Generalized Indefinite Strings

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joint work with **Jonathan Eckhardt** (Loughborough)

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Generalized indefinite strings

$$-f'' = z\,\omega f + z^2 \upsilon f \quad \text{on} \quad [0, L)$$

with

- $z \in \mathbb{C}$ - spectral parameter,
- $L \in (0, \infty]$,
- ω is a real H^{-1}_{loc} distribution on $[0, L)$,
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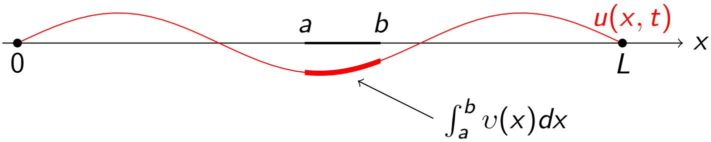
The object was introduced in 2014, however, the spectral problem (maybe, not with this class of coefficients) is not new and has a long history...

Vibrating inhomogeneous strings

Vibrating string with mass density given by υ:



Equations of motion:

$$\upsilon(x)u_{tt}(x,t) = u_{xx}(x,t), \qquad u(0,t) = u(L,t) = 0$$

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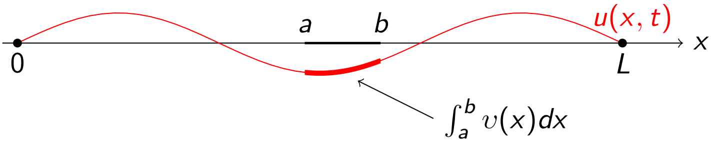

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Separation of variables: $u(x,t) = f(x)\,\mathrm{e}^{\mathrm{i}\lambda t} \;\to$

$$-f'' = \lambda^2\,\upsilon(x)\,f \quad \text{on } [0,L); \quad f(0) = f(L) = 0$$

Two-way diffusion equations (e.g., the Bothe equation...)

$$\omega(x)u_t(x,t) = u_{xx}(x,t), \qquad u(-L,t) = u(L,t) = 0$$

... with $x\omega(x) > 0$ a.e. on $(-L, L)$ (e.g., $\omega(x) = x$ or $\omega(x) = \mathrm{sgn}\,(x)$).

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History: Baouendi & Grisvard (1968), **R. Beals** (1977–...), ...

B. Ćurgus and B. Najman, *The operator* $-\mathrm{sgn}\,(x)\frac{d^2}{dx^2}$ *is similar to a selfadjoint operator in* $L^2(\mathbb{R})$, Proc. Amer. Math. Soc. **123**, 1125–1128 (1995).

The spectral problem for a string

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- The spectrum σ (the set of zeros of $s(\cdot, L)$) consists of simple and positive eigenvalues, $\sigma = \{\lambda_n\}_{n\in\mathbb{N}}$

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots; \qquad \lim_{n\to\infty} \frac{n}{\sqrt{\lambda_n}} = \frac{1}{\pi}\int_0^L \sqrt{\upsilon(x)}dx.$$

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“Can one hear the mass density of a string?”

No, “norming constants” are needed...
G. Borg, A.N. Tikhonov, V.A. Marchenko, I.M. Gelfand &B.M. Levitan...

M. G. Krein's spectral theory of strings

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- Existence & uniqueness,
- Analyticity w.r.t spectral parameter z,
- Fundamental system of solutions $c(z, x)$ and $s(z, x)$:

$$c(z, 0) = s'(z, -0) = 1, \quad c'(z, -0) = s(z, 0) = 0$$

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- m is a Stieltjes function, ($m \in \mathcal{N}_+$)

$$m(z) = \upsilon(\{0\}) - \frac{1}{Lz} + \int_{(0,\infty)} \frac{1}{\lambda - z} d\rho(\lambda), \quad z \in \mathbb{C}\backslash[0,\infty)$$

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- ρ is a spectral measure, which satisfies $\int_{(0,\infty)} \frac{d\rho(\lambda)}{1+\lambda} < \infty$.

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- For regular strings, i.e., $L + \upsilon([0, L)) < \infty$:

$$m(z) = \upsilon(\{0\}) - \frac{1}{Lz} + \sum_{\lambda \in \sigma} \frac{\gamma_\lambda}{\lambda - z}$$

  - σ is the Dirichlet spectrum,
  - $\{\gamma_\lambda\}_{\lambda \in \sigma}$ are the norming constants, $\gamma_n^{-1} = \|s(\lambda_n, \cdot)\|^2$.

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Let $\mathcal{S}_+ = \{(L, \upsilon) : \ L \in (0, \infty], \ \upsilon \in \mathcal{M}_+([0, L))\}$ be the set of strings.

Theorem (M. G. Krein '1951–1954)

The map $\Phi_+ : \begin{cases} \mathcal{S}_+ & \to & \mathcal{N}_+ \\ (L, \upsilon) & \mapsto & m \end{cases}$

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H. Dym & H. P. McKean, *Gaussian Processes, Function Theory and the Inverse Spectral Problem*, Academic Press, 1971.

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- A toy model in the theory of operators in Krein spaces.
- $(S1)$ appears in the study of forward-backward parabolic equations.
- **Relevant for particular nonlinear wave equations**:

Camassa–Holm: $\omega_t + \kappa u_x + 2u_x\omega + u_x\omega = 0, \quad \omega = u - u_{xx},$

Hunter–Saxton: $(u_t + uu_x)_x = \frac{1}{2}u_x^2,$

Dym: $u_t = u^3 u_{xxx}$

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Partial inverse spectral theory results (basically, uniqueness)

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Existence???

Rational Stieltjes functions and Stieltjes strings

Every **rational Stieltjes function** m is the Weyl–Titchmarsh function of the **Stieltjes string**

$$-f'' = z\,\omega f, \qquad x \in [0, L),$$

with $\omega = \sum \omega_k \delta_{x_k}$, $x_k - x_{k-1} := l_k$ and $L = \sum l_k$. It admits the expansion

$$m(z) = \omega_0 + \cfrac{1}{-l_0\,z + \cfrac{1}{\omega_1 + \cfrac{1}{\ddots + \cfrac{1}{-l_{n-1}\,z + \cfrac{1}{\omega_n + \cfrac{1}{-l_n z}}}}}}.$$

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Stieltjes formulas work for signed measures ω, however, there are Herglotz functions not having the above continued fraction expansion!

Rational Herglotz functions and Krein–Langer strings

On the other hand, every **rational Herglotz function** admits the expansion

$$m(z) = \omega_0 + \upsilon_0 z + \cfrac{1}{-l_0\, z + \cfrac{1}{\omega_1 + \upsilon_1 z + \cfrac{1}{\ddots + \cfrac{1}{-l_n\, z}}}}.$$

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It is the Weyl–Titchmarsh function of the **Krein–Langer string**

$$-f'' = z\,\omega f + z^2\,\upsilon f, \qquad x \in [0, L),$$

with $\omega = \sum \omega_k \delta_{x_k}$, $\upsilon = \sum \upsilon_k \delta_{x_k}$, $x_k - x_{k-1} := l_k$ and $L = \sum l_k$.

M. G. Krein & H. Langer, *On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems*, Beiträge Anal. **14**, 25–40 (1979); **15**, 27–45 (1980).

The Camassa–Holm Equation

$$u_t - u_{xxt} + 2\kappa u_x = 2u_x u_{xx} - 3uu_x + uu_{xxx}, \quad u|_{t=t_0} = u_0 \qquad \text{(CH)}$$

First appearance within a family of bi-Hamiltonian PDEs in

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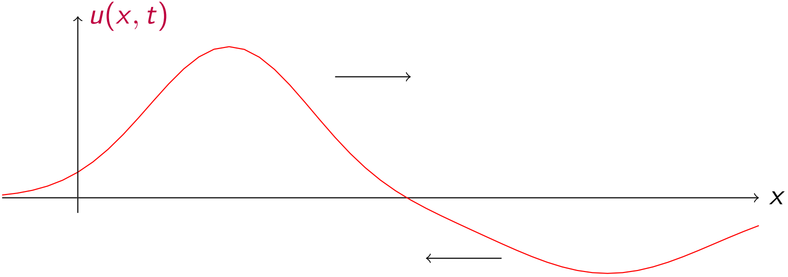
- R. S. Johnson, *Camassa–Holm, Korteweg–de Vries and related models for water waves*, J. Fluid Mech. **455** (2002)
- A. Constantin & D. Lannes, *The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations*, ARMA **192** (2009)

The Camassa–Holm equation models wave breaking:



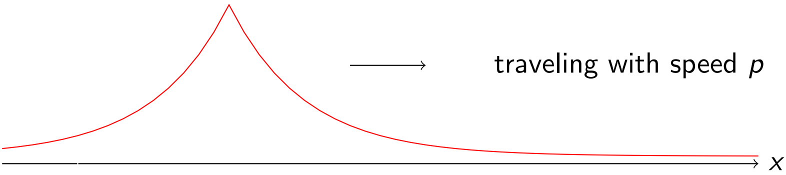
- Smooth initial data may develop singularities in finite time;
- Solutions stay bounded but their slope may become vertical;
- Wave breaking only happens when $\omega + \kappa = u - u_{xx} + \kappa$ changes sign!

The Camassa–Holm equation (weak formulation)

$$u_t + uu_x + p_x = 0, \qquad p = \frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-|x-s|}\Big(u^2 + \frac{1}{2}u_x^2\Big)ds, \qquad \text{(CHweak)}$$

has traveling wave solutions called peakons

$$u(x,t) = p\,\mathrm{e}^{-|x-pt-c|}, \quad x,\ t \in \mathbb{R}$$



More generally a multi-peakon is given by

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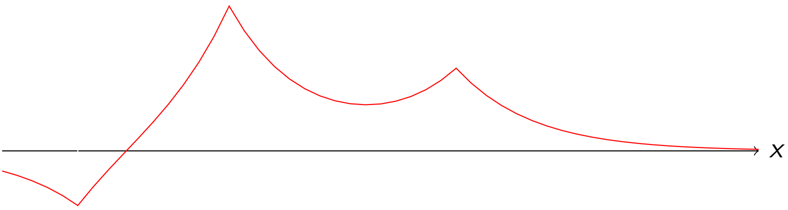

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with the Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{n,k=1}^{N} p_n p_k \mathrm{e}^{-|q_n-q_k|} = \frac{1}{4}\|u\|^2_{H^1(\mathbb{R})}$$

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R. Beals, D. H. Sattinger & J. Szmigielski, *Peakons, strings, and the finite Toda lattice*, Comm. Pure Appl. Math. **54** (2001).

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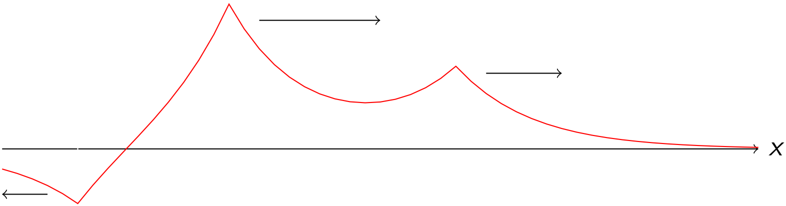

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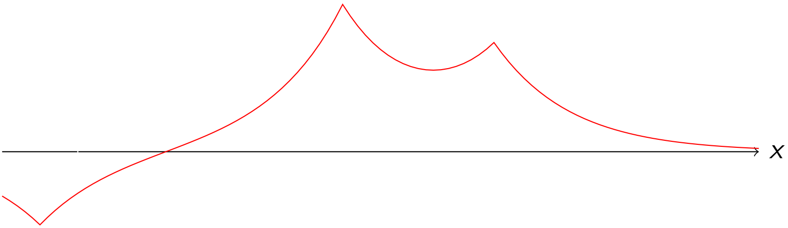

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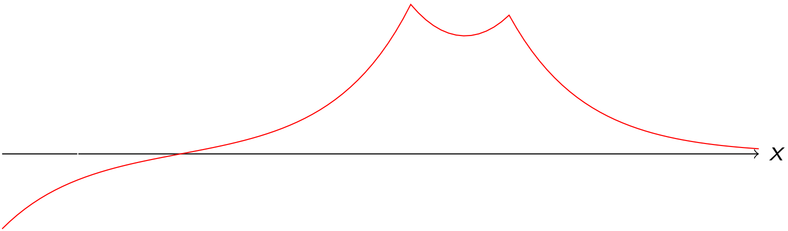

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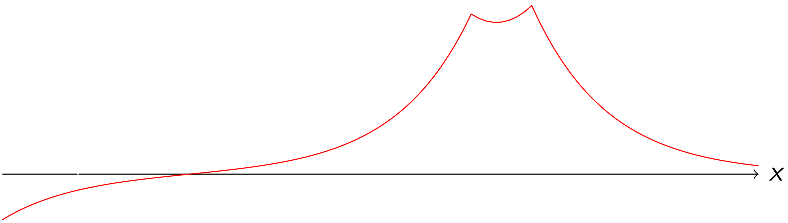

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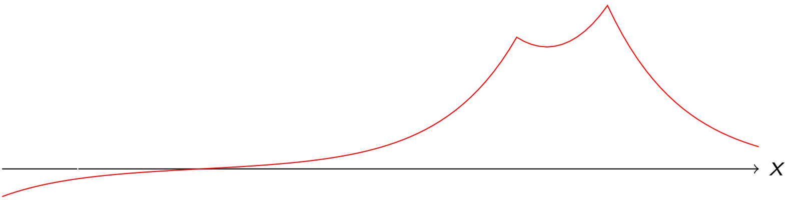

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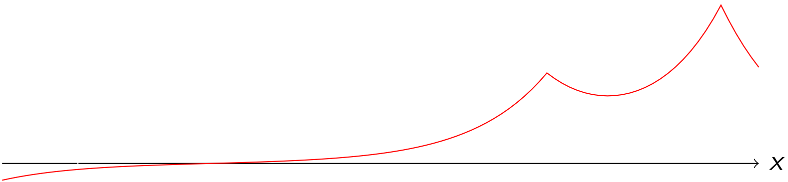

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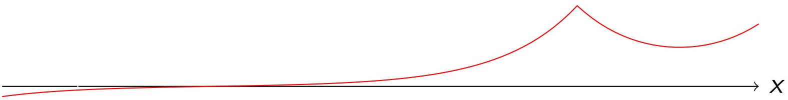

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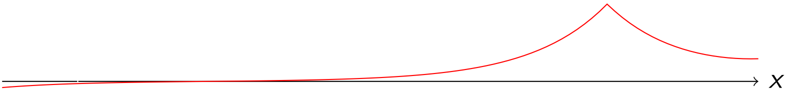

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$$u(x,t) = p(t)\mathrm{e}^{-|x-q(t)|} - p(t)\mathrm{e}^{-|x+q(t)|}; \qquad p(0) > 0, \quad q(0) < 0.$$

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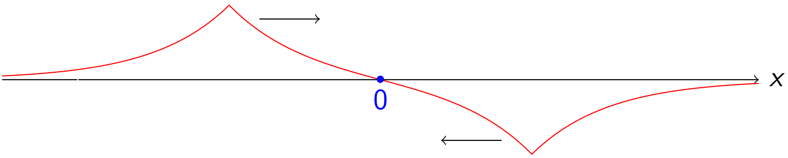

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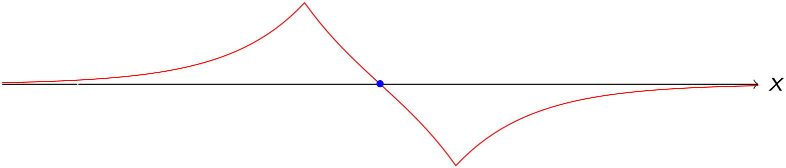

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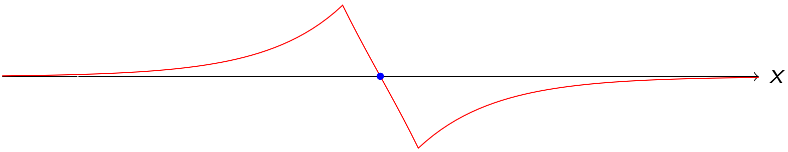

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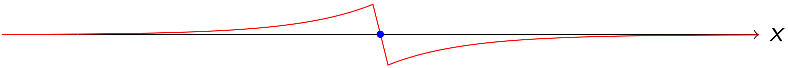

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Conservative solutions (u,μ): additional quantity μ measuring the loss of energy at the times of blow-ups . . . See also *H. Holden & X. Raynaud* (2007))

Inverse Spectral/Scattering Transform

Setting

$$\omega(x,t) := u(x,t) - u_{xx}(x,t)$$

consider the family of Sturm–Liouville problems ("Lax operators" for (CH))

$$-f'' + \frac{1}{4}f = z\,\omega(\,\cdot\,,t)f \quad \text{on} \quad \mathbb{R} \tag{Iso}$$

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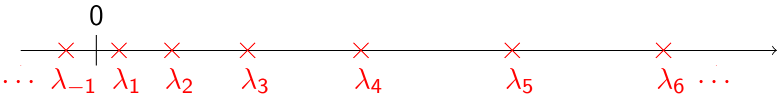

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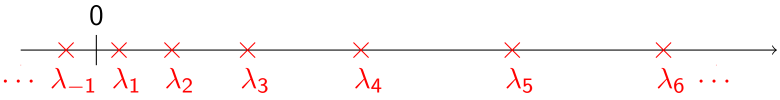

- Time evolution under the Camassa–Holm flow

$$\sigma(t) = \sigma(0) \quad \text{and} \quad \gamma_\lambda(t) = \mathrm{e}^{-\frac{t}{2\lambda}}\gamma_\lambda(0), \quad t \in \mathbb{R}$$

Peakon–Antipeakon Interaction: The Weyl function

$$M_t(z) = \cfrac{1}{-\ell_2(t)z + \cfrac{1}{m_2(t) + \cfrac{1}{-\ell_1(t)z + \cfrac{1}{m_1(t) + \cfrac{1}{-\ell_0(t)z}}}}},$$

where

$$m_1(t) = -m_2(t) = 8\cosh^2(q(t)/2)p(t),$$

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Take the limit as $t \to t^\times$: $\ell_0(t^\times) = \ell_2(t^\times) = \frac{1}{2}$ and $\ell_1(t^\times) = 0$.
However, $m_1(t^\times) = +\infty$ and $m_2(t^\times) = -\infty$!

Peakon–Antipeakon Interaction: The Weyl function

However, it turns out that for every z

$$\lim_{t\to t^\times} M_t(z) = M_{t^\times}(z) := \cfrac{1}{-z/2 + \cfrac{1}{4H_0^2\, z + \cfrac{1}{-z/2}}} = -\frac{1}{z} + \frac{H_0^2 z}{1 - H_0^2 z^2}.$$

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First of all, $M_{t^\times}$ is Herglotz. Moreover, $M_{t^\times}$ is the Weyl function for the quadratic spectral problem

$$-f'' + \frac{1}{4} f = z^2\, \upsilon\, f, \quad x \in \mathbb{R},$$

where $\upsilon = 4H_0^2\, \delta_0 = 4H(p,q)\, \delta_0$, a so-called dipole.

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M. G. Krein & H. Langer, *On some extension problems which are closely connected with the theory of Hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems*, Beiträge Anal. **14**, 25–40 (1979); **15**, 27–45 (1980).

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R. Beals, D. Sattinger, & J. Szmigielski, *Multipeakons and the classical moment problem*, Adv. Math. **154**, 229–257 (2000).

J. Eckhardt & A. Kostenko, *An isospectral problem for global conservative multi-peakon solutions of the Camassa–Holm equation*, Comm. Math. Phys. **329**, 893–918 (2014).

The conservative CH flow

Usually, the Cauchy problem is posed for

- Dispersionless CH ($\kappa = 0$) with decaying data on the line, the phase space $(u, \mu) \in H^1(\mathbb{R}) \times \mathcal{M}_+(\mathbb{R})$,
- CH with dispersion ($\kappa > 0$) with decaying data on the line, the phase space $(u, \mu) \in H^1(\mathbb{R}) \times \mathcal{M}_+(\mathbb{R})$,
- Periodic initial data (CH on the circle), the phase space $(u, \mu) \in H^1(\mathbb{T}) \times \mathcal{M}_+(\mathbb{T})$,
- Two-component Camassa–Holm equation ...

A. Bressan and A. Constantin, *Global conservative solutions of the Camassa–Holm equation*, ARMA **183** (2007).

H. Holden and X. Raynaud, *Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view*, Comm. PDE (2007).

H. Holden and X. Raynaud, *Periodic conservative solutions of the Camassa–Holm equation*, Ann. Inst. Fourier (Grenoble) **58** (2008).

Generalized Indefinite Strings

$$-f'' = z\,\omega f + z^2 \upsilon f \quad \text{on} \quad [0, L) \tag{S2}$$

...with $L \in (0, \infty]$, $\omega \in H^{-1}_{\mathrm{loc}}[0, L)$ and υ a positive Borel measure on $[0, L)$.

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- Existence & uniqueness,
- Analyticity w.r.t spectral parameter,
- Fundamental system of solutions $c(z, x)$ and $s(z, x)$:

$$c(z, 0) = s'(z, -0) = 1, \quad c'(z, -0) = s(z, 0) = 0$$

- ...

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L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice-Hall, 1968.
C. Remling, *Spectral Theory of Canonical Systems*, de Gruyter, 2018.

Summary:

- **Stieltjes strings** $\leftrightarrow$ the **Stieltjes moment problem**,
- **Krein strings** $\leftrightarrow$ **Stieltjes functions**,

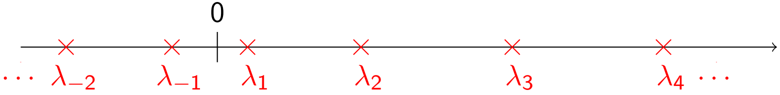
- **Krein–Langer strings** $\leftrightarrow$ the **classical moment problem**,
- **Generalized indefinite strings** $\leftrightarrow$ **Herglotz functions**.

Dispersionless CH with decaying data

If $(u, \upsilon) \in H^1(\mathbb{R}) \times \mathcal{M}_+(\mathbb{R})$, then for

$$-f'' + \frac{1}{4}f = z\,\omega f + z^2 \upsilon f \quad \text{on} \quad \mathbb{R} \qquad \text{(Iso)}$$

where $\omega = u - u_{xx} \in H^{-1}(\mathbb{R})$, the corresponding spectral picture is



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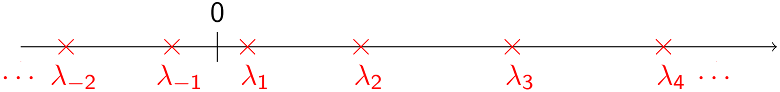

with $\sum_{\lambda\in\sigma} \lambda^{-2} < \infty$. To apply the IST approach additional decay assumptions are needed (e.g., one needs $\sum_{\lambda\in\sigma} |\lambda|^{-1} < \infty$) and this allows to prove the **soliton resolution conjecture** (McKean in 2003):

A weak solution (u, μ) of the conservative CH asymptotically splits into a train of single peakons, each corresponding to an e.v. $\lambda \in \sigma$ of (ISO)

J. Eckhardt and G. Teschl// Adv. Math. **235** (2013).

J. Eckhardt// Arch. Rat. Mech. Anal. **224** (2017).

$$-f'' = z\,\omega f + z^2 \upsilon f \quad \text{on} \quad [0, L) \tag{S2}$$

is considered as a **linear relation** T in $\mathcal{H} = \dot{H}^1_0([0,L)) \times L^2([0,L);\upsilon)$.
However, sometimes everything can be made more transparent!

Let $\chi \in H^{-1}_{\mathrm{loc}}([0,L))$ and consider

$$-f'' = z\,\chi f. \tag{S}$$

Define K_χ by saying $(f,g) \in \dot{H}^1_0([0,L)) \times \dot{H}^1_0([0,L))$ belongs to K_χ if

$$-f'' = \chi g.$$

In fact, K_ω is a graph of a closed densely defined operator. Moreover, $K_\chi^* = K_{\chi^*}$, and

$$\langle K_\chi f, g\rangle_{\dot{H}^1} = \chi(fg^*), \quad f, g \in \dot{H}^1_0 \cap C_c.$$

How to understand strings?

- $L = \infty$: If q is the normalized antiderivative of χ, $\chi(f) = -\int_0^L \mathrm{q} f'\, ds$, then K_χ is unitarily equivalent to J_q acting in $L^2(\mathbb{R}_{>0})$,

$$J_\mathrm{q}\colon f \mapsto \int_0^\infty \mathrm{q}(\max(x,s)) f(s) ds$$

In fact, $K_\chi = U^{-1} J_\mathrm{q} U$, $U\colon f \mapsto f'$ is unitary as $J_\mathrm{q}\colon \dot{H}^1_0 \to L^2(\mathbb{R}_{>0})$.

A. B. Aleksandrov, S. Janson, V. V. Peller, and R. Rochberg, *An interesting class of operators with unusual Schatten–von Neumann behavior*, in: "Function Spaces, Interpolation Theory and Related Topics", 61–149, de Gruyter, Berlin, 2002.

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Let σ be the spectrum of T, the linear relation associated in $\mathcal{H} = \dot{H}^1_0([0, L)) \times L^2([0, L); \upsilon)$ with

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Corollary

- T has purely discrete spectrum $\Leftrightarrow$ both K_ω and K_υ are compact.
- More generally, $\sum_{\lambda\in\sigma} |\lambda|^{-p} < \infty \Leftrightarrow K_\omega \in \mathfrak{S}_p$ and $K_\upsilon \in \mathfrak{S}_{p/2}$.

Discreteness and $\mathfrak{S}_p$ criteria

- $K_\upsilon \in \mathfrak{S}_{p/2}$:

I. S. Kac & M. G. Krein, *Criteria for the discreteness of the spectrum of a singular string*, Izv. VUZov 136–153 (1958).

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Thank you for your attention!!!

- J. Eckhardt & A. Kostenko, *An isospectral problem for global conservative multi-peakon solutions of the Camassa–Holm equation*, Comm. Math. Phys. **329**, 893–918 (2014).
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- J. Eckhardt & A. Kostenko, *Generalized indefinite strings with purely discrete spectrum*, to appear in S. Naboko memorial volume; arXiv:2106.13138.

Connection with canonical systems

Rewrite

$$-f'' = z\,\omega f + z^2 \upsilon f \quad \text{on} \quad [0, L) \tag{S2}$$

as a system

$$Y' = zJ\widetilde{H}(x)Y, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{H}(x) = \begin{pmatrix} 1 & \mathrm{w} \\ \mathrm{w} & \mathrm{w}^2 + \upsilon \end{pmatrix}, \tag{CSmeas}$$

where w is the normalized antiderivative of ω,

$$\omega(f) = -\int_0^L \mathrm{w}\, f' dx.$$

Next ,change variables to transform it into a canonical system on $[0, \infty)$ with trace normed Hamiltonian (similar to "from Krein string to CSs"),

$$Y' = zJH(x)Y, \quad H \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}), \quad \mathrm{tr}\, H \equiv 1 \text{ on } \mathbb{R}_{\geq 0}. \tag{CStr}$$

Linear relation

Definition

With

$$-f'' = z\,\omega f + z^2 \upsilon f \quad \text{on} \quad [0, L) \tag{S2}$$

we associate a linear relation T in $\mathcal{H} = \dot{H}^1_0([0, L)) \times L^2([0, L); \upsilon)$ by saying $(f, g) \in \mathcal{H} \times \mathcal{H}$ is in T exactly when

$$-f_1'' = \omega g_1 + \upsilon g_2, \qquad \upsilon f_2 = \upsilon g_1.$$

The first equality is understood in a distributional sense; the second one means that $f_2 = g_1$ a.e. with respect to υ on $[0, L)$.

Theorem

T is a self-adjoint linear relation and, moreover, its operator part is unitarily equivalent to the multiplication operator in $L^2(\mathbb{R}; \rho)$, where ρ is the measure from the integral representation of the Weyl function m.