On spectral properties of a class of compact Toeplitz operators on Bergman spaces and some applications.

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The plan of the talk

1. Some motivation.
2. Reminder - definitions and discussion.
3. What is known ?
4. Results of the talk and applications.
5. Open questions.

This is a joint work with S. Naboko† (St. Petersburg State University), M. Koita, B. Touré (University of Ségou, Mali).

Some motivation

- Bounded Jacobi matrices (compact perturbations of free Jacobi matrix J_0, compact perturbations of periodic Jacobi matrices, almost-periodic Jacobi matrices, isospectral evolutions, etc. etc.)

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- Unbounded Jacobi matrices - an example.

Some motivation

Let $p \in \mathbb{N}, p \geq 3$. Set

$$J_p = Jac\{n^p, 0, n^p\}$$

and fix a (self-adjoint) initial condition at $n = 0$.

Let $\sigma(J_p) = \{\lambda_n^\pm(J_p)\}$, or

$$\sigma(J_p) = \{\pm\lambda_n(J_p)\}, \ \lambda_n(J_p) \to +\infty.$$

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$$C_1 \leq \frac{\lambda_n(J_p)}{n^p} \leq C_2.$$
- **1997** : Valent's conjecture.

Some motivation

- **2010 - ...** : Berg-Szwarc ; **2015** - Romanov, **2018** - Bochkov - one has

$$\lim_{n\to+\infty} \frac{\lambda_n(J_p)}{n^p} = \tilde{C}_p,$$

with an explicit expression (suggested by Valent) for $\tilde{C}_p$.

Question 1

Can one come up with a (reasonable) functional model for J_p ?

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For instance, set
$$\phi_p(z) := \phi_1(e^{i\theta})\phi_{p0}(r) = \phi_1(e^{i\theta}) \cdot \frac{1}{(1-r^2)^p},$$
where $\phi_1(e^{i\theta}) = (e^{i\theta} + e^{-i\theta})$, and, as usual, $z = re^{i\theta} \in \mathbb{D}$.

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It turns out that $J_p \sim T_\phi$, a Toeplitz operator "on a Bergman space $A^2(\mathbb{D})$".

Bergman space

Reminder - definitions

As always, let $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{T} = \{z : |z| = 1\}$.

The Bergman space is defined as

$$A^2(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) : \|f\|_2^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty \right\}.$$

Above, $dA(z) = dxdy/\pi$.

Writing $f(z) = \sum_{n=0}^{\infty} a_n z^n$, one also has

$$A^2(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) : \|f\|_2^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty \right\}.$$

Bergman space
Reminder - definitions

- The reproducing kernel of the space has the property

$$f(z) = \langle f, K_z \rangle, \quad f \in A^2(\mathbb{D}),$$

and it is given by

$$K_z(w) = K(w, z) = \frac{1}{(1 - w\bar{z})^2}.$$

Bergman space

Reminder - definitions

- The Riesz projection on the space

$$P_+ : L^2(\mathbb{D}) \to A^2(\mathbb{D})$$

is defined by the operator

$$(P_+ f)(z) = \int_{\mathbb{D}} f(w) \frac{1}{(1 - z\overline{w})^2} dA(w) = \langle f, K_z \rangle, \quad f \in L^2(\mathbb{D}).$$

Toeplitz operators on Bergman spaces

Reminder - definitions

For a given function $\phi \in L^\infty(\mathbb{D})$, one sets T_ϕ as

$$T_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D}), \qquad T_\phi f = P_+(\phi f), \quad f \in A^2(\mathbb{D}). \tag{1}$$

T_ϕ is called a Toeplitz operator (on Bergman space) with symbol ϕ.

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In this talk, we'll be interested in compact operators T_ϕ.

Compactness properties of T_ϕ

Reminder - definitions and discussion

Let A be a (non-trivial) compact operator on a Hilbert space (*notation :* $A \in \mathcal{S}_\infty$), and $\{s_n(A)\}$ be the sequence of its singular values, *i.e.,*

$$s_n(A) \geq 0, \quad s_n(A) \geq s_{n+1}(A), \quad \lim_{n\to+\infty} s_n(A) = 0.$$

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It is often covenient to argue in terms of the counting function of s-values of the operator A, *i.e.,*

$$n(s, A) = \#\{s_n(A) : s_n(A) \geq s\}, \quad s > 0, \quad \lim_{s\to 0+} n(s, A) = +\infty.$$

Compactness properties of T_ϕ

Reminder - definitions and discussion

Proposition 2

Let $\phi \in C(\overline{\mathbb{D}})$, and T_ϕ be as above. Then, T_ϕ is compact ($T_\phi \in \mathcal{S}_\infty$) if and only if $\phi|_{\mathbb{T}} = 0$.

Equivalently, the last condition can be written as

$$\lim_{|z|\to 1-0} \phi(z) = 0.$$

Compactness properties of T_ϕ

Reminder - definitions and discussion

A hint of a part of the proof.

$(\Rightarrow)$ Set

$$k_z(w) = \frac{K_z(w)}{||K_z||} = \frac{(1-|z|^2)}{(1-\bar{z}w)^2}, \quad z, w \in \mathbb{D}.$$

Remind that $k_z \to 0$ weakly for $z, |z| \to 1-0$.

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Remind that $k_z \to 0$ weakly for $z, |z| \to 1-0$.

So, by the compacity, $||T_\phi k_z|| \to 0$ for these z, and

$$|\langle T_\phi k_z, k_z \rangle| \le ||T_\phi k_z|| \, ||k_z|| = ||T_\phi k_z|| \to 0.$$

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So, by the compacity, $||T_\phi k_z|| \to 0$ for these z, and

$$|\langle T_\phi k_z, k_z\rangle| \le ||T_\phi k_z||\,||k_z|| = ||T_\phi k_z|| \to 0.$$

On the other hand, by the properties of the 2D-Poisson kernel and the fact that $\phi \in C(\bar{\mathbb{D}})$, we get for a fixed $z_0 \in \mathbb{T}$

$$\phi(z_0) = \lim_{z \to z_0} \langle T_\phi k_z, k_z\rangle = 0.$$

What is known ?

A. Pushnitski's result

Let $\gamma > 0$ and put $\phi \in L^\infty(\mathbb{D})$ to be

$$\phi(z) := \phi_1(e^{i\theta})\phi_0(r) = \phi_1(e^{i\theta})(1-r)^\gamma, \qquad z = re^{i\theta} \in \mathbb{D}.$$

Suppose that $\phi_1 \in C(\mathbb{T})$.

Theorem 3

Let T_ϕ be the Toeplitz operator with the above symbol. Then

$$s_n(T_\phi) = n^{-\gamma}\left(\mathcal{C}_\gamma(\phi) + o(1)\right), \quad n \to +\infty,$$

or, equivalently,

$$\lim_{s\to 0+} s^{\frac{1}{\gamma}} n(s, T_\phi) = \mathcal{C}_\gamma(\phi)^{\frac{1}{\gamma}}.$$

What is known ?

A. Pushnitski's result

Moreover, one has the following expression for the constant $\mathcal{C}_\gamma(\phi)$

$$\mathcal{C}_\gamma(\phi) = 2^{-\gamma}\Gamma(\gamma+1)\left(\int_0^{2\pi} \mid \phi_1(e^{i\theta}) \mid^{\frac{1}{\gamma}} \frac{d\theta}{2\pi}\right)^\gamma .$$

****

We are interested in the case of the logarithmic decay of the symbol ϕ for z going to $\mathbb{T}$.

Namely, take $\gamma > 0$, consider $\phi_1 \in C(\mathbb{T})$, and

$$\phi(z) := \phi_1(e^{i\theta})\phi_0(r),$$

where

$$\phi_0(r) := \phi_{0,\gamma}(r) = \frac{1}{\left(1+\log\frac{1}{(1-r)}\right)^\gamma}, \qquad z = re^{i\theta} \in \mathbb{D}.$$

Results

The main result

Theorem 4

Once again, let ϕ be the symbol defined above, and T_ϕ is the corresponding Toeplitz operator. We have

$$\lim_{n\to+\infty}(\log(n+1))^\gamma s_n(T_\phi) = ||\phi_1||_{L^\infty(\mathbb{T})},$$

or, equivalently,

$$\lim_{s\to 0+} s^{1/\gamma}\log(n(s,T_\phi)+2) = ||\phi_1||^{1/\gamma}_{L^\infty(\mathbb{T})}.$$

Results

Ideas of the proof - some definitions

It is convenient to set

$$\tilde{n}(s, A) = n(s, A) + 2.$$

Results

Ideas of the proof - some definitions

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Then, one defines the following classes of compact operators (cf. Birman-Solomyak) :

$$\begin{aligned}
\Sigma_\gamma &= \left\{ A \in \mathcal{S}_\infty : s_n(A) = O\left(\frac{1}{(\log(n+1))^\gamma} \right) \right\}, \\
&= \left\{ A \in \mathcal{S}_\infty : \sup_{n \geq 1} (\log n)^\gamma s_n(A) < +\infty \right\}, \\
\Sigma_\gamma^0 &= \left\{ A \in \mathcal{S}_\infty : s_n(A) = o\left(\frac{1}{(\log(n+1))^\gamma} \right) \right\}, \\
&= \left\{ A \in \mathcal{S}_\infty : \lim_{n \to +\infty} (\log n)^\gamma s_n(A) = 0 \right\}.
\end{aligned}$$

Results

Ideas of the proof - some definitions

Of course, one can characterize these classes in terms of $\widetilde{n}(s, A)$, too.

For an $a \in \mathcal{S}_\infty$, the use of the following "functionals" will be quite convenient :

$$\Delta_\gamma(A) = \limsup_{s \to 0+} s^{\frac{1}{\gamma}} \, log \, \tilde{n}(s, A), \qquad \delta_\gamma(A) = \liminf_{s \to 0+} s^{\frac{1}{\gamma}} \, log \, \tilde{n}(s, A).$$

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Proposition 5 (Lemme de Ky-Fan)

Soient $A \in \Sigma_\gamma$ et $B \in \Sigma_\gamma^0$. Alors

$$\Delta_\gamma(A + B) = \Delta_\gamma(A), \qquad \delta_\gamma(A + B) = \delta_\gamma(A). \tag{2}$$

Results

Ideas of the proof - asymptotic orthogonality of a family of operators

Proposition 6

Let $A, A_k,\ k = 1, \dots, L,$ *and* $A = \sum_{k=1}^{L} A_k$.
Suppose that $A, A_k \in \Sigma_\gamma,\ \ k = 1, \dots, L$, *and*

$$A_k^* A_j, A_k A_j^* \in \Sigma_{2\gamma}^0, \quad j \neq k, \ \ j, k = 1, \dots, L.$$

Then

$$\Delta_\gamma(A) = \limsup_{s \to 0+} s^{1/\gamma} \log \left(\sum_{k=1}^{L} \tilde{n}(s, A_k) \right),$$

$$\delta_\gamma(A) = \liminf_{s \to 0+} s^{1/\gamma} \log \left(\sum_{k=1}^{L} \tilde{n}(s, A_k) \right).$$

Results

Ideas of the proof - asymptotic orthogonality of a family of operators

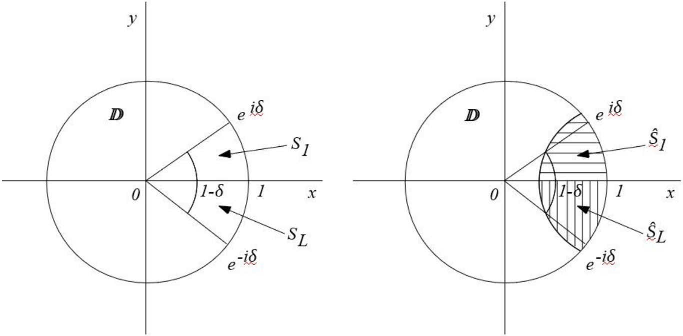

FIGURE – Asymptotic orthogonality of indicator functions $\chi_{S_1}\phi_0$ and $\chi_{S_L}\phi_0$.

Results

Ideas of the proof

Recall that

$$\begin{aligned} \phi(z) &= \phi_1(e^{i\theta})\phi_0(r) \\ &= \phi_1(e^{i\theta}) \cdot \frac{1}{\left(1 + \log \frac{1}{(1-r)}\right)^{\gamma}}, \qquad z = re^{i\theta} \in \mathbb{D}. \end{aligned}$$

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First, we look at the case where $\phi_1(e^{i\theta}) \equiv 1$, or $\phi(z) := \phi_0(z) = \phi_0(r)$.

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Lemma 7

We have

$$\lim_{n\to+\infty}(\log(n+1))^{\gamma}s_n(T_{\phi_0}) = ||\phi_1||_{L^\infty(\mathbb{T})} = 1.$$

Results

Ideas of the proof

Proof.
The symbol ϕ of the Toeplitz operator is radial, and one can compute the matrix of operator T_ϕ in the standard orthonormal basis of $A^2(\mathbb{D})$, $(e_n)_{n\in\mathbb{N}},\ e_n(z) = \sqrt{n+1}\, z^n$.

The matrix is diagonal, and we have

$$
\begin{aligned}
s_n(T_\phi) &= \langle T_\phi e_n, e_n\rangle = 2(n+1)\int_0^1 r^{2n+1}\phi_0(r)\,dr \\
&= 2(n+1)\,\frac{1}{(2n+1)(\log(2n+1))^\gamma}(1+o(1)) \\
&= \frac{1}{(\log(n+1))^\gamma}\,(1+o(1))\,, \quad n\to+\infty.
\end{aligned}
$$

□

Results

Ideas of the proof - a positive stair-function ϕ_1

We concentrate on the proof of the theorem in the case of a positive "stair-function" ϕ_1 on $\mathbb{T}$. The case of a general "stair-function" is similar, and the case for a $\phi_1 \in C(\mathbb{T})$ is obtained by an appropriate passing to the limit.

Let $L \in \mathbb{N}, L > 0$, and $I_j := [2\pi(j-1)/L, 2\pi j/L)$ be a n arc of $\mathbb{T}$, $j = 1, \dots, L$.
Set χ_{I_j} to be the indicator function of $I_j,\ j = 1, \dots, L$, and

$$\tilde{\chi}_j := \chi_{I_j}\phi_0, \quad T_{\tilde{\chi}_j} := P_+\tilde{\chi}_j = P_+(\chi_{I_j}\phi_0).$$

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$$\tilde{\chi}_j := \chi_{I_j}\phi_0, \quad T_{\tilde{\chi}_j} := P_+\tilde{\chi}_j = P_+(\chi_{I_j}\phi_0).$$

It is clear that $T_{\tilde{\chi}_j}$ is unitarily equivalent to $T_{\tilde{\chi}_1}$ by rotation, and so

$$\tilde{n}(s, T_{\tilde{\chi}_j}) = \tilde{n}(s, T_{\tilde{\chi}_1}),\ s > 0,$$

recall the Picture.

Results

Ideas of the proof - a positive stair-function ϕ_1

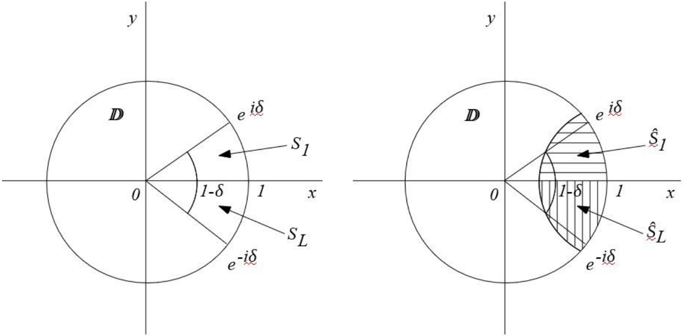

FIGURE – Asymptotic orthogonality of indicator functions $\chi_{S_1}\phi_0$ and $\chi_{S_L}\phi_0$.

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Theorem 8

For $L > 0$, let $\phi_1 : \mathbb{T} \longrightarrow \mathbb{R}_+$ be a positive stair-function

$$\phi_1 = \sum_{j=1}^{L} c_j \chi_j,$$

where $c_j \geq 0$. For $\phi = \phi_1 \phi_0$, we have

$$\Delta_\gamma(T_\phi) = \delta_\gamma(T_\phi) = ||\phi_1||^{1/\gamma}_{L^\infty(\mathbb{T})},$$

or, equivalently,

$$\lim_{s \to 0+} s^{1/\gamma} \log \tilde{n}(s, T_\phi) = ||\phi_1||^{1/\gamma}_{L^\infty(\mathbb{T})}.$$

Results

Ideas of the proof - a positive stair-function ϕ_1

Proof.

Step 1. To start with, let $\phi_1(e^{i\theta}) \equiv 1$ on $\mathbb{T}$, or $\phi = \phi_0$.

We see that

$$\phi_1 = \sum_{j=1}^{L} 1 \cdot \chi_j, \quad T_\phi = T_{\phi_0} = \sum_{j=1}^{L} 1 \cdot T_{\tilde{\chi}_j}.$$

By Lemma 7 above,

$$\Delta_\gamma(T_{\phi_0}) = \delta_\gamma(T_{\phi_0}) = ||1||_{L^\infty(\mathbb{T})}^{1/\gamma} = 1.$$

Results

Ideas of the proof - a positive stair-function ϕ_1

Operators $T^*_{\tilde{\chi}_j} T_{\tilde{\chi}_k}, T_{\tilde{\chi}_j} T^*_{\tilde{\chi}_k}$ lie in $\Sigma^0_{2\gamma}$ for $j \neq k,\ j, k = 1, \dots, L$.

** So, the family of operators $\{T_{\tilde{\chi}_j}\}_{j=1,\dots,L}$ is asymptotically orthogonal. **

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** So, the family of operators $\{T_{\tilde{\chi}_j}\}_{j=1,\dots,L}$ is asymptotically orthogonal. **

$$\begin{aligned}\Delta_\gamma(T_{\phi_0}) &= \limsup_{s\to 0+} s^{1/\gamma} \log\left(\sum_{j=1}^{L} \tilde{n}(s, T_{\tilde{\chi}_j})\right) \\ &= \limsup_{s\to 0+} s^{1/\gamma} \log(L\,\tilde{n}(s, T_{\tilde{\chi}_1})) = \Delta_\gamma(T_{\tilde{\chi}_1}).\end{aligned}$$

Similarly, $\delta_\gamma(T_{\phi_0}) = \delta_\gamma(T_{\tilde{\chi}_1})$, and

$$\Delta_\gamma(T_{\phi_0}) = \delta_\gamma(T_{\phi_0}) = \Delta_\gamma(T_{\tilde{\chi}_1}) = \delta_\gamma(T_{\tilde{\chi}_1}) = 1.$$

(Step 1) □

Step 2. Let now

$$\phi_1 = \sum_{j=1}^{L} c_j \cdot \chi_j, \quad T_\phi = \sum_{j=1}^{L} c_j \cdot T_{\tilde{\chi}_j},$$

where $c_j \geq 0, \ j = 1, \dots, L$.

It is convenient to suppose that $\max_{j=1,\dots,L} c_j = c_1 > 0$. In particular,

$$\tilde{n}(s, c_j T_{\tilde{\chi}_j}) \leq \tilde{n}(s, c_1 T_{\tilde{\chi}_1}), \quad j = 1, \dots, L.$$

Results

Ideas of the proof - a positive stair-function ϕ_1

Consequently,

$$
\begin{aligned}
\Delta_\gamma(T_\phi) &= \limsup_{s\to 0+} s^{1/\gamma} \log\left(\sum_{j=1}^{L} \tilde{n}(s, c_j T_{\tilde{\chi}_j})\right) \\
&= \limsup_{s\to 0+} s^{1/\gamma} \log\left(\tilde{n}(s, c_1 T_{\tilde{\chi}_1}) + \sum_{j=2}^{L} \tilde{n}(s, c_j T_{\tilde{\chi}_j})\right) \\
&= \limsup_{s\to 0+} s^{1/\gamma} \log \tilde{n}(s, c_1 T_{\tilde{\chi}_1}) \left\{1 + \frac{\sum_{j=2}^{L} \tilde{n}(s, c_j T_{\tilde{\chi}_j})}{\tilde{n}(s, c_1 T_{\tilde{\chi}_1})}\right\} \\
&= \limsup_{s\to 0+} s^{1/\gamma} \log \tilde{n}(s, c_1 T_{\tilde{\chi}_1}) = c_1^{1/\gamma} \Delta_\gamma(T_{\phi_0}) = c_1^{1/\gamma}.
\end{aligned}
$$

Idem for $\delta_\gamma(T_\phi)$. (Step 2*) □

Thank you !

Applications

Let $D^0 : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be a compact operator. Let

$$D^0 := [d^0_{i,j}]_{i,j=0,\dots,\infty}$$

be its matrix in the standard basis of $\ell^2(\mathbb{Z}_+)$.

Applications

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For a given $N \in \mathbb{N}$, we say that D^0 is $(2N+1)$-banded matrix (or, $(2N+1)$-banded operator)

$$\text{iff} \quad d^0_{i,j} = 0 \text{ for } |i-j| > N+1.$$

Applications

Take $\{b_{-N}, \ldots, b_N\}$, a set of $(2N+1)$ complex coefficients.

For a $\gamma > 0$, we say that $(2N+1)$-banded matrix D_0 has a logarithmic decay, iff

$$d^0_{m,m+j} = \frac{b_j}{(\log m)^\gamma}(1 + o(1)), \qquad m \to +\infty,\ j = -N, \ldots, N.$$

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$$d^0_{m,m+j} = \frac{b_j}{(\log m)^\gamma}(1+o(1)), \quad m \to +\infty,\ j = -N, \dots, N.$$

Put

$$\phi_{1,b}(e^{i\theta}) := \sum_{j=-N}^{N} b_j e^{ij\theta},$$

as well as

$$D := [d_{i,j}]_{i,j=0,\dots,\infty}, \quad d_{m,m+j} = \frac{b_j}{(\log m)^\gamma}, \quad j = -N, \dots, N.$$

Applications

Corollary 9

We have the following spectral asymptotics for the $(2N+1)$*-banded matrix* D

$$\lim_{n\to+\infty}(\log n)^\gamma s_n(D) = ||\phi_{1,b}||_{L^\infty(\mathbb{T})}.$$