, \mathbb{D}^d, \dots$.

E.g. $\mathcal{H} = H^2(\mathbb{D}^2) =$

$$= \{ f(z,w) = \sum_{n,m \geq 0} a_{nm} z^n z^m : \sum |a_{nm}|^2 < \infty \}.$$

Or $\mathcal{H} = \mathcal{D}_{s_1, s_2}(\mathbb{D}^2) =$

$$= \{ f(z,w) : \sum_{n,m \geq 0} (1+n)^{s_1} (1+m)^{s_2} |a_{nm}|^2 < \infty \}.$$

$s_1 = s_2 = 1$ = Dirichlet space.

Carleson measures for $\mathcal{H}$ are measures on $\mathbb{D}^2$ (or $\mathbb{D}^3, \dots, \mathbb{D}^d$) such that

$$id : \mathcal{H} \longrightarrow L^2(\mu) \text{ is bounded.}$$

That is $\exists\, C < \infty$ $\forall f \in \mathcal{H}$

$$\int_{\mathbb{D}^2} |f|^2 \, d\mu(z,w) \leq C \|f\|^2_{\mathcal{H}}.$$

↑ Embedding.

Theorem μ is Carleson measure for Dirichlet space $\Longleftrightarrow$ natural $\nu = \nu(\mu)$ on T^2 is embedding ($w \equiv 1$, ν) as before. ●

There is the same results for $0 < s_i \leq 1$.

But NOT for $s_i = 0$.

Embedding measures for $H^2(\mathbb{D}^2)$ are NOT described.

Conj: These are Chang-Carleson measures.

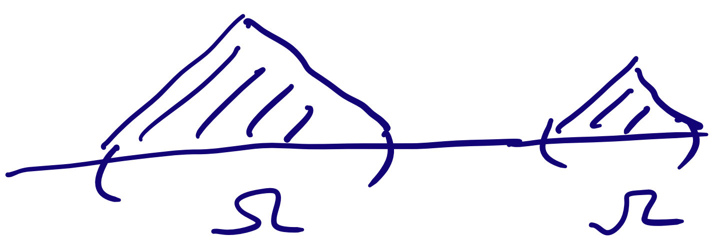
"Proof". First Chang–Carleson measures on $\mathbb{D}^2$.

We saw them on $\mathbb{T}^2$, now on $\mathbb{C}_+^2$

Tents:

$\forall\, \Omega \subset \mathbb{R}\times\mathbb{R}$ open, tent $T(\Omega)$ is

$$\{z=(z_1,z_2) : \underbrace{(x_1-y_1,\, x_1+y_1)}_{\text{interval}} \times \underbrace{(x_2-y_2,\, x_2+y_2)}_{\text{interval}}$$



but in $\mathbb{R}^4$

ν is Chang – Carleson $\Leftrightarrow$ (*)

$\nu(T(\Omega)) \le C|\Omega| \quad \forall$ open Ω.

Def Involutions: $\left.\begin{array}{l}(z_1, z_2) \to (-\bar{z}_1, z_2) \\ (z_1, z_2) \to (z_1, -\bar{z}_2)\end{array}\right\} \mathbb{C}_+^2 \to \mathbb{C}_+^2$

Def Property $\mathcal{P}$ of measures on $\mathbb{C}_+^2$ is inv. inv. property if $\nu^{*1}, \nu^{*2} \in \mathcal{P}$ if $\nu \in \mathcal{P}$.

(*) is involution inv. property.

Question (OPEN). Is

$id: H^2(\mathbb{C}_+^2) \to L^2(\mathbb{C}_+^2, \nu)$ inv. inv.?

Answer is YES if

$\Gamma_{\bar{\varphi}}: H^2(\mathbb{C}_+^2) \to H^2(\mathbb{C}_-^2) \Leftrightarrow \varphi \in BMO_{ChF}$

$\Leftrightarrow \left|\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\varphi(z_1, z_2)\right|^2 \operatorname{Im} z_1 \operatorname{Im} z_2 \, dm_2(z_1)\, dm_2(z_2)$

satisfies (*) ($\equiv$ is Chang-Fefferman measure).

Explanation. If $\Rightarrow$ were true, then

$\Gamma_{\bar{\varphi}}: H^2(\mathbb{C}_+^2) \longrightarrow H^2_-(\mathbb{C}_+^2, L^2(\nu))$

$\Leftrightarrow \left\|\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\varphi(z_1, z_2)\right\|^2_{L^2(\nu)} \operatorname{Im} z_1 \operatorname{Im} z_2 \, dm_2(z_1)\, dm_2(z_2)$

satisfies (*). Here $\varphi \in H^\infty(L^2(\nu))$.

<u>Lemma</u> ν : $id : H^2(\mathbb{C}^2_+) \to L^2(\nu)$ (1)

$$\Phi(w, x) := \frac{(\mathrm{Im}\, w_1)^{1/2}}{x_1 - \bar{w}_1} \cdot \frac{(\mathrm{Im}\, w_2)^{1/2}}{x_2 - \bar{w}_2}.$$

Denote $\varphi(x) := \Phi(\cdot, x) : \mathbb{R} \times \mathbb{R} \to L^2(\mathbb{C}^2_+, \nu)$

Then (1) $\Leftarrow$ $\Gamma_{\bar{\varphi}} : H^2(\mathbb{C}^2_+) \to H^2_-(\mathbb{C}^2_+, L^2(\nu))$. (2)

<u>Proof</u>. Easy computation. ●

Theorem/Question. $id : H^2(\mathbb{C}^2_+) \to L^2(\nu)$ bdd $\Longrightarrow$ ν is inv. under involutions

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$$\Longrightarrow \int |f(z, \bar{w})|^2 \, d\nu \le C \|f\|^2_{H^2(\mathbb{C}^2_+)}$$

$$\int |f(\bar{z}, w)|^2 \, d\nu \le C \|f\|^2_{H^2(\mathbb{C}^2_+)}$$

$$\int |f(\bar{z}, \bar{w})|^2 \, d\nu \le C \|f\|^2_{H^2(\mathbb{C}^2_+)}$$

$\Longrightarrow$ $\mathrm{Poisson}_{1,2} : L^2(\mathbb{R}^2) \to L^2(\mathbb{C}^2_+, \nu)$ is a bounded operator

$\overset{\text{Chang}}{\Longrightarrow}$ ν is Chang-Carleson measure. ●

Why Question $\Rightarrow$ ν inv. under involutions

$$\nu \longrightarrow \nu_1^* \quad (w_1, w_2) \longrightarrow (-\bar{w}_1, w_2)$$

$$\varphi \longrightarrow \varphi_1^* = \frac{1}{(x_1 + w_1)(x_2 - \bar{w}_2)}$$

$$\operatorname{Im} z_1 \operatorname{Im} z_2 \| \partial_{z_1} \partial_{z_2} \varphi_1^* \|^2_{L^2(\nu)} = \int_{\mathbb{D}^2} \frac{\operatorname{Im} z_1 \operatorname{Im} z_2 \, \operatorname{Im} w_1 \operatorname{Im} w_2 \, d\nu(w_1, w_2)}{|z_1 + w_1|^4 |z_2 - \bar{w}_2|^4}$$

Is it Carleson - Chang measure density ?

Yes, becaus C-C is invariant under $(x_1, x_2) \to (-x_1, x_2)$

$$(z_1, z_2) \to (-\bar{z}_1, z_2)$$

So to check C-C property is the same as to check if for

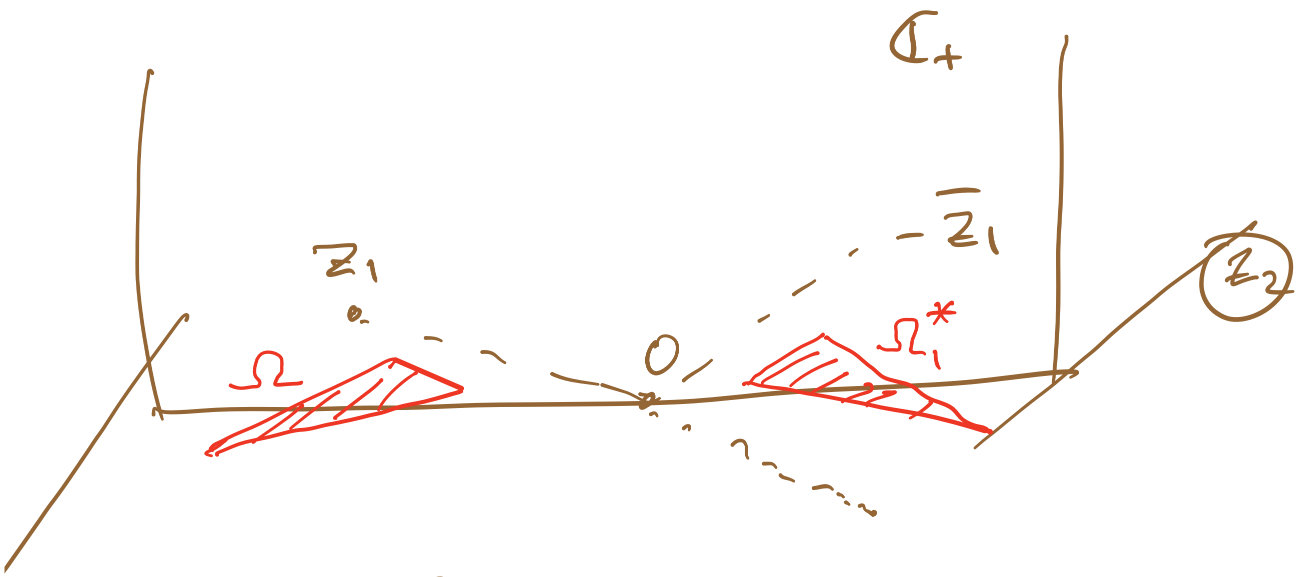
$$\int_{\mathbb{D}^2} \frac{\operatorname{Im} w_1 \operatorname{Im} w_2 \operatorname{Im} z_1 \operatorname{Im} z_2 \, d\nu(w_1, w_2)}{|-\bar{z}_1 + w|^4 |z_2 - \bar{w}_2|^4} .$$

But this is Carleson because this is the same as

$$\int_{\mathbb{D}^2} \frac{\operatorname{Im} z_1 \operatorname{Im} z_2 \, \operatorname{Im} w_1 \operatorname{Im} w_2 \, d\nu(w_1, w_2)}{|z_1 - \bar{w}_1|^4 |z_2 - \bar{w}_2|^4} \equiv$$

$$\| \partial_{z_1} \partial_{z_2} \varphi(z_1, z_2) \|^2_{L^2(\nu)} \ \mathrm{Im}\, z_1 \, \mathrm{Im}\, z_2 .$$

Hence to disprove Lacey-Ferguson: it would be enough to construct embedding measure $d\nu(w_1, w_2)$ for $H^2(\mathbb{D}^2)$ s.t. it is not Carleson-Chang measure.



$\mathbb{C}_+ \times \mathbb{C}_+$

$$\mathbb{R} \times \mathbb{R} \supset \Omega \xrightarrow{(x_1, x_2) \to (-x_1, x_2)} \Omega^*_1$$