Analysis I

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Contents

1 The natural numbers

1.1 Peano's axioms

We denote by $\mathbf{N}$ the set of the natural numbers i.e.

$$\mathbf{N} = \{1, 2, 3, \ldots\}.$$

If we include the zero element 0, then we denote the resulting set by $\mathbf{N}_0$. Hence

$$\mathbf{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

$\mathbf{N}$ is characterised by the following properties:

Peano's Axioms

1. $1 \in \mathbf{N}$;
2. Every natural number has a uniquely determined successor n';
3. Every natural number n, with the exception of 1, is the successor of a uniquely determined natural number.
4. Suppose that A is a subset of $\mathbf{N}$ which is such that $1 \in A$ and if $n \in A$, then $n' \in A$. It follows that $A = \mathbf{N}$.

1.2 Mathematical induction

The third property above is often expressed in the following way:

Proposition 1 *(The Principle of mathematical induction) Let $A(n)$ be a statement which depends on the natural number n. Then if*

1. *$A(1)$ holds;*
2. *for each $n \in \mathbf{N}$ we have: if $A(n)$ holds, then so does $A(n')$,*

Then $A(n)$ is true for each $n \in \mathbf{N}$.

PROOF. Put $A = \{n \in \mathbf{N} : A(n) \text{ holds}\}$ and use property 3).

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There are several useful variants of this principle, for example

Proposition 2 *(Mathematical Induction — Variant I) Let $A(n)$ be a statement which depends on the natural number n. If*

1. *$A(n_0)$ holds;*
2. *for each $n \geq n_0$ we have: If $A(n)$ holds, then so does $A(n')$.*

Then $A(n)$ is true for each $n \geq n_0$.

Proof. Put $B(n) = A(n_0 + (n - 1))$ and use the original form.

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Proposition 3 *(Mathematical Induction — Variant II) If $A(n)$ is a statement which depends on the natural number n and if*

1. *$A(1)$ is valid;*
2. *for each $n \in \mathbf{N}$ we have: whenever $A(1), \ldots, A(n)$ hold, so does $A(n')$.*

Then we have $A(n)$ for each n.

Proof. Put $B(n) = A(1) \wedge A(2) \wedge \cdots \wedge A(n)$.

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With the aid of mathematical induction, we can define:
I. Addition: We define the sum $m + n$ of two natural numbers as follows:

1. For $m \in \mathbf{N}$ we put $m' := m + 1$;
2. If a natural number is the successor n' of the natural number n, then we put $m + n' := (m + n)'$.

One then sees that $m + n$ is defined for each n. (put $A = \{n : m + n \text{ is defined}\}$).

As expected, the familiar laws of addition hold:

1. $m + n = n + m$ $(m, n \in \mathbf{N})$ (commutativity);
2. $m + (n + p) = (m + n) + p$ $(m, n, p \in \mathbf{N})$ (associativity).

Proof. We prove the associativity. For this we use induction on p.

For $p = 1$ we have the statement: $(m + n') = m + n'$. $p \to p + 1$: We assume that $m + (n + p) = (m + n) + p$. Then

$$(m+n)+p' = [(m+n)+p]' = [m+(n+p)]' = m+(n+p)' = m+(n+p').$$

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The proofs of the other facts are similar. The reader should work some of them out if only for achieving the kind of self-discipline of proving facts which appear to be obvious.

II. Multiplication: We also define the product of two natural numbers inductively:

1. $m.1 = m$ $(m \in \mathbf{N})$;
2. $m.n' = m.n + m$ $(m, n \in \mathbf{N})$.

One can then show as for addition that mn is defined for each pair m and n as for addition.

Once again, the familiar laws hold:

1. $m.n = n.m$ $(m, n \in \mathbf{N})$;
2. $m.(n.p) = (m.n).p$ $(m, n, p \in \mathbf{N})$;
3. $m.(n + p) = m.n + m.p$ $(m, n, p \in \mathbf{N})$ (the distributive law);

Notation Suppose that for each k with $m \leq k \leq n$ we are given a number a_k. Then one puts

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

$$\prod_{k=m}^{n} a_k = a_m.a_{m+1} \ldots a_n.$$

Once again, we are tacitly using induction. A more formal version of this definition would be:

$$\sum_{k=m}^{m} a_k = a_m$$

$$\sum_{k=m}^{n'} a_k = \sum_{k=m}^{n} a_k + a_{n'}.$$

The fact that this concept is well-defined can be shown as follows: If we put $A = \{n \geq m : \sum_{k=m}^{n} a_k \text{ is defined}\}$, then it follows that $A = \{n : n \geq m\}$). We bring some examples of simple and useful facts which are proved with induction:

Proposition 4

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

for each natural number n.

Proof. Induction. The case $n = 1$ is clear.
$n \to n+1$: Suppose that $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$. Then

$$\begin{aligned}\sum_{k=1}^{n'} k &= (\textstyle\sum_{k=1}^{n} k) + n' \\ &= \tfrac{n(n+1)}{2} + (n+1) \\ &= \tfrac{1}{2}(n+1)(n+2) = \tfrac{1}{2}n'(n'+1).\end{aligned}$$

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Proposition 5

$$\sum_{k=1}^{n}(2k-1) = n^2$$

for all natural numbers n.

Proof. Exercise.

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Notation If $n \in \mathbf{N}$ we put

$$n! = \prod_{k=1}^{n} k.$$

(We use the convention $0! = 1$).

Proposition 6 *The number of possible permutations of a set $\{a_1, \dots, a_n\}$ with n elements is $n!$.*

Proof. We prove the formally more general statement that if S, S_1 n are sets with n elements then there are $n!$ bijections from S onto S_1. Once again we use induction. The case $n = 1$ is clear.
$n \to n+1$: We fix one element a in S. There are $n+1$ possible images for a in S_1. Each such choice of image provides us with $n!$ bijections by the induction hypothesis. Hence altogether we have $(n+1)!$.

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Notation If k and n are natural numbers with $k \leq n$, we denote by $\binom{n}{k}$ the number of subsets of a set with n elements which contain k elements. (We use the convention $\binom{n}{k} = 1$, in the case where $k = 0$).

Proposition 7 *1.* $\binom{n}{1} = n$, $\binom{n}{n} = 1$*;*

2. for $1 \leq k \leq n$ *we have the relation*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

3. for $0 \leq k \leq n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. 1) is clear.
2) We decompose the family of subsets with k elements of

$$\{a_1, \dots, a_n\}$$

into two disjoint classes.

a) those which contain a_1. There are $\binom{n-1}{k-1}$ such sets.

b) those which do not contain a_1. There are $\binom{n-1}{k}$ such subsets.

3) The proof is by induction on n. It is clearly true for $n = 0$. $n \to n+1$: Suppose that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Then

$$\binom{n+1}{k} \qquad \frac{=\binom{n}{k-1} + n}{k}$$

$$= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{n-k+1} + \frac{1}{k}\right)$$

$$= \frac{n!}{(k-1)!(n-k)!}\left(\frac{n+1}{(n-k+1)k}\right)$$

$$= \frac{(n+1)!}{(n-k+1)!k!} = \frac{(n')!}{k!(n'-k)!}.$$

∎

In the following Proposition we use the notion of a real number. This will be discussed in detail in the next section (we now only use the facts that multiplication and addition of real numbers obeys the usual basic laws).

Proposition 8 *(Binomial theorem) Let x, y be real numbers and n a natural number. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof. By induction on n.
$n = 1$ is clear.
$n \to n+1$: Suppose that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. Then

$$\begin{aligned}
(x+y)^{n'} &= (x+y)^{n+1} \\
&= (x+y)\textstyle\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\
&= \textstyle\sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1} \\
&= \textstyle\sum_{k=0}^{n'} \text{n'-1} k x^{n'-k} y^k + \sum_{k=0}^{n'} \binom{n'-1}{k-1} x^{n'-k} y^k \\
&= \textstyle\sum_{k=0}^{n'} \left[\binom{n'-1}{k} + \binom{n'-1}{k-1} \right] x^{n'-k} y^k \\
&= \textstyle\sum_{k=0}^{n'} \binom{n'}{k} x^{n'-k} y^k.
\end{aligned}$$

(We use the convention $\binom{n}{k} = 0$, if $k < 0$).

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Proposition 9 *If $x \neq 1$ then*

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Proof. Exercise.

■

1.3 Exercises

Exercise Show that $m.n = n.m$ $(m, n \in \mathbf{N})$. (Another example of proving an obvious fact. If the reader is puzzled by the necessity of such a proof, he should rethink the contents of this chapter).

Exercise Prove:

$$\sum_{k=1}^{n} k^2 = \tfrac{n(n+1)(2n+1)}{6}$$
$$\sum_{k=1}^{n} k^3 = \tfrac{n^2(n+1)^2}{4}.$$

Exercise Calculate

$$1 + \cos\theta + \cdots + \cos n\theta$$

and

$$\sin\theta + \cdots + \sin n\theta.$$

(Hint: use the formula for $\sin A \sin B$ und $\sin A \cos B$ and the telescope principle)

Exercise Show that $2^n \geq n^2 \quad (n \geq 4)$.

Exercise Calculate

$$\sum_{k=0}^{n} \binom{n}{k} \quad \text{bzw.} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

Exercise Show that

$$\sum_{j=0}^{n} \binom{k+j}{j} = \binom{k+n+1}{n}.$$

Exercise Show that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots = n2^{n-1}.$$

Exercise Show that

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots = 0.$$

Exercise Show that

$$2^{n-1} \leq n! \leq n^n \quad (n \geq 1).$$

Exercise Show that

$$N!(N+1)^{n-N} \leq n! \leq N!n^{n-N} \quad (N \geq 1, n \geq N).$$

Exercise Show that

$$\sqrt{\frac{\frac{5}{4}}{4n+1}} \leq \frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots.2n} \leq \sqrt{\frac{\frac{3}{4}}{2n+1}}.$$

Exercise (The Bernoulli-inequality) Show that

$$(1+x)^n \geq 1+nx \quad (x \geq -1, n \in \mathbf{N}).$$

Exercise Let $x_1, \ldots, x_n$ $(n \geq 2)$ be real numbers, all of which are either positive or all negative and all > -1. Show that

$$(1+x_1)\ldots(1+x_n) > 1+x_1+\cdots+x_n.$$

Exercise Show that

$$2^n \leq n! \quad (n \geq 4).$$

Exercise Let $m, n \in \mathbf{N}$. Show that there unique non-negative numbers q and r so that

$$n = qm + r, \qquad 0 \leq r < m.$$

("Division with remainder").

Exercise Show that

$$\binom{n}{k}\frac{1}{n^k} \leq \frac{1}{k!} \quad (k \in \mathbf{N}_0).$$

Exercise Show that

$$(1+\frac{1}{n})^n \leq \sum_{k=0}^{n}\frac{1}{k!} \leq 3.$$

Exercise Show that

$$\binom{n}{3} \leq \frac{1}{3}n!.$$

Exercise Show that
$$\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k}.$$

Exercise Let p be the polynomial $t^n + a_1 t^{n-1} + \cdots + a_n$, with zeroes $\lambda_1, \ldots \lambda_n$. Put $s_k = \sum_i \lambda_i^k$ and show that
$$ka_k = -\sum_{i=0}^{k-1} a_i s_{k-i}.$$

Exercise Let x be a real number and k a natural number. Put
$$\binom{x}{k} = \prod_{m=1}^{k} \frac{x-m+1}{m}.$$
Show that
$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{n-k}\binom{y}{k}.$$

Exercise There are
$$\frac{n!}{n_1! \ldots n_k!}$$
possible ways of distributing $n = n_1 + \cdots + n_k$ balls amongst k urns $K_1, \ldots, K_k$ so that n_1 are placed in K_1, $\ldots$, n_k in K_k.

Exercise (Abel partial Summation) Show that
$$\sum_{k=1}^{n} a_k b_k = A_n b_n + \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}),$$
whereby $A_n = \sum_{k=1}^{n}$.

2 The field of real numbers

Starting with the natural numbers, we can construct successively the whole numbers, the rational numbers and finally the real numbers. The first two constructions are purely algebraic and will be sketched briefly. The third step is discussed in the appendix.

2.1 The whole numbers:

We define the set of whole numbers $\mathbf{Z}$ to be the quotient space $\mathbf{N} \times \mathbf{N}|_\sim$, whereby

$$(m, n) \sim (\bar{m}, \bar{n}) \iff m + \bar{n} = \bar{m} + n.$$

($\sim$ is an **equivalence relation**. $\mathbf{N} \times \mathbf{N}|_\sim$ denotes the corresponding quotient space.

We extend the algebraic operations to $\mathbf{Z}$ in the natural way:

$$[(m, n)] + [(\overline{m}, \overline{n})] = [(m + \overline{m}, n + \overline{n})]$$

and

$$[(m, n)][(\overline{m}, \overline{n})] = [(m.\overline{m} + n.\overline{n}, m.\overline{n} + \overline{m.n})].$$

($[(m, n)]$ denotes the equivalence class determined by (m, n)).

Once again, the familiar laws of multiplication and adddition hold:

2.2 The rational numbers:

The set $\mathbf{Q}$ of rational numbers is defined as $\mathbf{Z} \times (\mathbf{Z} \setminus \{0\})|_\sim$, whereby

$$(m, n) \sim (\overline{m}, \overline{n}) \iff m.\overline{n} = \overline{m.n}.$$

The algebraic operations are extended as follows:

$$[(m, n)] + [(\overline{m}, \overline{n})] = [(m.\overline{n} + \overline{m.n}, n.\overline{n})]$$

and

$$[(m, n)].[(\overline{m}, \overline{n})] = [(m.\overline{m}, n.\overline{n})].$$

It is a matter of routine to verify that $\mathbf{Q}$ satisfies the following **field axioms**:

Axioms of addition:

1. $x + (y + z) = (x + y) + z$ für x,y, $z \in \mathbf{Q}$;
2. $x + y = y + x$ $(x, y \in \mathbf{Q})$;
3. $x + 0 = 0 + x = x$ $(x \in \mathbf{Q})$;
4. for each $x \in \mathbf{Q}$ there exists a Zahl y, so that$x + y = 0$.

Axioms of multiplication:

1. $(xy)z = x(yz)$ $(x, y, z \in \mathbf{Q})$;
2. $xy = yx$ $(x, y \in \mathbf{Q})$;
3. $x.1 = 1.x = x$ $(x \in \mathbf{Q})$;
4. for each $x \neq 0$ in $\mathbf{Q}$ there exists an element x^{-1}, so that $x.x^{-1} = 1$.
5. (the distributive law) $x(y + z) = xy + xz$ $(x, y, z \in \mathbf{Q})$.

We express this by saying that $\mathbf{Q}$ is a field. The reader will meet several further examples of fields for example, the field of complex numbers. We will also discuss more general structures, which satisfy some, but not all of the above axioms. For example a skew field satisfies all of the above, except the commutativity of multiplication. The best known example is the skew field of quaternions.

A second important example is that of a ring. They satisfy the axioms of addition, the distributive law and axiom (1) of multiplication.

In addition to its algebraic structure, $\mathbf{Q}$ is an **ordered field**. In order to see this, we define successively the natural order on $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$:

1. in $\mathbf{N}$: $m < n \iff$ there exists $p \in \mathbf{N}$ with $n = m + p$;
2. in $\mathbf{Z}$: $[(m, n)] < [(\overline{m}, \overline{n})] \iff m + \overline{n} < \overline{m} + n$.

Before defining the ordering on $\mathbf{Q}$ we first note that each element therein has a representation of the form $[(m, n)]$ with $n > 0$. In this case i.e. if $n, \overline{n} > 0$, we define

$$[(m, n)] < [(\overline{m}, \overline{n})] \iff m.\overline{n} < \overline{m.n}.$$

$\mathbf{Q}$ satisfies the following axioms:

2.3 The ordering axioms:

1. for each $x \in \mathbf{Q}$ exactly one of the folloiwng holds:
$$x > 0, x < 0, x = 0.$$
2. $x > 0, y > 0$ implies $x + y > 0$;
3. $x > 0, y > 0$ implies $xy > 0$.

We then write $x < y$, if $y - x > 0$.

In order to differ between $\mathbf{Q}$ and the family $\mathbf{R}$ of real numbers, we require a final axiom:

The axiom of completeness: We postulate the existence of an ordered field $\mathbf{R}$, which satisfes the following axiom: Suppose that $A \neq \emptyset$ is a subset of $\mathbf{R}$, which is bounded above (i.e. there exists a y so that $x \leq y$ for each $x \in A$). Then there exists a *smallestl* upper bound x_0 for A (called its **supremum**). Symbolically: there exists x_0 so that

$$1) \quad x \in A \text{ implies } x \leq x_0$$

and

$$2) \quad \text{for each } \epsilon > 0 \text{ there exists } x \in A \text{ so that } x_0 - \epsilon < x.$$

We write then $x_0 = \sup A$. Then it follows easily each set A which is boundjed from below has an Infimum. In fact, we have $\inf A = \sup(-A)$, where $-A = \{-x : x \in A\}$.

This is the axiom which distinguishes between the reals and the rationals. For $\mathbf{Q}$ is clearly not complete in this sense. (For a sketch of a possible construction of the reals, see the appendix).

We remark that it is of interest that the axioms for the real numbers are categorical i.e. there is essentially one model for them (see Appendix).

2.4 Exercises

Exercise Verify that the axiom of Archimedes holds for the real numbers: for each $x \in \mathbf{R}$ there exists an $n \in \mathbf{N}$ with $n \geq x$. This implies that for each $\epsilon > 0$ in $\mathbf{R}$ there exists $n \in \mathbf{N}$ with $\frac{1}{n} \leq \epsilon$.

Exercise Show that $\sqrt{2}$ and $\sqrt{2} + \sqrt{3}$ are irrational.

Exercise Take $x > 0$. Show that there exists an irrational number y with $0 < y < x$.

Exercise Let $p_1, \dots, p_n$ be positive numbers with $p_1 + \cdots + p_n = 1$. Then

$$\min(a_, \dots, a_n) \leq p_1 a_1 + \cdots + p_n a_n \leq \max(a_1, \dots, a_n).$$

Exercise If $a, b > 0$, show

$$\sqrt{(ab)} \leq \frac{a+b}{2}.$$

Exercise Let $a_1, \ldots, a_n$ be positive. Show

$$\sqrt[n]{a_1 \ldots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Exercise Let a and b be the zeroes of the polynomial $x^2 - x - 1$ and put $x_n = \dfrac{(a^n - b^n)}{a - b}$. Show that $x_1 = 1$, $x_2 = 1$ and $x_{n+1} = x_n + x_{n-1}$.

Exercise Let $(a_k)_{k_1}^n$ and $(b_k)_{k=1}^n$ be finite sequences of real numbers. Calculate the discriminant of the quadratic function

$$t \mapsto \sum_{k=1}^{n} (a_k t + b)^2.$$

and use this to prove the **Cauchy-Schwarz inequality**

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 \leq \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right).$$

Exercise Let $x_1, \ldots, x_n$ be positive numbers. Show

$$\left(\sum_{k+1}^{n} x_k \right) \left(\sum_{k=1}^{n} \frac{1}{x_k} \right) \geq n^2.$$

Exercise Prove Lagrange's identity:

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 = \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.$$

(this can be used to give an alternative proof of the Cauchy-Schwarz inequality).

Exercise If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, show that

$$\left(\sum_k a_k \right) \left(\sum_k b_k \right) \leq n \sum_k a_k b_k.$$

Exercise Show that the set of all real numbers of the form $a + b\sqrt{2}$ ($a, b \in \mathbf{Q}$) is a field.

Exercise Show that the family of all real polynomials is a ring. The family of all rational functions is a field. The set of all $n \times n$ matrices is a ring.

Exercise If x is a real number, put $|x| = x$ when $x \geq 0$ and $= -x$ when $x \leq 0$. Show that

$$|x+y| \leq |x| + |y| \quad |xy| = |x|.|y| \text{ and } \sup\{x,y\} = \frac{1}{2}(x+y+|x-y|).$$

What is the corresponding formula for $\inf\{x,y\}$?

Exercise Show that

$$|a| - |b| \leq ||a| - |b|| \leq |a-b|$$

resp.

$$||a| - |b|| \leq |a+b|$$

$(a, b \in \mathbf{R})$.

Exercise Show that

$$\left|\frac{a}{b} + \frac{b}{a}\right| \geq 2$$

$(a, b \in \mathbf{R},\ a \neq 0,\ b \neq 0)$.

Exercise Let n be a natural number which is not the square of a $p \in \mathbf{N}$. Show that $\sqrt{n}$ is irrationial.

Exercise Let a, b, c, d be real numbers with $b > 0$, $d > 0$, so that $\frac{a}{b} < \frac{c}{d}$. Show that $\frac{a+c}{b+d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

Exercise Calculate $\sup A$, $\inf B$, where

$$A = \{\frac{1}{n} + (-1)^n : n \in \mathbf{N}\}$$

resp.

$$A = \{(-1)^n(1 + \frac{1}{n}) : n \in \mathbf{N}\}.$$

Exercise Let A, B be subsets of $\mathbf{R}$. Show that

$$\sup(A \cup B) = \max(\sup A, \sup B).$$

What is the corresponding expression for $\inf(A \cup B)$? Is there a similar formula for $\sup(A \cap B)$?

Exercise Let A and B be subsets of $\mathbf{R}$. Show that $\sup(A + B) = \sup A + \sup B$ but not necessarily $\sup(AB) = \sup A . \sup B$. For which subsets does this formula hold? Find a formula which is valid in the general case.

Exercise Let a and b be positive numbers. Show that $\sqrt{2}$ lies between $\dfrac{a}{b}$ and $\dfrac{a+2b}{a+b}$. (Which of the two numbers is closer to $\sqrt{2}$?)

3 Sequences and limits

3.1 Sequences

Definition 1 *A* **sequence** *of real numbers is a mapping from* $\mathbf{N}$ *into* $\mathbf{R}$ *i.e. we associate to each* $n \in \mathbf{N}$ *a real number* a_n*. We write*

$$(a_n)_{n\in\mathbf{N}} \text{ or } (a_1, a_2, a_3, \dots)$$

for such a sequence.

Examples:

1. the constant sequence $(a, a, \dots)$ i.e. $a_n = a$ for each n;
2. $a_n = \frac{1}{n}$ for each n i.e. the sequence $(1, \frac{1}{2}, \frac{1}{3}, \dots))$;
3. recursively defined sequences. The most famous example is the Fibonacci sequence
$$(1, 1, 2, 3, 5, 8, 13, 21, \dots).$$
This is the sequence (a_n) which is defined by stipulating
$$1) \quad a_1 = a_2 = 1$$
$$2) \quad a_n = a_{n-1} + a_{n-2} \quad (n > 2).$$

3.2 Convergence

Sequence often arise in the practice as successive approximations to the solutions of a problem.

The success of such an approach is documented in the following

Definition 2 *A sequence* (a_n) *of real numbers converges to a (symbolically* $\lim_{n\to\infty} a_n = a$ *or* $a_n \to a$*), if for each* $\epsilon > 0$ *there exists* $N = N(\epsilon)$*, so that* $|a_n - a| < \epsilon$*, whenever* $n \geq N$*.*

Examples: The constant sequence $(a, a, \dots)$ converges to a. The sequence $\left(\frac{1}{n}\right)$ converges to 0. The sequence $(-1)^n$ does not converge. We collect some trivial properties of limits in

Proposition 10 *1. The limit of a sequence is unique i.e.* $a_n \to a$ *and* $a_n \to b$ *implies* $a = b$*;*

2. *The limit is additive i.e.* $a_n \to a$ *and* $b_n \to b$ *implizies* $a_n + b_n \to a + b$;

3. *The limit is multiplicative i.e.* $a_n \to a$ *and* $b_n \to b$ *implies* $a_n.b_n \to a.b$.

4. *If a sequence* (a_n) *of non-zero real numbers converges to* $a \neq 0$, *then* $\lim \frac{1}{a_n} = \frac{1}{a}$.

As a consequence of the order completeness every sequence of real numbers which should converge does in fact converge. More precisely

Definition 3 *A sequence* (a_n) *is* **Cauchy**, *if for each* $\epsilon > 0$ *there exists an* $N \in \mathbf{N}$, *so that*

$$|a_n - a_m| < \epsilon \text{ whenever } n, m \geq N.$$

It is clear that every convergent sequence is Cauchy.
PROOF. Put $\lim a_n = a$. Choose $N \in \mathbf{N}$, so that $|a_n - a| < \frac{\epsilon}{2}$, whenever $n \geq N$. Then for $m, n \geq N$, we have

$$|a_m - a_n| = |(a_m - a) - (a_n - a)| \leq |a_m - a| + |a_n - a| \leq \epsilon.$$

■

Example: consider the infinite decimal expansion

$$N, a_1 a_2 ..... = N + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots \qquad (0 \leq a_i \leq 9).$$

Then the approximands

$$A_n := N + \sum_{k=1}^{n} \frac{a_k}{10^k},$$

form a Cauchy sequence.

The main reason why one does analysis in **R** and not in **Q** is the so-called **completeness** of **R**:

Proposition 11 *Every Caucy sequence in* **R** *converges.*

We shall prove this result shortly.

We now extend the concept of convergence in order to include the notion of convergence to infinity.

Definition 4 *A sequence* (a_n) **converges to** ∞ *(in symbols,* $a_n \to \infty$ *or* $\lim_{n\to\infty} a_n = \infty$*), if*

$$\text{for each } K > 0 \text{ there exists } N \in \mathbf{N}, \text{ so that } a_n \geq K \text{ if } n \geq N.$$

$a_n \to -\infty$ *is defined similarly.*

Examples: For the sequence (x^n) we have: if $|x| < 1$, then the sequence converges to 0. If $x = 1$, then it converges to 1. If $x = -1$ then the sequence does not converge. If $x > 1$, then it converges to ∞. If $x < -1$, the sequence again fails to converge. With respect to convergence, the behaviour of **monotone** sequences is particularly simple:

Definition 5 *a sequence (a_n) is*

1. **increasing**, *if $a_n \leq a_{n+1}$ for each $n \in \mathbf{N}$;*
2. **strictly increasing**, *if $a_n < a_{n+1}$ for each $n \in \mathbf{N}$;*
3. **decreasing**, *if $a_n \geq a_{n+1}$ for each $n \in \mathbf{N}$;*
4. **strictly decreasing**, *if $a_n > a_{n+1}$ for each $n \in \mathbf{N}$.*

Definition 6 *A sequence (a_n) is*

1. **bounded**, *if there exists $K > 0$, so that if $n \in \mathbf{N}$, then $|a_n| < K$;*
2. **bounded above**, *if there is a $K > 0$ so that whenever $n \in \mathbf{N}$, then $a_n < K$;*

Proposition 12 *Let (a_n) be an increasing sequence. If (a_n) is bounded from above, then it converges to $\sup\{a_n\}$. If (a_n) is not bounded from above, then it converges to infinity.*

Proof. We show that if (a_n) is increasing and bounded from above, then $a_n \to a = \sup\{a_n\}$. For take $\epsilon > 0$. There exists $N \in \mathbf{N}$ with $a_N > a - \epsilon$. Then we have, for $n \geq N$,

$$a - \epsilon \leq A_N \leq a_n \leq a < a + \epsilon$$

i.e. $|a_n - a| < \epsilon$.

■

Examples: I. ***b*-adic representations:** Let b be a natural number which is ≥ 2. A b-adic fraction is a limit of the form $\lim A_n$, where

$$A_n = N + \sum_{k=1}^{n} a_k b^{-k}$$

Here (a_k) is a sequence of natural numbers, so that $0 \leq a_k \leq b - 1$.

It is clear that (A_n) is Cauchy. By the completeness, if converges to a real number x. Conversely,

Proposition 13 *Each real number x can be represented as a b-adic fraction.*

The most important cases are

1. $b = 10$ —the decimal expansion
2. $b = 2$—the dyadic expansion:
3. $b = 60$—sexagesimal expansion:
4. $b = 12$—duodecimal expansion.

Example: (Algorithm for calculating square roots).

Let a be a given positive real number. We choose an initial value $x_0 > 0$ and define a sequence (x_n) recursively as follows:

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}).$$

Proposition 14 *(x_n) is decreasing, the sequence (y_n) (whereby $y_n = \dfrac{a}{x_n}$) increasing and $0 < y_n \leq x_n$ $(n \in \mathbf{N})$.*

PROOF. We prove the following statements by induction: 1) $x_n > 0$ for each n (trivial). 2) $x_n^2 - a \geq 0$ for each $n \geq 1$. For

$$\begin{aligned} x_n^2 - a &= \tfrac{1}{4}(x_{n-1} + \tfrac{a}{x_{n-1}})^2 - a \\ &= \tfrac{1}{4}x_{n-1}^2 + \tfrac{a}{2} + \tfrac{1}{4}\tfrac{a^2}{x_{n-1}^2} - a \\ &= \tfrac{1}{4}(x_{n-1} - \tfrac{a}{x_{n-1}})^2 \geq 0. \end{aligned}$$

3) $y_n^2 - a \leq 0$ $(n \geq 1)$. For $x_n^2 \geq a$ implies that $\dfrac{1}{x_n^2} \leq \dfrac{1}{a}$. Hence

$$y_n^2 = \left(\frac{a}{x_n}\right) \leq a.$$

4) $x_{n+1} \leq x_n$. For

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) = \frac{1}{2x_n}\left(x_n^2 - a\right) \geq 0.$$

5) $y_{n+1} \geq y_n$. This follows from the definition and 4). 6) $x_n \geq y_n$. For if $x_n < y_n$, then we would have $x_n^2 < y_n^2$ and this contradicts 2) and 3).

We now know that the sequence (x_n) is increasing and bounded from above. Hence it converges. Let L be the limit. Taking the limit in the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

we get $L = \frac{1}{2}(L + \frac{a}{L})$ i.e. $L^2 = a$.

Thus we have proved

■

Proposition 15 *The limit L of the sequence (x_n) satisfies the condition $L^2 = a$.*

We then say that L is a (the) **square root** of a.

Example As a further application of this method, we note the fact that

$$\lim \left(1 + \frac{1}{n}\right)^n$$

exists. This follows from the facts that the sequence is increasing (proof by induction) since it is obviously bounded (for example by 3). The limit is, by definition, the Euler number e which will be discussed below.

Definition 7 *Let (a_n) be a bounded sequence of real numbers. Then we define*

$$\begin{aligned}\limsup_{n\to\infty}(a_n) &:= \lim_{k\to\infty} \sup(\{a_k, a_{k+1}, \dots\}) \\ \liminf_{n\to\infty}(a_n) &:= \lim_{k\to\infty} \inf(\{a_k, a_{k+1}, \dots\}).\end{aligned}$$

The existence of $\limsup a_n$ and $\liminf a_n$ is a consequence of the order completeness of **R**. It is clear that the following properties hold:

Proposition 16 *1.* $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$*;*

2. $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ *if and only if* $\lim a_n$ *exists. The limit is then the common value of* $\liminf$ *and* $\limsup$*.*

Now it is clear that if (a_n) is Cauchy, then

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

This proves our statement about the convergence of Cauchy sequences in **R**.

We now bring an application of completeness— the method of nested intervals.

Proposition 17 *Let I_n be a decreasing sequence of closed, bounded intervals. Then their intersection $\bigcap_{n=1}^{\infty} I_n$ is non-empty. Further if $\lim_{n\to\infty} diam\, I_n = 0$, then there is precisely one point in the intersection. (If I is an interval, thendiam I is the length of I).*

PROOF. Put $I_n = [a_n, b_n]$. By the assumptions (a_n) is increasing and (b_n) is decreasing. Hence gilt $[a, b] \subset \bigcap I_n$, where $a = \lim a_n$, $b = \lim b_n$. The second part is simple.

∎

As an application, we bring a proof of the fact that the set $[0, 1]$ is uncountable. (compare the proof in the appendix). We first introduce some definition:

Definition 8 *A set A is* **countable***, if there is a surjective mapping from* **N** *onto A i.e. A is the range $\{a_n\}$ of a sequence (a_n) (such a mapping is called a numeration of A). Otherwise A is uncountable.*

Examples: Evey finite set A is countable. **N** is countable. **Z** and **Q** are countable. If (A_n) is a sequence of countable sets, then their union $\bigcup_{n\in\mathbf{N}} A_n$ is also countable.

Using the diagonal method, Canto showed that the real numbers are not countable (see appendix). We now prove the same result using nested intervals.

Once again this is proved by contradiction. We suppose that $[0, 1]$ is countable i.e. we have a numeration $x_1, x_2, \ldots$. We construct a nested sequence (I_n) of intervals as follows. We choose some non-degenerate closed interval I_1 which does not contain x_1. Then we choose a second one $I_2 \subset I_1$, which does not contain x_2. Continuing in the obvious way, we obtain a nested sequence (I_n), with $x_n \notin I_n$. Now the intersection is non-empty. However, this intersection does not contain any x_n and this leads to a contradiction.

3.3 Compactness

Definition 9 *Let (a_n) be a sequence. A* **subsequence** *of (a_n) is one of the form*

$$(a_{n_0}, a_{n_1}, a_{n_2}, \ldots),$$

whereby

$$n_0 < n_1 < n_2 < \ldots$$

If (a_n) converges, then so does each subsequence.

Proposition 18 *(Proposition of Bolzano-Weierstraß) Every bounded sequence* (a_n) *has a convergent subsequence.*

PROOF. We can assume that the sequence lies in the interval $[0,1]$. We consider now the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Put

$$A_1 = \{n : a_n \in [0, \frac{1}{2}]\} \text{ resp. } A_2 = \{n : a_n \in [\frac{1}{2}, 1]\}.$$

Since $\mathbf{N} = A_1 \cup A_2$, either A_1 or A_2 is infinite. Hence we obtain a subsequence, which we denote by $(a_1^1, a_2^1, \dots)$ so that its elements lie in a subinterval of length $\frac{1}{2}$.

We repeat this method and obtain a sequence $(a_k^n)_{k=1}^\infty$ of sequences, so that

1. for each n $(a_k^{n+1})_k$ is a subsequence of $(a_k^n)_k$;
2. $|a_r^n - a_s^n| \leq 2^{-n}$ for $r, s \in \mathbf{N}$.

Consider now the **diagonal sequence** (a_n^n) obtained by going down the diagonal of the array of subsequences:

$$\mathbf{a_1^1} \quad a_2^1 \quad a_3^1 \quad a_4^1 \dots$$

$$a_1^2 \quad \mathbf{a_2^2} \quad a_3^1 \quad a_4^1 \dots$$

and

$$a_1^3 \quad a_2^3 \quad \mathbf{a_3^3} \quad a_4^3 \dots$$

etc. This is

1. a subsequence of (a_n);
2. Cauchy and so convergent.

∎

Remark: This method is called the **diagonal method**. Variants of it are used frequently in mathematics (cf. the proof (due to Cantor) of the uncountability of the real line).

Definition 10 *A real number* a *is* **a cluster point** *of a sequence* (a_n)*, if a subsequence* (a_{n_k}) *exists which converges to* a*.*

The theorem of Bolzano and Weierstraß states that each bounded sequence x_n has a cluster point. In fact, the reader can check that both $\liminf x_n$ and $\limsup x_n$ are cluster points. **Example:** The sequence $(-1)^n$ is not convergent and has two cluster points 1 and -1.

3.4 Exercises

Exercise Is the following statement true?

$$\frac{1}{n^2} + \frac{1}{n} \to \frac{1}{n}.$$

Exercise Calculate

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{n-m}{n+m}$$

and

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{n-m}{n+m}.$$

Exercise Calculate $\lim_{n\to\infty}$ for the following sequences:

$$\frac{n^2-2n+1}{n^2-6}, \quad \frac{6(-1)^n n+11}{n^2-5}, \quad \frac{3n^2-20n}{n+1}.$$

Exercise Show that if $a_n \to a$, $b_n \to b$, then $|a_n| \to |a|$ and $\max\{a_n, b_n\} \to \max\{a, b\}$.

Exercise Let p be a non-constant polynomial. Show that

$$\lim \frac{p(n+1)}{p(n)} = 1.$$

Exercise Let (a_n) be a sequence which converges to a. Show that

$$\frac{1}{n}(a_1 + \cdots + a_n) \to a.$$

Exercise Calculate the following limits:

$$\lim \frac{x^n}{n!} \quad \lim a^{1/n} \quad \lim n^{k/n}.$$

(x is a real number a is positive and $k \in \mathbf{N}$).

Exercise Calculate $\lim(1+\frac{a}{n})^n$ $(a \geq 0)$.

Exercise Let (a_n) be a sequence which converges to a. Show that

$$\frac{na_1 + (n-1)a_2 + \cdots + a_n}{\frac{1}{2}n(n+1)} \to a.$$

Exercise Let (a_n) be a sequence such that

$$a_n < \frac{1}{2}(a_{n-1} + a_{n+1}) \quad (n > 1).$$

Show that (a_n) converges (possibly to $-\infty$ or ∞).

Exercise Let (a_n) be a sequence so that $a_{n+1} = \dfrac{3a_n + 1}{a_n + 3}$ and $a_1 > -1$. Show that $a_n \to 1$.

Exercise Let (a_n) be such that

$$a_{n+1}^2 = a_n + 6 \quad (a_{n+1} \geq 0).$$

Show that if $a_1 \geq -6$, then $a_n \to 3$.

Exercise Let a and b be real numbers. Investigate whether the sequence

$$a_n = \frac{an^4 + 13n^2}{bn^4 + 4n^2 + 1}$$

converges or diverges.

Exercise Let a and b be real numbers. The sequence (a_n) is defined recursively as follows:

$$a_1 = a, a_2 = b, a_k = \frac{1}{2}(a_{k-1} + a_{k-2}).$$

Calculate the limt.

Exercise Calculate the limit of the partial sums of the series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

Exercise Calculate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$$

i.e. the limit of the sequence

$$p_k = \prod_{n=2}^{k} \frac{n^3 - 1}{n^3 + 1}.$$

Exercise Let a and x_0 be positive numbers and (x_n) be defined recursivley as follows:

$$x_{n+1} = \frac{1}{k}((k-1)x_n + \frac{a}{x_n^{k-1}}).$$

Then (x_n) is Cauchy. The limit is a positive number b so that $b^k = a$.

Exercise Show that a sequence of real numbers is convergent if and only if if is bounded and has exactly one cluster point.

4 Limits and continuity for functions

4.1 Notation

We now consider function between suitable susbets of the real line. We shall often require the following special sets: $[a,b] = \{x \in \mathbf{R} : a \leq x \leq b\}$, $[a,b[= \{x \in \mathbf{R} : a \leq x < b\}$, $(a,b] = \{x \in \mathbf{R} : a < x \leq b\}$, $]a,b[= \{x \in \mathbf{R} : a < x < b\}$, $[a,\infty[= \{x \in \mathbf{R} : a \leq x\}$, $]a,\infty[= \{x \in \mathbf{R} : a < x\}$, $]-\infty,a] = \{x \in \mathbf{R} : x \leq a\}$, $]-\infty,a[= \{x \in \mathbf{R} : x < a\}$. We write further

$$\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}.$$

We consider real-valued functions, more exactly functions defined on subsets D of $\mathbf{R}$ with values in $\mathbf{R}$. D is **the domain of definition** of f and the **graph** of f is the set

$$\Gamma_f = \{(x,y) \in D \times \mathbf{R} : y = f(x)\}.$$

Some examples of functions are
I. The constant functions. These are functions of the form $x \mapsto a$ for a fixed $a \in \mathbf{R}$.
II. The identical function. We write id_D for the function $x \mapsto x$ on D.
III. Abolute value. This is the mapping $x \mapsto |x|$.
IV. entier: This function maps x onto the largest whole number which is smaller than or equal to x.
V. Polynomials. These are functions of the form

$$x \mapsto a_0 + a_1 x + \cdots + a_n x^n.$$

VI. Rational functions: These are functions which are quotients $\frac{p}{q}$ of polynomials. The domain of definition is $\{x \in \mathbf{R} : q(x) \neq 0\}$).
VII. Step functions: Functions of the form: $\sum_{i=1}^n a_i \chi_{I_i}$, where $I_1, \ldots I_n$ are intervals and $a_1, \ldots, a_n$ are real numbers. The most famous example is the **Heaviside function** $\chi_{[0,\infty[}$.
VIII. The exponential function and the trigonometric functions: These are defined by the the power series:

$$\begin{aligned} \exp(x) &= \sum_{n=0}^\infty \frac{x^n}{n!} \\ \sin x &= \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{(2k)!} \\ \cos x &= \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

power series will be studied in detail below).

Rational operations on functions: If f, g are functions on D, then we define $f+g$, cf $(c \in \mathbf{R})$, fg and f/g in the natural way. For example $f+g$ is the function

$$x \mapsto f(x)+g(x).$$

($\dfrac{f}{g}$ is defined on the set $\{x \in D : g(x) \neq 0\}$).

Composition of functions: If $f : D \to \mathbf{R}$ and $g : E \to \mathbf{R}$ are functions with $f(D) \subset E$, then the function

$$g \circ f : D \to \mathbf{R}$$

is defined by $(g \circ f)(x) = g(f(x))$ for $x \in D$.

Examples: We can construct the rational functions from the constants and the identity by means of repeated applications of the elementary arithmetical operations. Other functions which can be obtained from simpler ones with these methods are

$$\begin{aligned}
\cosh x &= \tfrac{1}{2}(\exp(x)+\exp(-x)) \\
\sinh(x) &= \tfrac{1}{2}(\exp(x)-\exp(-x)) \\
\tan(x) &= \tfrac{\sin x}{\cos x} \\
\operatorname{cotan}(x) &= \tfrac{\cos x}{\sin x} \\
\sec(x) &= \tfrac{1}{\cos x} \\
\operatorname{cosec}(x) &= \tfrac{1}{\sin x}.
\end{aligned}$$

4.2 Limits:

Let $f : D \to \mathbf{R}$ be a function and $a \in \mathbf{R}$ such that a is a cluster point of $D \setminus \{a\}$ i.e. there exists a sequence (a_n) in D, which converges to a.

We write

$$\lim_{x \to a} f(x) = c,$$

if for each sequence (a_n) in $D \setminus \{a\}$ with $a_n \to a$ we have

$$\lim_{n \to \infty} f(a_n) = c.$$

Example: 1).
$$\lim_{x\to 0} \exp x = 1.$$
For if $1 > x > 0$, then we have the estimate:

$$\begin{aligned}\exp(x) - 1 \qquad &= x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \dots \\ &\leq x + x^2 + x^3 + \cdots = \tfrac{x}{1-x}.\end{aligned}$$

If $x < 0$ we use the identity $\exp(-x) = (\exp(x))^{-1}$.
2). In the case of the Heaviside function $\lim_{x\to 0} H(x)$ fails to exist.

One-sided limits: The last example is the motivation for the following definition:

Definition 11 *If $f : D \to \mathbf{R}$ and a is as above we put $\lim_{x\to a^+} f(x) = c$, whenever c the limit of the restriction of f to $D \cap [a, \infty[$. $\lim_{x\to a^-} f(x)$ is defined correspondingly. We then have*

$$\lim_{x\to 0^-} H(x) = 0 \text{ and } \lim_{x\to 0^+} H(x) = 1.$$

4.3 Continuity:

Definition 12 *Let $f : D \to \mathbf{R}$ be a function, a a point in D. f is* **continuous at** *a, if $f(x_n) \to f(a)$ for each sequence (x_n) in D with $x_n \to a$.*

f is **continuous on** *D, if f is continuous at each point of D.*

Left and right continuity are defined in a similar fashion.

Example: 1) It is clear that each constant function is continuous;
2) The Heaviside function is continuous on the right but not on the left at the origin.
3) The exponential function is continuous on the real line. For

$$\lim_{h\to 0} \exp(x+h) - \exp(x) = \exp(x) \lim_{h\to 0}(\exp h - 1) = \exp(x).$$

(Here we use the functional equation $\exp(x + y) = \exp x. \exp y$ which will be proved later).

Proposition 19 *Let $f, g : D \to \mathbf{R}$ be functions which are continuous at $a \in D$ and suppose that $c \in \mathbf{R}$. Then the functions $f + g, cf, fg$ are also continuous at a. If g has no zeroes, then f/g is continuous.*

Since the identity and constants are continuous, ti follows from this result that raional functions (and hence also polynomials) are continuous. Further examples of functions whose continuity can be deduced imeediately are the hyperbolic functions sinh, cosh.

Proposition 20 *If* $f : D \to \mathbf{R}$ *and* $g : E \to \mathbf{R}$ *are continuous and* $f(D) \subset E$*, then* $g \circ f$ *is also continuous.*

PROOF. Exercise.

■

Proposition 21 *Every continuous function* $f : [a, b] \to \mathbf{R}$ *is bounded i.e. the set* $\{f(x) : x \in [a, b]\}$ *is bounded (there exists a constant* M*, so that* $x \in [a, b]$ $|f(x)| \leq M$ $(x \in [a, b])$*).*

PROOF. Proof by contradiction. Suppose that the continuous function f is not bounded from above. Then there exists for each n and point x_n with $f(x_n) > n$. According to the result of Bolzano and Weierstraß, the sequence (x_n) has a convergent subsequence (x_{n_k}). ut $x = \lim x_{n_k}$. By the continuity of f and the fact that the sequence (x_{n_k}) converges, we see that the image sequence $(f(x_{n_k}))$ is also convergent. This is obviously inconsistent with that fact that the latter sequence is not bounded.

■

Example: The function $f : x \mapsto \dfrac{1}{x}$ on $]0, 1]$ is continuous, but not bounded.

Proposition 22 *Each function which is continuous on a closed bounded interval i.e.*

$$f : [a, b] \to \mathbf{R}$$

attains its maximum and minimum i.e. there exist points x_0 *and* x_1*, so that*

$$f(x_0) = \sup\{f(x) : x \in [a, b]\}$$

and

$$f(x_1) = \inf\{f(x) : x \in [a, b]\}.$$

PROOF. Choose a sequence (x_n) so that

$$f(x_n) \geq \sup\{f(x) : x \in [a, b]\} - \frac{1}{n}.$$

The limit x_0 of a convergent subsequence satisfies the condition.

■

Once again, this proposition fails for function which are defined on open intervals or on non-bounded ones.

Proposition 23 *Intermediate value theorem Let* $f : [a, b] \to \mathbf{R}$ *be a continuous function with* $f(a) < y$, $f(b) > y$. *Then there exists an* $x_0 \in]a, b[$ *with* $f(x_0) = y$.

PROOF. Put $x_0 = \sup\{x \in [a, b] : f(x) < y\}$.

■

In many situations it is convenient to use a variant of the definition of continuity. This is the $\epsilon - \delta$ definition:

Definition 13 *Let* $f : D \to \mathbf{R}$ *and let* $x_0 \in D$. f *is continuous at* x_0 *if and only if for each* $\epsilon > 0$ *there exists a* $\delta > 0$ *so that*

$$|f(x) - f(x_0)| < \epsilon \text{ for each } x \in D \text{ with } |x - x_0| < \delta.$$

This is equivalent to the original definition.
PROOF. Exercise.

■

There are two useful strengthenings of the concept of continuity.

Definition 14 $f : D \to \mathbf{R}$ *is* **uniformly continuous**, *if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *so that for each* x *and* y *in* D *with* $|x - y| < \delta$, *we have* $|f(x) - f(y)| < \epsilon$. $f : D \to \mathbf{R}$ *is* **Lipschitz continuous**, *if there is* $K > 0$ *so that*

$$|f(x) - f(y)| \le K|x - y|$$

for each pair x, y *in* D.

It is clear that Lipschitz continuity implies uniform continuity. while each uniformly continuous function is continuous. The converse is not true in general.

Example: The function $x \mapsto x^2$ on $\mathbf{R}$ is continuous, but not uniformly continuous. The function $x \mapsto \sqrt{x}$ on $[0, 1]$ is uniformly continuous, but not Lipschitz continuous. (The latter function will be defined rigorously below). However, we do have:

Proposition 24 *Every continuous function on a closed, bounded interval is uniformly continuous.*

Proof. Once again, we prove this by contradiction, using the result of Bolzano and Weierstraß. Suppose that the continuous function f is not uniformly continuous. Then there is a positive ϵ for which the defining condition fails. Hence we can find for each n points x_n and y_n with $|x_n - y_n| \leq \frac{1}{n}$, but $|f(x_n) - f(y_n)| > \epsilon$. There exist convergent subsequences (x_{n_k}) and (y_{n_k}). (There is a small subtlety involved. Why can we assume that these subsequences are both indexed by the same set?) From the first condition we see that $\lim_k x_{n_k} = \lim_k y_{n_k}$. Let a be the limit. Then $\lim_k f(x_{n_k}) = \lim_k f(y_{n_k}) = f(a)$, and this clearly contradics the second estimate above.

■

Inverse functions: $f : D \to \mathbf{R}$ is

1. **increasing**, falls $x \leq y$ implies $f(x) \leq f(y)$;
2. **strictly increasing**, if $x < y$ implies $f(x) < f(y)$;
3. **decreasing**, if $x \leq y$ implies $f(x) \geq f(y)$;
4. **strictly decreasing**, if $x < y$ implies $f(x) > f(y)$.

If $f : [a, b] \to \mathbf{R}$ is continuous and strictly increasing, then its image is the interval $[A, B]$, where $A = f(a)$, $B = f(b)$ (this is a consequence of the intermediate value theorem). f is then a bijection from $[a, b]$ onto $[A, B]$. Hence we can define the **inverse function** f^{-1} from $[A, B] \to [a, b]$ as follows:

$$x = f^{-1}(y) \iff y = f(x).$$

Proposition 25 *In the above situation, the inverse function is continuous.*

Proof. Once again we a use a proof by contradiction. If the inverse is not continuous, then we could find a sequence (y_n) with the following properties: $y_n \to d$ in $[A, B]$ but $|x_n - c| > \epsilon$ for a fixed $\epsilon > 0$, whereby $d = f(c)$ (why?). (x_n) is the sequence $x_n = f^{-1}(y_n)$. We can then pass over to a convergent subsequence (x_{n_k}). Let c_1 be the limit of this sequence. Then $c \neq c_1$ but $f(c) = f(c_1)$. This contradicts the injectivity of f.

■

We can now extend our list of special functions:

The root functions: We define a continuous function $x \mapsto x^r$ for each rational number r. x^n is defined for $r = n \in \mathbf{N}$. For $r = \frac{1}{n}$ ($n \in \mathbf{N}$) we define the function $y \mapsto y^{1/n}$ as the inverse of $x \mapsto x^n$ i.e.

$$x = y^{1/n} \iff y = x^n.$$

This function is defined on $\mathbf{R}_+$ and is continuous (dsince $x \mapsto x^n$ is a bijection from $\mathbf{R}_+$ onto $\mathbf{R}_+$). If $r = \frac{p}{q}$ is rational with p and q natural numbers we define $x \mapsto x^r$ to be $x^r = \left(x^{1/q}\right)^p$. For $r < 0$, we put

$$x^r = \frac{1}{x^{-r}}.$$

The logarithm function: The function ln is defined to be the inverse of exp i.e. $x = \ln y \iff y = \exp x$. Since exp is a bijection from $\mathbf{R}$ onto $\mathbf{R}_+$, ln is continuous from $\mathbf{R}_+$ onto $\mathbf{R}$. It follows from the functional equation of the exponential function that $\ln xy = \ln x + \ln y$.

The generalised power functions: We can now define the function $x \mapsto x^\alpha$ for a general $\alpha \in \mathbf{R}$ as follows: $x^\alpha = \exp(\alpha \ln x)$.

Remark: Since $\ln e = 1$, we can now write $\exp(x)$ as e^x.

Further interesting function which can be defined as the inverses of elementary functions are the inverse trigonometric functions (arcsin, arccos etc.). For more detaisl see below. We conclude this section with some extensions of the limit concept:

Definition 15 *Let f be a function whose domain of definition D contains an interval $[K, \infty[$. We write:* $\lim_{x\to\infty} f(x) = a$, *whenever*

for each $\epsilon > 0$ there exists $K_1 > K$, so that $|f(x) - a| < \epsilon$ if $x > K_1$.

The limit as x tends to $-\infty$ is defined analogously.

We write $\lim_{x\to a} f(x) = \infty$, *if*

for every $K > 0$ there exists $\epsilon > 0$ with $f(x) > K$, whenever $0 < |x - a| < \epsilon$.

4.4 Exercises

Exercise Calculate $\lim_{x\to 0} \cos \frac{1}{x}$. (It is understood that in such examples that one first determines whether the limit exists).

Exercise Calculate

$$\lim_{x\to 0} \frac{\sqrt{1+3x^2} - \sqrt{1-3x^2}}{x}.$$

Exercise Calculate

$$\lim_{x\to\infty} \frac{3x^4 - 7x^2 - 1}{x^4 + 7}.$$

Exercise At which points is the function $x \mapsto x$ when $x \in \mathbf{Q}$) and $\mapsto -x$ otherwise continuous?

Exercise On which intervals are the following functions continuous resp. Lipschitz continuous?

$$f : x \mapsto \sqrt{x} \quad (x \in \mathbf{R}_+) \qquad g : x \mapsto \frac{1}{x} \quad (x \in \mathbf{R} \setminus \{0\}).$$

a) $[1, 2]$; b) $]0, 1]$.

Exercise Which of the following statements is true? a) $f : D \to \mathbf{R}$ is Lipschitz continuous and D bounded implies f bounded. b) $f, g : \mathbf{R} \to \mathbf{R}$ Lipschitz continuous implies fg Lipschitz continuous; c) $f, g : \mathbf{R} \to \mathbf{R}$ Lipschitz continuous and bounded implies fg Lipschitz continuous.

Exercise Let f and g be continuous functions on $[0, 1]$, so that $f(x) = g(x)$, whenever x is rational. Show that $f = g$.

Exercise Let p be a polynomial of odd order. Show that p has a zero.

Exercise Let $A \subset \mathbf{R}$ be bounded, $f : \mathbf{R} \to \mathbf{R}$. Does it follow that $f(A)$ is bounded

1. if f is continuous;
2. if f is uniformly continuous;
3. if f is Lipschitz continuous?

Exercise Calculate the limits

$$\lim_{x\to 0} \exp\left(\frac{\sin x}{x}\right) \qquad \lim_{x\to 0} \frac{\tanh x}{x}.$$

Exercise Suppose that the function $f : \mathbf{R} \to \mathbf{R}$ satisfies the condition

$$|f(x) - f(y)| \leq 2(e^{|x-y|} - 1).$$

Show that f is uniformly continuous.

Exercise Suppose that f is continuous on $[a, b]$. Show that the funciton

$$x \mapsto \sup\{f(t) : t \in [a, x]\}$$

is also continuous.

Exercise Let f be continuous and injective on $]0, 1[$. Show that f is monotone i.e. either increasing or decreasing.

Exercise $f : [0, 1] \to [0, 1]$ is continuous. Show that f has a fix point.

Exercise Let f and g be defined on $\mathbf{R}_+$ and satisfy the condition $g(x) = f(x^2)$. Show that $\lim_{x\to\infty} f(x) = A \iff \lim_{x\to\infty} g(x) = A$.

Exercise Let $f(x) = x$ when x is rational and $= x^2$ otherwise. Show that

$$\lim_{x\to 0+} f(x) = 0 \qquad \lim_{x\to 1-} f(x) = 1.$$

For which $c \in]0, 1[$ does $\lim_{x\to c} f(x)$ exist?

Exercise Prove directly that the function $x \mapsto x^2 - 7x + 6$ is continuous on $\mathbf{R}$.

Exercise Let f be bounded on $[0, 1]$ and suppose that

$$f(ax) = bf(x) \quad (0 \le x \le \frac{1}{a}),$$

where $a, b > 1$. Show that f is continuous at 0.

Exercise Let $f : [a, b] \to \mathbf{R}$, $x \in [a, b]$. Put

$$\operatorname{osc}(f; x_0) = \inf_n \sup\{|f(x) - f(y)| : |x - x_0| < \frac{1}{n}, |y - x_0| < \frac{1}{n}\}.$$

Show that f is continuous at $x_0 \iff \operatorname{osc}(f; x_0) = 0$. Define:

$$\operatorname{osc}_u(f) = \inf_n\{\sup |f(x) - f(y)| : |x - y| < \frac{1}{n}\}.$$

What is the meaning of the condition $\operatorname{osc}_u(f) = 0$?

Exercise Let (a_n) be a sequence and a a real number. Define a function $f : D \to \mathbf{R}$ as follows: $D = \{\frac{1}{n} : n \in \mathbf{N}\} \cup \{0\}$, $f(\frac{1}{n}) = a_n$, $f(0) = a$. Show that

$$a_n \to a \iff f \text{ continuous.}$$

Exercise Let $f : [a, b] \to \mathbf{R}$ be increasing. Show that for each $x_0 \in]a, b[$ there exists $f(x_0^+) = \lim_{x \to x_0^+} f(x)$ and $f(x_0^-) = \lim_{x \to x_0^-}$. $f(x_0^+) - f(x_0^-)$ is called the **jump** in f at the point x_0. Prove that there exists at most countable many points x_0, where the jump is non-zero.

Exercise Let $f : [0, 1[\to \mathbf{R}$ be continuous on the right at 0 and such that $f(x^2) = f(x) (x \in [0, 1[)$. Show that f is constant.

Exercise Let $f : \mathbf{R} \to \mathbf{R}$ be such that $f(x + y) = f(x) + f(y)$. Show that if f is continuous at one point, then it is of the form $x \mapsto cx$.

Exercise Show that if $f : [0, 1] \to \mathbf{R}$ is continuous and injective, then it is strictly monotone.

Exercise Let f be a continuous function on $\mathbf{R}$ and (x_n) be a sequence so that $x_{n+1} = f(x_n)$ for each n. Show that if the sequence (x_n) converges, then the limit is a fixed pointof f.

5 Differentiation

5.1 Definitions

Definition 16 *Let $f : D \to \mathbf{R}$. Then f is differentiable at the point x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists on the set $D \setminus \{x_0\}$. The limit is then the **derivative** *of f at x_0, written $f'(x_0)$. If the derivative exists at each point of D, then f is* **differntiable** *and the function*

$$x \mapsto f'(x)$$

is the **derivative** *of f. If this function is continuous, then f is said to be* **continuously differntiable***.*

We then define the n-th derivative recursively. f is n-times differentiable if f' exists and is $(n-1)$-times differentiable. The n-th derivative $f^{(n)}$ of f is then the derivative of $f^{(n-1)}$. If further $f^{(n)}$ is continuous, then f is said to be n-times continuously differentiable.

It is clear that a differentiable function is continuous.

There are various alternative forms of the above defintions which are often useful.

1. there exists a real number a, so that the function

   $$x \mapsto \frac{f(x) - f(x_0)}{x - x_0}$$

   when $x \neq x_0$ and $\mapsto a$ when $x = x_0$

   is continuous on D. (a is then the derivative of f at x_0).

2. there exist a real number a and a function ρ on $D \setminus \{x_0\}$, so that

   $$f(x) = f(x_0) + (x - x_0)a + \rho(x)$$

   and

   $$\lim_{x \to x_0} \frac{\rho(x)}{x - x_0} = 0.$$

   Once again a is the derivative $f'(x_0)$.

3. (Carathéodory) There exists a function ϕ, which is continuous at x_0 so that

   $$f(x) = f(x_0) + (x - x_0)\phi(x).$$

   $a = \phi(x_0)$ is then the derivative of f.

One verifies easily that the familiar laws with respect to the algebraic operations are valid:

1. the sum $f+g$ of two differentiable functions is differentiable and $(f+g)' = f' + g'$;
2. the product $f.g$ of two differentiable functions is differentiable and $(f.g)' = f'.g + f.g'$.
3. the reciprocal of a differentiable function g without zeroes is differentiable and
$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}.$$

The following fact is less trivial

Proposition 26 *(the chain rule) Let f and g be differentiable functions so that the composition $g \circ f$ exists. Then the latter is differentiable and*

$$(g \circ f)' = (g' \circ f).f' \quad \textit{d.h.}\ (g \circ f)'(x) = g'(f(x)).f'(x).$$

PROOF. We consider a point x and put $y = f(x)$. Using the differentiablilty of f and g we can write

$$\begin{aligned} f(x+h) &= f(x) + h.f'(x) + \rho(h) \\ g(y+k) &= g(y) + k.g'(y) + \sigma(k), \end{aligned}$$

with $\lim_{h\to 0} \dfrac{\rho(h)}{h} = 0$ and $\lim \dfrac{\sigma(k)}{k} = 0$. If we put $k(h) = h.f'(x) + \rho(h)$, then we get

$$\begin{aligned} g \circ f(x+h) - g \circ f(x) &= g\left(f(x) + k(h)\right) - g(f(x)) \\ &= k(h).g'(y) + \sigma(k(h)) \\ &= h.g'(y)f'(x) + \rho(h)g'(y) + \sigma(k(h)), \end{aligned}$$

and the remainder term $\tau(h) = \rho(h)g'(y) + \sigma(h.f'(x) + \rho(h))$ satisfies the condition $\lim_{h\to 0} \dfrac{\tau(h)}{h} = 0$.

■

Examples: I. Constant functions: it is clear that the derivative of a constant function is the constant function 0.
II. $f : x \mapsto cx$ on $\mathbf{R}$. In this case, the derivative is the constant function c.
III. For the function $f : x \mapsto x^n$ we have $f'(x) = nx^{n-1}$. (Use the above and the product rule)
IV. Die exponential function: We have

$$\begin{aligned} \exp'(x) &= \lim_{h\to 0} \tfrac{\exp(x+h)-\exp(x)}{h} \\ &= \lim_{h\to 0} \exp(x) \tfrac{\exp(h)-1}{h} \\ &= \exp(x) \lim_{h\to 0} \tfrac{\exp(h)-1}{h} \\ &= \exp(x). \end{aligned}$$

V. The sine function:

$$\begin{aligned} \sin'(x) &= \lim_{h\to 0} \tfrac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \tfrac{2\cos(\frac{2x+h}{2})\sin\frac{h}{2}}{h} \\ &= \left(\lim_{h\to 0} \cos(x+\tfrac{h}{2})\right) . \left(\lim_{h\to 0} \tfrac{\sin\frac{h}{2}}{h/2}\right) \\ &= \cos x. \end{aligned}$$

Similarly, $\cos'(x) = -\sin x$. VI. The absolute value function. This function is not differntiable at 0.

One-sided derivatives: The last function above is a typical example of a function which is not differentiable but has one-sided derivatives.

Definition 17 *Let $x \in D$ and $f : D \to \mathbf{R}$. f is said to be* **differentiable on the right** *at x if*

$$f'_+(x) = \lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}$$

exists. f is said to be **differentiable on the left** *at x, if*

$$f'_-(x) = \lim_{h\to 0-} \frac{f(x+h) - f(x)}{h}$$

exists.

For example $\text{abs}'_+(0) = 1$, $\text{abs}'_-(0) = -1$.

Piecewise continuously differentiable functions: The function $f : [a,b] \to \mathbf{R}$ is called n-times continuously differentiable if there are finitely many points $a_1 < a_2 < \cdots < a_k$ so that

1. f is n-times continuously differentiable on $[a,b] \setminus \{a_1, \dots, a_k\}$;
2. for each $i \leq n$ and $r \leq k$ the one-sided derivatives $f_+^{(i)}(a_r)$ and $f_-^{(i)}(a_r)$ exist.

For example, Step functions have this property.

Proposition 27 *Let D be a closed interval and $f : D \to \mathbf{R}$ a continuous, strictly monotone function. We denote by $\phi = f^{-1} : E \to \mathbf{R}$ the inverse function, where $E = f(D)$. If f is differentiable and $f'(x) \neq 0$ ($x \in$*

$$\phi'(y) = \frac{1}{f'(x)} = \frac{1}{f'(\phi(y))} \qquad (y = \phi(x)).$$

Proof. We fix x_0 and $y_0 = f(x_0)$. Consider the difference quotient

$$\frac{y - y_0}{x - x_0} \text{ and } \frac{x - x_0}{y - y_0},$$

where $y = f(x)$. Since f and its inverse are continuous,

$$x \to x_0 \iff y \to y_0.$$

This gives the required result on passing to the limit as $x \to x_0$ in the equation

$$\frac{x - x_0}{y - y_0} = \left(\frac{y - y_0}{x - x_0} \right)^{-1}.$$

■

Examples: I. Since ln is the inverse of exp we have

$$\ln'(x) = \frac{1}{\exp'(\ln x)} = \frac{1}{\exp(\ln x)} = \frac{1}{x}.$$

II. arcsin arccos, arctan: These are the inverse functions of sin, cos and tan. More precisely we consider the inverses of the functions

$$\sin \text{ with domain of definition } [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\cos \text{ with domain of definition } [0, \pi]$$

and

$$\tan \text{ with domain of definition }]-\frac{\pi}{2}, \frac{\pi}{2}[.$$

Then

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \quad \arccos'(x) = -\frac{1}{\sqrt{1 - x^2}} \quad \arctan'(x) = \frac{1}{1 + x^2}.$$

5.2 Some basic results

Definition 18 *Let $f :]a,b[\to \mathbf{R}$ be a function. f has a* **local minimum** *at $x \in]a,b[$ if there exists $\epsilon > 0$ so that for each y with $|x-y| < \epsilon$ $f(x) \geq f(y)$. A* **local maximum** *is defined similarly. x is a* **lokal extremum**, *if it is either a local maximum or a local minimum.*

Proposition 28 *A function $f : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and differentiable on $]a,b[$. Then we have for each local extremum $x \in]a,b[$ $f'(x) = 0$.*

Proof. Exercise.

■

Proposition 29 *(Rolle's theorem) Let $f : [a,b] \to \mathbf{R}$ be differentiable with $f(a) = f(b)$. Then there exists a point $x_0 \in]a,b[$, so that $f'(x_0) = 0$.*

Proof. If the function is constant then the claim is trivially valid. Otherwise there exists a point x_1, with either $f(x_1) > f(a)$ or $f(x_1) < f(a)$. In the first case we choose x_0 so that $f(x_0) = \sup\{f(x) : x \in [a,b]\}$.

■

Proposition 30 *(Intermediate theorem of differential calculus) Let $f : [a,b] \to \mathbf{R}$ be continuous and differenzierbar on $]a,b[$. Then there exists ξ in $]a,b[$, so that*

$$f(b) - f(a) = f'(\xi)(b-a).$$

Proof. We prove this for the case where $a = 0$, $b = 1$. We apply use Rolle's Proposition to the function

$$f_1(x) = f(x) - ((1-x)f(0) + x(f(1)).$$

■

Corollar 1 *Let f and g be continuous functions on $[a,b]$, both differentiable on the open interval $]a,b[$. Then there is a ξ in $]a,b[$, so that*

$$f'(\xi)(g(b) - g(a)) = g'(\xi)(f(b) - f(a)).$$

Proof. Exercise.

■

Proposition 31 *Let $f : [a,b] \to \mathbf{R}$ be continuous and differentiable on $]a,b[$. If $f' = 0$ on $]a,b[$, then f is constant.*

PROOF. Exercise.

∎

A similar argument shows that if a function f on an interval say is differentiable and its derivative is bounded, then it is Lipschitz continuous. Hence if f is continuously differentiable on a closed, bounded interval, then it is automaticall Lipschitz continuous.

Proposition 32 *Let $f : [a, b] \to \mathbf{R}$ be continuous and differentiable on $]a, b[$. If $f' > 0$ on $]a, b[$, then it is strictly increasing. If $f' < 0$, then f is strictly decreasing.*

PROOF. Exercise.

∎

Proposition 33 *(L'Hospital's rule:) Let f and g be differentiable near a point c and*

1. *$f(c) = g(c) = 0$;*
2. *g and g' have no zeroes in a neighbourhood of c;*
3. *$\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists.*

Then $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ exists and we have

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

PROOF. We prove this for the one-sided limit $\lim_{x\to c+}$. Put $L = \lim_{x\to c+} \dfrac{f'(x)}{g'(x)}$. Choose $\delta > 0$, so that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon,$$

whenever $x \in]c, c+\delta[$. Choose $0 < h < \delta$. There exists $\xi \in]c, c+h[$, so that

$$\frac{f(c+h)}{g(c+h)} \left(= \frac{f(c+h) - f(c)}{g(c+h) - g(c)} \right) = \frac{f'(\xi)}{g'(\xi)}.$$

Then

$$\left| \frac{f(c+h)}{g(c+h)} - L \right| < \epsilon.$$

∎

Proposition 34 *Taylor's theorem Let $f : [a, b] \to \mathbf{R}$ be n-times differentiable and let x_0 be a point in $]a, b[$. Then there exists to each x in $[a, b]$ an ξ between x_0 and x, so that*

$$f(x) = f(x_0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n.$$

Proof. Once again we assume that $x_0 = 0$, $x = 1$. (Otherwise we consider the function

$$g(t) = f(x_0 + t(x - x_0)).)$$

Wir define functions F and G on $[0, 1]$ as follows:

$$F(t) = f(1) - \sum_{k=1}^{n} \frac{f^{(k-1)}(x_0)}{(k-1)!}(1 - t)^{n-1}$$

resp.

$$G(t) = (1 - t)^n.$$

There exists ξ with $\frac{F(1)-F(0)}{G(1)-G(0)} = \frac{F'(\xi)}{G'(\xi)}$. This is the required result.

■

This last result is the motivation for the following definition. Suppose that f is infinitely differentiable near x_0. Then we call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

the **Taylor series** of f at x_0. Note that at this point we make no assumptions about the convergence of this series.

5.3 Exercises

Exercise Calculate the derivate of the functions

$$\sqrt{\exp x^{\cos\sqrt{x}}}, \quad x^{\sqrt{x}}.$$

Exercise Show that

$$x^\alpha y^\beta \leq \alpha x + \beta y,$$

where x, y, α, β are positive and $\alpha + \beta = 1$. (Calculate the extremum of the function $x \mapsto x^\alpha y^\beta - \alpha x$).

Exercise Prove Hölder's inequality

$$\sum_{j=1}^{n} x_j y_j \le \left(\sum_{j=1}^{n} x_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} y_j^q\right)^{\frac{1}{q}},$$

whereby $x_j > 0, y_j > 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.)

Exercise Let f be a continuously differentiable function on $[a, b]$. Show that f is Lipschitz continuous.

Exercise Verify Leibniz' rule:

$$(f.g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Calculate $(x^2 \sin x)^{(9999)}$..

Exercise Determine the Taylor series of $\ln x$ at $x_0 = 1$. For which $x > 0$ does the series converge?

Exercise Consider the function $f(x) = e^{-\frac{1}{x^2}}$ when $x \neq 0$ and $= 0$ when $x = 0$. Show that $\lim_{x\to 0} p(\frac{1}{x}) \exp(\frac{-1}{x^2}) = 0$ for each polynomial p. Consider the Taylor series for f at 0. Is f represented by this series?

Exercise Let f be twice continuously differentiable. Show that

$$\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a).$$

Exercise Show that

$$\lim_{x\to 0} x^{-2}[b(1+ax)^{\frac{1}{3}} - a(1+bx)^{\frac{1}{3}} + (a-b)(1+abx^2)^{\frac{1}{2}}] = \frac{7}{18}ab(a-b).$$

Exercise Let f be twice differentiable with $|f(x)| \le A$, $|f''(x)| \le B$ $(x \ge K)$. Show that $|f'(x)| \le \sqrt{2(AB)}$ $(x \ge K)$.

Exercise Let f be a function on $[a, b]$, so that

$$|f(x) - f(y)| \le A|x-y|^2 \quad (x, y \in [a, b]).$$

Show that f is constant.

Exercise Calculate the extrema of the function

$$x \mapsto x^m(1-x)^n$$

on $[0, 1]$.

Exercise Let f, g, h be continuously differentiable functions on $[a, b]$. Show that there is a point ξ, so that the determinant of

$$\begin{bmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(\xi) & g'(\xi) & h'(\xi) \end{bmatrix}$$

vanishes.

Exercise Use power series representations of sin and cos to show that

$$\sin'(x) = \cos x \qquad \cos'(x) = -\sin x.$$

Exercise Show that the Carathéodory definition of differentiability is equivalent to the original one and use it to give proofs of of the chain rule resp. the result on the differetiability of inverse functions.

6 The Riemann Integral:

6.1 The definitions

Recall that a function $h : [a, b] \to \mathbf{R}$ is a step function if there is a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of the intervall so that h is constant on $]x_{k-1}, x_k[$ for each k.

It is clear that the family of all such function forms a vector space i.e. that the sum $h_1 + h_2$ and a product lh are again step functions whenever h_1, h_2, h are step functions and l is a scalar. If c_k is the value of h on the interval $]x_{k-1}, x_k[$, then the **Integral** $\int_a^b h(x)\,dx$ of h is the sum

$$\sum_{k=1}^{n} c_k(x_k - x_{k-1}).$$

This integral is positive and additive i.e.

$$\begin{aligned} \int_a^b (h_1 + h_2)(x)\,dx &= \textstyle\int_a^b h_1(x)\,dx + \int_a^b h_2(x)\,dx \\ \int_a^b \lambda h(x)\,dx &= \textstyle\lambda \int_a^b h(x)\,dx. \end{aligned}$$

Further

$$h \leq h_1 \text{ impliziert } \int_a^b h(x)\,dx \leq \int_a^b h_1(x)\,dx.$$

One attampts to define resp. calculate the integral of a more general function via approximation from above and below by step functions.

Definition 19 *Let $f : [a, b] \to \mathbf{R}$ be an arbirtary bounded function. We put $\int_a^{b*} f(x)\,dx$ equal to*

$$\inf\{\int_a^b h(x)\,dx : h \text{ a step function with } f \leq h\}$$

and $\int_{a*}^{b} f(x)\,dx$ equal to

$$\sup\{\int_a^b h(x)\,dx : h \text{ a step function with } f \geq h\}.$$

Examples: If f is a step function, then the upper and lower integrals of f concide. If f is the function $x \mapsto 0$ (x rational) and $\mapsto 1$ (x irrational) then the lower integral is 0 and the upper one is 1.

A bounded function is defined to be **Riemann-integrable**, if

$$\int_a^{b*} f(x)\,dx = \int_{a\ *}^{b} f(x)\,dx.$$

The common value is then **the integral** of f—written

$$\int_a^b f(x)\,dx.$$

We can reformulate the definition as follows:

Proposition 35 *A bounded function $f : [a,b] \to \mathbf{R}$ is integrable if and only if for each $\epsilon > 0$ there are step functions h_1 and h_2 with*

$$h_1 \leq f \leq h_2 \quad \text{and} \quad \int_a^b h_2(x)\,dx - \int_a^b h_1(x)\,dx \leq \epsilon.$$

6.2 Continuous functions

Proposition 36 *Every continuous function is integrable.*

Proof. Let

$$a = t_0 < t_1 < \cdots < t_n = b$$

be a partition of $[a,b]$. For $k = 1, \ldots, n$ put

$$m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} \qquad M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\}.$$

We define the **Riemann-Sum**:

$$\begin{aligned} s(f) &= \textstyle\sum_{i=1}^n m_i(t_i - t_{i-1}) \\ S(f) &= \textstyle\sum_{i=1}^n M_i(t_i - t_{i-1}). \end{aligned}$$

It is clear that f is Riemann integrable if one can find, for each ϵ a partition so that $|S(f) - s(f)| < \epsilon$ for this partition.

If f is continuous, then there exists for each ϵ a δ, so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \dfrac{\epsilon}{b-a}$. We thus choose a partition so that $|t_i - t_{i-1}| < \delta$ for each i. Then $|S(f) - s(f)| < \epsilon$.

■

In addition we have

Proposition 37 *Every monotone function $f : [a, b] \to \mathbf{R}$ is integrable.*

PROOF. Exercise.

■

Once again, it is clear that $f + g$ and λf are integrable, if this is the case for f and g. Further, the integral is monotone i.e. if $f \le g$ for integrable functions f and g, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

6.3 Basic results

Proposition 38 *(Intermediate value theorem of integral calculus) Let $f, h : [a, b] \to \mathbf{R}$ be continuous functions, whereby $h \ge 0$. Then there exists and $\xi \in [a, b]$, so that*

$$\int_a^b f(x)h(x)\,dx = f(\xi)\int_a^b h(x)\,dx.$$

PROOF. We put m for the infimum and M for the supremum of f on the intervall. Then

$$mh(x) \le f(x)h(x) \le Mh(x)$$

and so

$$m\int_a^b h(x)\,dx \le \int_a^b f(x)h(x)\,dx \le M\int_a^b h(x)\,dx.$$

Hence there is $\eta \in [m, M]$, so that

$$\int_a^b f(x)h(x)\,dx = \eta\int_a^b h(x)\,dx.$$

It follows from the intermediate value theorem that $\eta = f(\xi)$ for an $\xi \in [a, b]$.

■

Now let I be an interval (which is either open, half-open or closed), which contains a. If $f : I \to \mathbf{R}$, then the function

$$F : x \mapsto \int_a^x f(t)\,dt$$

is a **primitive** of f. (N.B. If $b < a$, then we define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$). The origin of this name lies in the following fact:

Proposition 39 *F is continuously differentiable and we have $F' = f$.*

PROOF. Let x_0 be a point of $[a,b]$. We calculate

$$\lim_{h\to 0}\frac{1}{h}(F(x_0+h)-F(x_0)) = \lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h} f(x)\,dx.$$

By the intermediate value theorem, we have

$$\lim_{h\to 0}\frac{1}{h}f(\xi_h).h,$$

where ξ_h lies between x_0 and x_0+h. It is clear that the limit is $f(x_0)$.

■

In general we call any function F with the property that $F' = f$ a primitive f. In fact, we have that if F_1 and F_2 are primitives of f then $F_1 - F_2$ is constant. In other words, $\int_a^x f(t)\,dt$ is up to a constant the only primitive of f. We sometimes write $\int f$ for a primitive of f.

Proposition 40 *(The fundamental theorem of differential and integral calculus) Let $f : I \to \mathbf{R}$ be a continuous function with primitive F. Then we have for each $a, b \in I$,*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF. Exercise.

■

Thus we have the following method for calculating $\int_a^b f(x)\,dx$. Find a function F with f as derivative. Then the value of the integral is $F(x)\big|_a^b (= F(b) - F(a))$.

Proposition 41 *(Integration by parts) Let f and g be continuous functions with primitives F and G. Then $FG - \int fG$ is a primitive for Fg. Hence*

$$\int_a^x F(t)g(t)\,dt = F(x)G(x) - \int_a^x f(t)G(t)\,dt.$$

Proposition 42 *(The substitution rule) Let $\phi : [a,b] \to \mathbf{R}$ be continuously differentiable and $f : I \to \mathbf{R}$ continuous with $\phi([a,b]) \subset I$. Then*

$$\int_a^b f(\phi(t))\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

PROOF. This follows from the chain rule. For if F is a primitive for f, then $(F \circ \phi).\phi'$ is a primitive for $f \circ \phi$.

■

6.4 Exercises

Exercise Calculate the following integrals

$$\int_a^b x^\alpha \, dx, \quad \int_a^b \frac{1}{x} \, dx, \quad \int_a^b \sin x \, dx.$$

Exercise Calculate primitives for

$$\cos x, \quad \exp x, \quad \frac{1}{\sqrt{(1-x^2)}}, \quad \frac{1}{1+x^2}.$$

Exercise Calculate primitives for

$$x^\alpha, \quad e^{\lambda x}, \quad \sin(\lambda x), \quad \sinh(\lambda x), \quad \int \frac{dx}{x}.$$

Exercise Calculate the indefinite integrals

$$\int \frac{f'(x)\, dx}{f(x)}, \quad \int \frac{dx}{ax+b}, \quad \int \tan x \, dx.$$

Exercise Calculate

$$\int \sec(x) \, dx, \quad \int \frac{dx}{\sqrt{(1-x^2)}}, \quad \int \frac{dx}{1+x^2}.$$

Exercise Show how to calculate a primitive for functions of the form $\frac{1}{x+a}$ bzw. $\frac{1}{x^2+ax+b}$ Using this and the representation of a general rational function as a linear combination of such functions, skethc how one can in principle calculate a primitive for such a function.

Exercise Calculate primitives for

$$\cos \alpha x \sin \underline{x}, \quad \sqrt{a^2 - x^2}, \quad \frac{1}{\sqrt{(x-\alpha)(\underline{-}x)}}.$$

Exercise Calculate by means of integraion by parts

$$\int x^2 e^{ax}\,dx, \quad \int e^{ax}\,dx, \quad \int e^{ax}\sin bx\,dx, \quad \int \sin^m x \cos^n x\,dx.$$

Exercise A function $f : [a, b] \to \mathbf{R}$ is of bounded variation, if there exists $K > 0$ so that

$$\sum_{k=0}^{n-1} |f(x_{i+1}) - f(x_i)| \leq K$$

for each partition $a = x_0 < x_1 \cdots < x_n = b$. The smallest such K K is called the **variation** of f (written $\mathrm{Var}\,(f)$). Show

1. each monotone function is of bounded variation;
2. if f is of bounded variation, then the function

$$x \mapsto \mathrm{Var}\,(f|_{[a,x]})$$

   is increasing;
3. each function of bounded variation is the difference of two increasing functions;
4. there exist non-cintuous functions of bounded variation;
5. if f is continuous and of bounded variation, then the function

$$x \mapsto \mathrm{Var}\,(f|_{[a,x]})$$

   is continuous.

6.5 Criteria for the convergence of series and improper integrals:

We now consider convergence properties for infinite series $\sum_{n=1}^{\infty} a_n$, whereby (a_n) is a sequence of real numbers. The series **converges** to s (once also says that s is the sum of the series) if $s_n \to s$, where s_n is the n-th partial sum $\sum_{k=1}^{n} a_k$.

We begin by recasting the Cauchy criterium in this context:

Proposition 43 *$\sum_{n=1}^{\infty} a_n$ converges if and only if for each $\epsilon > 0$ we can find $N \in \mathbf{N}$, so that*

$$\left| \sum_{k=m}^{n} a_k \right| < \epsilon \text{ four all } m \geq n \geq N.$$

If the series is not convergent, we say it diverges. In the case where the a_n are non-negative, the sequence of partial sums is increasing. Hence in this case there are only two possibilities:

1. either the partial sums are bounded in which case the series converges
2. or the sum are unbounded and the series diverges. In this case one somtimes writes $\sum_n a_n = \infty$).

It is clear that the series $\sum_{n=1}^{\infty} a_n$ can only converge when $a_n \to 0$. For

$$a_n = s_{n+1} - s_n \to s - s = 0.$$

However, there are divergent series $\sum_{n=1}^{\infty} a_n$, with $a_n \to 0$ as the following example shows.

Example: $\sum \frac{1}{n}$ diverges. For we can estimate the partial sums $s_{2^{k+1}}$ from below as follows:

$$\begin{aligned} s_{2^{k+1}} &= 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + \cdots + (\textstyle\sum_{m=2^k+1}^{2^{k+1}} \tfrac{1}{m}) \\ &\geq 1 + \tfrac{1}{2} + \tfrac{1}{2} + \cdots + \tfrac{1}{2} > 1 + \tfrac{k}{2}. \end{aligned}$$

Hence the partial sums are non bounded.

Example:

The series $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges for each natural number $k > 1$. We prove this for the case $k = 2$. The general case will be proved below. We have the estimate

$$\begin{aligned} \sum_{n=1}^{2^{k+1}-1} \frac{1}{n^2} &= 1 + (\tfrac{1}{2^2} + \tfrac{1}{3^2}) + \cdots + \left(\textstyle\sum_{n=2^k}^{2^{k+1}-1} \tfrac{1}{n^2}\right) \\ &\leq \textstyle\sum_{r=0}^{k} \tfrac{2^r}{(2^r)^2} \leq 2. \end{aligned}$$

Remark: In fact

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

(this will be demonstrated below).

We now bring some general criteria for the convergence or divergence of series.

Proposition 44 *(Leibniz' criterium for alternating series) Let (a_n) be a decreasing sequence of positive numbers which converges to 0. Then the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent.*

PROOF. For the partial sums s_k we have

1. s_{2k} decreases:
2. s_{2k-1} increases;
3. $s_{2k-1} \leq s_{2k}$.

Hence: $L = \lim_k s_{2k}$ and $L' = \lim_k s_{2k-1}$ exist. Furthermor

$$L - L' = \lim s_{2k} - \lim s_{2k-1} = \lim(s_{2k} - s_{2k-1}) = \lim a_{2k} = 0.$$

This clearly implies the result.

■

We remark that it follow easily from the proof that the limit lies between s_n and s_{n-1} for each n. This means that the error caused by breaking off the sum after n-terms is at most $_{n+1}$.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. (In fact the sum is $\ln 2$).

Definition 20 *The series $\sum_{n=1}^{\infty} a_n$ is* **absolutely convergent**, *if $\sum_{n=1}^{\infty} |a_n|$ converges.*

It follows from the Cauchy criterium that the series then converges. For

$$\left| \sum_{k=m}^{n} a_n \right| \leq \sum_{k=m}^{n} |a_n| \,.$$

The example $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ shows that a series can converge, without being absolutely convergent. We then say that it is **conditionally convergent**.

Proposition 45 *Let $\sum_{n=1}^{\infty} c_n$ be a convergent series with non-negative terms. konvergente Reihe mit nicht negativen Gliedern. Then the sum $\sum_{n=1}^{\infty} a_n$ converges whenever the sequence (a_n) is majorised by (c_n) (i.e. $|a_n| \leq c_n$ for each n).*

Proof. In this case we have

$$|\sum_{k=m}^{n} a_n| \leq \sum_{k=m}^{n} c_n.$$

This implies that the Cauchy condition is satisfied.

■

As a consequence we have

Proposition 46 *Let (a_n) and (b_n) be two sequences with positive terms. Suppose that $l > 0$ exists, so that $\frac{a_n}{b_n} \to l$. Then $\sum_n a_n$ converges $\iff$ $\sum_n b_n$ converges.*

Proof. Exercise.

■

Since the convergence or divergence of a series is not influenced by a change in finitely many terms it suffices when there is an $N \in \mathbf{N}$ and $K > 0$, so that $|a_n| \leq K c_n$, if $n \geq N$.

Proposition 47 *Let $\sum a_n$ be a series with non-zero terms. Suppose that there exists a λ with $0 < \lambda < 1$, so that*

$$\left|\frac{a_{n+1}}{a_n}\right| \leq \lambda \text{ for each } n.$$

Then the series converges (absolutely).

Proof. It is a simple conseqnece of the hypothesis that one can majorise the series by a convergent geometric series.

■

Proposition 48 *Let $\sum a_n$ be an infinite series such that $\lim |a_n|^{1/n} = \lambda$, where $\lambda < 1$. Then the series converges absolutly.*

Proof. Once again the series can be compared with a convergent geometric series.

■

Remark: In fact, the condition $\limsup_n |a_n|^{1/n} = \lambda < 1$ is sufficient to guarantee convergence. Similarly the condition

$$\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$$

is sufficient in applications of the quotient criterium.

Examples: I. $\sum_{n=1}^{\infty} \dfrac{n^2}{2^n}$ converges (by the quotient criterium).
II. $\sum \dfrac{1}{n}$. We know that this series diverges. However, $\left| \dfrac{a_{n+1}}{a_n} \right| < 1$ for each n resp. $|a_n|^{1/n} < 1$ for each n.
III. The series $\sum \dfrac{1}{n}$ and $\sum \dfrac{1}{n^2}$ show that the the quotient and root criteria provide no information in the cases where the corresponding limit is 1.

Criteria for divergence: There are corresponding criteria for the divergence of series which we quote without proof.

Proposition 49 *If the series $\sum_n a_n$ of non-zero terms diverges, then so does $\sum_n b_n$ whenever $b_n \geq a_n$ for each n.*

If the series $\sum_n a_n$ satisfies the condition $\lim \left| \frac{a_{n+1}}{a_n} \right| \to l$ with $l > 1$, then it diverges.

If the series $\sum_n a_n$ satisfies the condition $\lim |a_n|^{1/n} \to l$ with $l > 1$, then it diverges.

Proposition 50 *Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent. Then each series which arises from the former by permutation of the terms is also convergent (to the same sum). More precisely, if $\sigma : \mathbf{N} \to \mathbf{N}$ is a bijection, then the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges to $a = \sum_{n=1}^{\infty} a_n$.*

PROOF. Let $\epsilon > 0$ be given. We choose N_1 so that $\sum_{k=n}^{\infty} |a_n| < \epsilon$, whenever $n \geq N_1$. Choose N so that $\{\sigma(1), \dots, \sigma(N)\} \supset \{1, \dots, N_1\}$. Then for $m \geq N$:

$$\begin{aligned} \left| \sum_{k=1}^{m} a_{\sigma(k)} - a \right| &\leq \left| \sum_{k=1}^{m} a_{\sigma(k)} - \sum_{n=1}^{N_1} a_n \right| + \left| \sum_{n=1}^{N_1} a_n - a \right| \\ &\leq \sum_{n=N_1+1}^{\infty} |a_n| + \epsilon < 2\epsilon. \end{aligned}$$

■

This result is false for conditionally convergent series. In fact we can rearrange such a series so that it *diverges* or so that it converges to any previously determined value. For let $\sum a_n$ be such a series. We decompose it into two series with non-negative terms as $\sum a_n^+$ and $\sum a_n^-$, whereby $a_n^+ = a_n$ $(a_n \geq 0)$ and $= 0$ otherwise and a_n^- is defined correspondingly. It is clear that $\sum a_n$ is absolutely convergent if and only if these two series converge. Hence both of them diverge if $\sum a_n$ is conditionally convergent. Let L be an arbitrary number. We construct a rearrangement which converges to L as follows. We first choose sufficiently many positive terms so that the sum is larger then L. We then take negative terms until the sum is less thant L. We continue in this way and obtain a rearrangement with the required properties.

We now consider the behavious of series with respect to algebraic operations:

Proposition 51 *If the series $\sum_n a_n$ and $\sum_n b_n$ converge, then so does the series of sums $\sum_n (a_n + b_n)$ (the sum being of course $\sum_n a_n + \sum_n b_n$).*

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then so is the formal product $\sum_{n=1}^{\infty} c_n$ (whereby $c_n = \sum_{i+j=n} a_i b_j$) absolutely-convergent. In addition

$$\sum_{n=1}^{\infty} c_n = \left(\sum_{i=1}^{\infty} a_i \right) \left(\sum_{k=1}^{\infty} b_k \right).$$

PROOF. Let a and b be the sums of the corresponding series. resp. A and B the sums of the absolute values. We shall show that $\lim_{n\to\infty} \sum_{n=1}^{\infty} c_n = ab$. Choose N so that

1. $|(\sum_{k=0}^{n} a_k)(\sum_{k=0}^{n} b_k) - ab| < \epsilon$ for $n \geq N$;
2. $\sum_{k=n}^{\infty} |a_k| < \epsilon$ for $n \geq N$;
3. $\sum_{k=n}^{\infty} |b_k| < \epsilon$ for $n \geq N$.

Then if $n \geq \max(\dfrac{N}{2}, \dfrac{N}{2B}, \dfrac{N}{2A})$,

$$\left| \sum_{k=0}^{n} c_k - ab \right| < \epsilon.$$

■

Example (the exponential sequence). Consider the series $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$. It follows easily from the quotient criterium that the seres converges absolutely for each $x \in \mathbf{R}$. We write (as above) $\exp x$ for the sum. In particular, $e = \exp(1) = \sum_{n=0}^{\infty} \dfrac{1}{n!}$ is the **euler number**.

From the formula for the product of two absolutely convergent series, one calculate

Proposition 52 *(The functional equation for the exponential function) For $x, y \in \mathbf{R}$ we have*

$$\exp(x+y) = \exp(x).\exp(y).$$

PROOF. Let $a_i = \dfrac{x^i}{i!}$, $b_j = \dfrac{y^j}{j!}$. Then

$$c_n = \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!}\frac{y^k}{k!} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = \frac{1}{n!}(x+y)^n.$$

∎

From this we can immediately obtain some elementary properties of the exponential function:

1. for $x \in \mathbf{R}$, we have $\exp x > 0$;
2. $\exp(-x) = (\exp(x))^{-1}$;
3. $\exp n = e^n$ for $n \in \mathbf{Z}$.

Further examples of functions which we define by means of infinite series are:

The trigonometric functions: We define

$$\begin{aligned} \cos x &= \textstyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} \\ \sin x &= \textstyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

The series converge absolutely for each $x \in \mathbf{R}$ (quotient criterium). We can deduce from this definition that

1. $\cos 0 = 1$, $\sin 0 = 0$;
2. $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$;

3. $\cos(x+y) = \cos x \cos y - \sin x \sin y$;

4. $\sin(x+y) = \sin x \cos y + \cos x \sin y$;

5. $\cos^2(x) + \sin^2(x) = 1$;

6. $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$;

7. $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$.

Proof. (1) and (2) are trivial. (3) and (4) are proved using the Cauchy product as for the exponential function. (5), (6), (7) follow from (3) and (4). ∎

Further trigonometric functions: Using the functions sin and cos we can define further trigonometric functions as follows:

$$\begin{aligned}\tan x &= \tfrac{\sin x}{\cos x};\\ \operatorname{cosec}(x) &= \tfrac{1}{\sin x};\\ \operatorname{cotan}(x) &= \tfrac{\cos x}{\sin x};\\ \sec(x) &= \tfrac{1}{\cos x}.\end{aligned}$$

Of course, these are defined for those values of x for which the corresponding denominators are non-zero.

We can deduce elementary properties of these functions from those of sin and cos. For example we have the following sum formula tan:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

6.6 Improper integrals:

I. Integrals over unbounded intervals: Let $f : [a, \infty[\to \mathbf{R}$ be bounded. If

$$\lim_{c\to\infty} \int_a^c f(x)\,dx$$

exists, then we say that the integral $\int_a^\infty f(x)\,dx$ **converges** and we define

$$\int_a^\infty f(x)\,dx = \lim_{c\to\infty} \int_a^c f(x)\,dx.$$

Similarly, we define the existence and value of $\int_{-\infty}^{a} f(x)\,dx$ resp. $\int_{\mathbf{R}} f(x)\,dx$ for a function f, which is defined on $]-\infty, a]$ resp on $\mathbf{R}$.
II. Integrals of functions with singularities resp of unbounded functions: Let $f :]a, b] \to \mathbf{R}$ be such that for each $\epsilon > 0$ the restriction of the function to $[a + \epsilon, b]$ is bounded and integrable. IF

$$\lim_{\epsilon \to 0+} \int_{a+\epsilon}^{b} f(x)\,dx$$

exists, then we define

$$\int_{a}^{b} f(x)$$

to be this limit.

We remark here that there are criteria for the convergence and divergence of improper integrals which are analogues of corresponding criteria for series. In addition there is an analogous distinction between integrals which converges absolutely (i.e. such that the corresponding integral of the absolute value of the function converges) and those which converge conditionally. For example, the integarl $\int_0^\infty \frac{\sin x}{x}\,dx$ converges (its value is $\frac{\pi}{2}$—see the exercises). However, $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges.

Examples: Consider the integrals

$$\int_1^\infty \frac{1}{x^\alpha}\,dx$$

and

$$\int_0^1 \frac{1}{x^\alpha}\,dx.$$

The first integral converges if and only if $\alpha > 1$, the second if and only if $\alpha < 1$.

Proposition 53 *Integral criterium for the convergence of series Let*

$$f : [1, \infty[\to \mathbf{R}$$

be a non negative, decreasing function. Then the series $\sum_{n=1}^\infty f(n)$ *converges if and only if the improper integral*

$$\int_1^\infty f(x)\,dx$$

does so.

PROOF. Suppose that the series converges. Then so does the integral since $\sum_{n=2}^{N} f(n)$ is the integral of a step function which is majorised by f.

On the other hand, $\sum_{n=1}^{\infty} f(n)$ is the integral of a step function which majorises f.

■

Example: The series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$. Using the above result we immediately see that $\sum \frac{1}{n^\alpha}$ converges for $\alpha > 1$ and diverges for $\alpha \leq 1$.

6.7 Exercises

Exercise For which $x \in \mathbf{R}$ do the following series converge?

$$\sum_{k=1}^{\infty} \frac{(k+2)^4}{k^5+6)} x^{2k} \qquad \sum_{k=0}^{\infty} k^2 x^k.$$

Exercise Show that

$$\sum_{k=1}^{\infty} \frac{k+3}{k(k+1)(k+2)} = \frac{5}{4}, \quad \sum_{k=2}^{\infty} \frac{1}{k(k^2-1)} = \frac{1}{4}.$$

Exercise Which of the following series converge?

$$\sum_{n=0}^{\infty} \frac{n^3}{1+n^4}, \quad \sum_{n=0}^{\infty} \frac{n^4}{1+n^4}, \quad \sum_{n=1}^{\infty} \frac{1}{(n!)^{1/n}}.$$

Exercise Which of the following statements are valid?

1. if $\frac{a_{n+1}}{a_n} < 1$, then $\sum a_n$ converges;
2. if $a_n - b_n \to 0$ and $\sum b_n$ converge, then so does $\sum a_n$;
3. if $\sum a_n$ converges, then so does $\sum a_n^2$;
4. if $\sum a_n^2$ converges, then so does $\sum a_n$;
5. if $a_{n+1} + \cdots + a_{2n} \to 0$, then $\sum a_n$ converges.

Exercise Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\dots(n+k)} = \frac{1}{k!k}.$$

Exercise Show that

$$\sum_{r=1}^{n} r x^{r-1} = \frac{1-(n+1)x^n + n x^{n+1}}{(1-x)^2} \quad (x \neq 1).$$

For which x does the infinite series $\sum_{r=1}^{\infty} r x^{r-1}$ converge?

Exercise Let k be the sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{3}{4}k.$$

Exercise For which a, b, does the series

$$(a-b) + (a^2 - b^2) + (a^3 - b^3) + \ldots$$

converge?

Exercise Let f be a nicht-negative, decreasing function from $\mathbf{R}_+ \to \mathbf{R}$. Show that: $\sum_{n=1}^{\infty} f(n)$ converges $\iff$ $\sum_{n=1}^{\infty} 2^n f(2^n)$ converges. (Cauchy density theorem).

Exercise Use the last exercise to investigate the convergence of series $\sum \dfrac{1}{n^{\alpha} \ln n}$.

Exercise For which x does the series

$$\sum \binom{\alpha + n - 1}{n} x^n$$

converge?

Exercise For which x does the series

$$\sum \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\gamma(\gamma+1)\ldots(\gamma+n-1)} x^n$$

converge?

Exercise Formulate a version of the comparison test for series which is valid for imporper integrals.

Exercise Show that the imporper integral $\int_0^\infty \dfrac{\sin x}{x}\, dx$ converges conditionally.

Exercise Show that
$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$
(Differentiate the expression $F(y) = \int_0^\infty e^{-xy} \frac{\sin x}{x}\, dx$ with respect to y).

Exercise (Raabe's criterium) Show that if
$$\left|\frac{a_{n+1}}{a_n}\right| \leq 1 - \frac{\beta}{n},$$
whereby $\beta > 1$, then the series converges $\sum a_n$ (absolutely).

7 Convergence of sequences and series of functions

7.1 Definitions

Definition 21 *Let* (f_n) *be a sequence of functions from* $D \subset \mathbf{R}$ *with values in* $\mathbf{R}$. *We say that* f_n **converges pointwise fo a function** f, *if :* $f_n(x) \to f(x)$ *for each* $x \in D$.

In a certain sense this is the natural notion of convergence for functions. However, in many situations it is too weak a notion and we therefore introduce a more subtle one.

Definition 22 *Let* (f_n) *and* f *be as above. Then* f_n *converges* **uniformly** *to* f, *if for each positive* $\epsilon > 0$ *there exists an* N, *so that* $|f_n(x) - f(x)| < \epsilon$ *for each* $x \in D$ *and each* $n \geq N$.

Examples: It is clear that uniform convergence implies pointwise convergence. The following is an example of a sequence of functions which converges pointwise, but not uniformly.

$$f_n(x) = x^n \quad (0 \leq x \leq 1).$$

We can reformulate the definition of uniform convergence as follows:

Definition 23 *Let* f *be a bounded function on* D. *We define*

$$\|f\|_\infty = \sup\{|f(x)| : x \in D\}.$$

If f *is unbounded, then we put* $\|f\|_\infty = \infty$.

Proposition 54 f_n *converges uniformly to* f *if and only if*

$$\|f_n - f\|_\infty \to 0$$

.

Proposition 55 *Let* (f_n) *be a sequence of continuous function on* D, *which converges uniformly to* f *on* D. *Then* f *is also continuous.*

PROOF. We fix a point x_0 in D and an $\epsilon > 0$. There exists $N \in \mathbf{N}$ with $\|f_n - f\|_\infty < \frac{\epsilon}{3}$, whenever $n \geq N$. Since f_N is continuous, there is a ¿0, so that $|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$, falls $|x - x_0| < \delta$. Then for $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \leq \epsilon.$$

■

7.2 Criteria for convergence

Proposition 56 *(The WeierstraßM-test) Let $f_n : D \to \mathbf{R}$ be a sequence of functions on D with the preoprty that $\sum_n \|f_n\|_\infty < \infty$. Then the series $\sum_n f_n$ converges absolutely and uniformly to a function f. Hence if each f_n is continuous, then so is f.*

PROOF. Exercise.

■

Power series: A power series is a series of functions of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

In order to simplify the notation, we shall usually assume that $x_0 = 0$.

Typical examples are the series which we used to define the exponention function and the trigonometric functions. In working with these functions we often tacitly assumed that such power series have pleasant properties and we shall now prove that this is indeed the case.

Proposition 57 *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series which converges for $x_0 \neq 0$. Then the series converges absolutely for each x with $|x| < |x_0|$. Further if converges uniformly on any interval of the form $[-a, a]$ with $a < |x_0|$. Hence the function*

$$g(x) = \sum_{n=o}^{\infty} a_n x^n$$

is defined on $]-|x_0|, |x_0|[$ and continuous there.

PROOF. Since $\sum_n a_n x_0^n$ converges, the sequence $(a_n x_0^n)$ is bounded. Let $K > 0$ be such that $|a_n x_0^n| \leq K$ for each n.

We can now estimate the term $a_n x^n$ as follows:

$$|a_n x^n| = \left| a_n \left(\frac{x^n}{x_0^n} \right) x_0^n \right| \leq \left| \frac{x}{x_0} \right|^n K.$$

Hence by comparison with a geometric series, we see that the series converges absolutely.

The proof of the second claim is similar.

■

If we put

$$R = \sup\{x > 0 : \sum_n a_n x^n \text{ converges}\},$$

then: $\sum_n a_n x^n$ converges for each x with $|x| < R$. Further, the convergence is absolute and uniform on any interval $[-a, a]$ with $a < R$. For $|x| > R$ the series diverges. (In case $|x| = R$ then we get no information in general. The behaviour there depends on the specific form of the series).

R is called **the radius of convergence** of the series. Using the root test, we can get the following explicit formula for R:

$$R = \frac{1}{\limsup_n |a_n|^{1/n}}.$$

Examples: We have already mentioned the power series representations for exp, sin and cos. Further examples are the binomial series

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad (-1 < x < 1, \alpha \in \mathbf{R})$$

resp. the hyperbolic functions

$$\sinh x = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}.$$

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

Proposition 58 *If (f_n) is a sequence of continuous functions on $[a, b]$, which converges uniformly to f, then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Proof. The claim follows immediately from the estimate

$$\left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| \le (b-a)\|f-f_n\|_\infty.$$

■

The examples above show that a similar claim for pointwise convergence is false.

Proposition 59 *Let (f_n) be a sequence of continuously differentiable functions on $[a,b]$, which converges pointwise to f. Further suppose that the sequence (f_n') of derivatives is uniformly convergent. Then f is (continuously)-differentiable and*

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Proof. We put $g = \lim f_n'$ and fix $x \in [a,b]$. We have the relationship

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)\,dt.$$

We now let n go to ∞ and get

$$f(x) = f(a) + \int_a^x g(t)\,dt.$$

Since g is continuous, we have

$$f'(x) = g(x) = \lim_n f_n'(x).$$

■

Example: The example

$$f_n(x) = \frac{1}{n}\sin nx$$

shows that a sequence of functions can converge uniformly without the derived sequence converging.

As a consequence of this result, we have

Proposition 60 *Let $\sum_{n=0}^\infty a_n x^n$ be a power series with positive radiou of convergence R. Then the function $f : x \mapsto \sum_{n=0}^\infty a_n x^n$ auf $]-R,R[$ is (infinitely) differentiable and*

$$f'(x) = \sum_{n=1}^\infty n a_n x^{n-1}.$$

PROOF. The radius of convergence of the series $\sum n a_n x^{n-1}$ coincides with that of the derived series $\sum a_n x^n$ (why?).

■

7.3 Taylor series:

We return to the topic of Taylor series. Recall that for a $(n+1)$-times continuously differentiable function f on the interval I we have

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

whereby a lies in the interior of I. The remainder term has the form $\frac{f^{(n+1)}(\xi)}{n!}(x-a)^{(n+1)}$. In applications one is interested in estimates for the remainder term. For this the following explicit formula is often useful:

Proposition 61 *We have*

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt.$$

PROOF. This is proved by induction: $n = 1$: Then

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

$n-1 \to n$: Suppose that

$$R_n(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)\, dt.$$

Integrating by parts, we see that

$$\begin{aligned} R_n(x) &= -f^{(n)}(t)\tfrac{(x-t)^n}{n!}\Big|_a^x + \int_a^x \tfrac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt \\ &= \tfrac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \tfrac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt \\ &= \tfrac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x). \end{aligned}$$

■

Remark: From this it follows immediately that the remainder satisfies the growth condition
$$\lim_{x\to a}\frac{R_{n+1}(x)}{(x-a)^n}=0.$$

Definition 24 *Let $f: I \to \mathbf{R}$ be an inifinitely differentiable function and $a \in I$. Then*
$$T_f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$
is called the **Taylor series** *of f.*

We do not claim that this series necessarily represents f in any sense. In fact it can happen

1. that the series fails to converge (except trivially at the point a)
2. that the series converges, but not to f.

Examples:
$$f(x)=e^{-\frac{1}{x^2}}$$
where we set $f(0)=0$.

The Taylor series of this function is the trivial series with each term vanishing.

Of course this example is non-typical. Normally the Taylor series converges to f in a neighbourhood of a. For example is a function defines as a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ around a, then the Taylor series of f at a is just this power series and so converges to f.

7.4 Exercises

Exercise Let (f_n) and (h_n) be uniformly converging sequences of functions. Show that a) it is not in general true that $f_n g_n$ converges uniformly; b) if (f_n) and (g_n) are, in addition uniformly bounded, then $f_n g_n$ converges uniformly.

Exercise Let (f_n) be a sequence of continuous functions on $[0,1]$, which converges pointwise to f. Show that the convergence uniform if and only if f is continuous and (f_n) is equicontinuous.

Exercise Prove Dini's theorem: Let (f_n) be a sequence of continuous functions on $[0,1]$, which converges pointwise to the continuous function f. If the sequence is decreasing i.e. $f_{n+1} \leq f_n$ for each n, then it converges uniformly.

Exercise Let (g_n) be a decreasing sequence of functions on $[a, b]$, which converges uniformly to 0. Show that the series $\sum(-1)^n g_n$ converges uniformly.

Exercise Let (g_n) be a uniformly bounded decreasing sequence of functions $[a, b]$, resp. $\sum f_n$ a uniformly converging series of functions. Show that the series $\sum f_n g_n$ is also uniformly convergent.

Exercise Let (a_n) be a positive, decreasing sequence. Show that the series $\sum a_n \sin nx$ converges uniformly on $\mathbf{R}$ if and only if $na_n \to 0$.

Exercise Suppose that the series $\sum a_n$ converges. Show that the Dirichlet series $\sum a_n n^{-s}$ converges uniformly on $[0, \infty[$.

8 Fourier Series

8.1 The definitions

Definition 25 *Let f and g be, for example, piecewise continuous functions on the interval $[a, b]$. We define the scalar product $(f|g)$ of f and g as follows:*

$$(f|g) = \int_a^b f(x)g(x)\,dx.$$

Then as in Linear Algebra we see that

$$(f|g) = (g|f) \quad (cf|g) = c(f|g) \quad (f_1 + f_2|g) = (f_1|g) + (f_2|g)$$

for functions f, g, f_1, f_2 and real numbers c. Further $(f|f) > 0$, if $f \neq 0$. Hence we define

$$\|f\| = \sqrt{(f|f)}.$$

Then

$$|(f|g)| \leq \|f\|\|g\| \text{ (the Cauchy-Schwarz inequality)}$$

and

$$\|f + g\| \leq \|f\| + \|g\| \text{ (the Minkowski inequality).}$$

Definition 26 *A sequence (ϕ_n) of continuous functions on $[a, b]$ is called* **orthogonal***, if $(\phi_m|\phi_n) = 0$, whenever $m \neq n$. The sequence is* **orthonormal***, if, in addition, $\|\phi_n\| = 1$ for each n.*

Example The sequence (ϕ_n) is orthonormal on $[0, 2\pi]$, whereby

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}, \quad \phi_{2n-1}(x) = \frac{\cos nx}{\sqrt{\pi}}, \quad \phi_{2n}(x) = \frac{\sin nx}{\sqrt{\pi}} \quad (n = 1, 2, \ldots).$$

If (ϕ_n) is an orthonormal system, and f is Riemann integrable, then the series

$$\sum_n c_n\phi_n \quad \text{wobei } c_n = (f|\phi_n)$$

is called the **Fourier series** of f with respect to (ϕ_n). If (ϕ_n) is the sequence of trigonometric functions as above, then one speaks simply of the Fourier series.
The classical Fourier series is thus

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

whereby

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt.$$

The definition is motivated by the following fact:

Proposition 62 *If the series $\sum_n c_n\phi_n(x)$ is unformly convergent on $[a,b]$, then its sum is a continuous function with Fourier coefficients (c_n).*

PROOF. Exercise.

■

Proposition 63 *Let (ϕ_n) be orthonormal on $[a,b]$ and let f be continuous with Fourier series $\sum_{n=0}^{\infty} c_n\phi_n$ and with partial sums s_n. Put*

$$t_n = \sum_{k=0}^{n} b_k\phi_k,$$

whereby $(b_0,\dots,b_n)$ is an arbitrary sequence of numbers. Then

$$\int_a^b |f(x)-t_n(x)|^2\,dx = \int_a^b |f(x)|^2\,dx - \sum_{k=0}^{n}|c_k|^2 + \sum_{k=0}^{n}|c_k-b_k|^2.$$

Hence

$$\int_a^b |f(x)-s_n(x)|^2\,dx \le \int_a^b |f(x)-t_n(x)|^2\,dx.$$

PROOF. It suffices to prove the expression for $\int_a^b |f(x)-t_n(x)|^2\,dx$. This is analogue to the corresponding calculation for euclidean spaces as in Linear Algebra.

■

Proposition 64 *Let f, (ϕ_n) and (c_n) be as above. Then*

1)

$$\sum_{n=0}^{\infty}|c_n|^2 \le \int_a^b |f(x)|^2\,dx \quad \textit{(the Bessel inequality)}$$

and so the series $\sum |c_n|^2$ converges;

2) We have

$$\sum_{n=0}^{\infty}|c_n|^2 = \int_a^b |f(x)|^2\,dx \quad \textit{(Parseval's formula)}$$

if and only if $\lim_{n\to\infty}\int_a^b |f(x)-s_n(x)|^2\,dx = 0$.

PROOF. Exercise.

■

We say that the system (ϕ_n) is **complete**, if the second condition above holds.

For example, the trigonometric system above is complete on $[0, 2\pi]$. This will be proved below. A much more general result will be proved in analysis III.

8.2 Convergence

We now investigate the question of the pointwise convergence of Fourier series. We use the concept of a piecewise Lipschitz-continuous functions. That is a piecewise continuous function with singularities $a_1, \dots, a_k$ so that the restrictions of f to $]a, a_1[,]a_1, a_2[, \dots$ are Lipschitz continuous.

Proposition 65 *(The Lemma of Riemann-Lebesgue) Let f be Riemann integrable on $[a, b]$. Then*

$$\lim_{\alpha \to \infty} \int_a^b f(x) \sin(\alpha x + \beta) \, dx = 0$$

for each $\beta \in \mathbf{R}$.

PROOF. Step 1: The result holds if f is the characteristic function $\chi_{[a,b]}$ of an interval (direct computation).

Step 2: The result holds if f is a step function. This holds by step 1 and the linearity of the integral.

The result now follows from 1 and 2 using an approximation argument.

■

Dirichlet Integrals We now investigate integrals of the form

$$\int_0^\delta g(x) \frac{\sin \alpha x}{x} \, dx.$$

Proposition 66 *Proposition (Jordan) Let g be piecewise Lipschitz continuous on $[0, \delta]$. Then*

$$\lim_{\alpha \to \infty} \int_0^\delta g(x) \frac{\sin \alpha x}{x} \, dx = g(0+).$$

Proof. We put

$$\int_0^\delta g(t)\frac{\sin\alpha t}{t}\,dt = \int_0^h [g(t)-g(0+)]\frac{\sin\alpha t}{t}\,dt + g(0+)\int_0^h \frac{\sin\alpha t}{t}\,dt + \int_h^\delta g(t)\frac{\sin\alpha t}{t}\,dt.$$

By the Lemma of Riemann-Lebesgue the third term converges to 0 as $\alpha \to \infty$.

The second term converges to

$$\lim_{\alpha\to\infty} g(0+)\int_0^\delta \frac{\sin\alpha t}{t}\,dt = \lim_{\alpha\to\infty} g(0+)\int_0^{h\alpha}\frac{\sin t}{t}\,dt = \frac{\pi}{2}g(0+).$$

The first term converges to 0 and this completes the proof.

■

In order to apply this theorem, we use the following explicit formula for the partial sums

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(\alpha_k \cos kx + b_k \sin kx)$$

of the Fourier series of f.

We have

$$s_n(x) = \frac{2}{\pi}\int_0^{\pi/2} \frac{f(x+2t)+f(x-2t)}{2}\frac{\sin(2n+1)t}{\sin t}\,dt.$$

Proof. This follows directly from the formula

$$\frac{1}{2} + \sum_{k=1}^{m}\cos kt = \frac{\sin(2n+1)t}{2\sin t}.$$

We denote the sum by $D_n(t)$ (the Dirichlet kernel).

■

Proposition 67 *Let f be piecewise continuous on $[0,2\pi]$. Then the Fourier series of f converges at x if and only if the limit*

$$\lim_{n\to\infty}\frac{2}{\pi}\int_0^\delta \frac{f(x+2t)+f(x-2t)}{2}\frac{\sin(2n+1)t}{t}\,dt$$

exists for some $\delta > 0$. In this case $s_n(x)$ converges to this limit.

From this we easily deduce

Proposition 68 *Let f be piecewise Lipschitz continuous on $[0,2\pi]$. Then $s_n(x)$ converges pointwise to the value $\frac{1}{2}[f(x+)+f(x-)]$.*

PROOF. We put

$$g(t) = \frac{f(x+2t) - f(x-2t)}{2}$$

on $[0, \frac{\delta}{2}]$ and apply the above result.

■

Remark The Fourier series converges tokonvergiert die $\frac{1}{2}[f(0+) + f(2\pi-)]$ at the points 0 und 2π .

8.3 Césaro summability of the Fourier series

The convergence properties of the Fourier series is improved by employing the so-called Césaro method. We define

$$\sigma_n(x) = \frac{s_0(x) + \cdots + s_n(x)}{n+1}.$$

Then

$$\sigma_n(x) = \frac{2}{(n+1)\pi} \int_0^{\pi/2} \frac{f(x+2t) + f(x-2t)}{2} \left(\frac{\sin(n+1)t}{\sin t} \right)^2 dt.$$

(Proof—Exercise. One calculates the Césaro means $K_n(t)$ of the Dirichlet kernel).

Proposition 69 *Fejér) Let f be piecewise continuous. Then $\sigma_n(x)$ converges for each x (and the sum is as above). The sequence converges uniformly on each closed subinterval of $[0, 2\pi] \setminus \{a_1, \ldots, a_n\}$. In particular, it converges uniformly on $[0, 2\pi]$, provided that f has a continuous extension to the real line which is 2π-periodic.*

It is a consequence of this result that the trigonometric system is complete on $[0, 2\pi]$.

8.4 Exercises

Exercise Let $\phi(x) = \dfrac{1}{2^n n!} f_n^{(n)}(x)$, whereby $f_n(x) = (x^2-1)^n$. Show that (ϕ_n) is orthonormal on $[-1,1]$. Calculate $\phi_1, \ldots, \phi_4$.

Exercise Let $f : [-\pi, \pi] \to \mathbf{R}$ be even. Show that

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

whereby $a_n = \frac{2}{\pi}\int_0^\pi f(t)\cos nt\, dt$. What is the corresponding statement for odd f?

Exercise Show that

$$x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad (0 < x < 2\pi)$$

$$\frac{x^2}{2} = \pi x - \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \quad (0 \leq x \leq 2\pi).$$

Exercise Show that

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} \quad (0 < x < \pi);$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} \quad (0 \leq x \leq \pi).$$

Exercise Show that

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n} \quad (-\pi < x < \pi);$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} \quad -\pi \leq x \leq \pi).$$

Exercise Shos that

$$x^2 = \frac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n}\right) \quad 0 < x < 2\pi).$$

Exercise Show that

$$\cos x = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n\sin 2nx}{4n^2-1} \quad (0 < x < \pi).$$

What is the corresponding expression for $\sin x$?

Exercise Calculate the Fourier series on $[-\pi, \pi]$ for $x\cos x$, $x\sin x$.

Exercise Calculate the Fourier series of the 2π-periodic functions $\ln\left|\sin\frac{x}{2}\right|$, $\ln\left|\cos\frac{x}{2}\right|$, $\ln\left|\tan\frac{x}{2}\right|$.

Exercise Use Parseval's formula to prove the following identities:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \qquad \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}.$$

Exercise (the Gibbs phenomenon): Let f be the function $2H - 1$ (morep precisely the 2π-periodic extension of its restriction to $[-\pi, \pi]$). Show that

a) $f(x) = \frac{4}{\pi}\sum_{n+1\infty}\frac{\sin(2n-1)x}{2n-1}$;

b) $s_n(x) = \frac{2}{\pi}\int_0^x \frac{\sin 2nt}{\sin t}\,dt$;

c) the function s_n on $]0, \pi[$ has local maxima and minima at the points $x_m = \frac{1}{2}\frac{m\pi}{n}$ $(m = 1, 2, \ldots, 2n-1)$;

d) $s_n\left(\frac{\pi}{2n}\right)$ is an absolute maximum of the function;

e) $\lim_{n\to\infty} s_n\left(\frac{\pi}{2n}\right) = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt \sim 1.179$.

9 The construction of the real numbers

We proved in the first section that the set $\mathbf{Q}$ of rational numbers is an ordered field. We now sketch two methods for extending $\mathbf{Q}$ to a complete ordered field.

Dedekind cuts (motivation: each real number x generates a decomposition

$$\{y \in \mathbf{Q} : y < x\} \text{ resp. } \{y \in \mathbf{Q} : y \geq x\}$$

of $\mathbf{Q}$ into two disjoint sets). We define: a Dedekind cut is a decomposition $\mathbf{Q} = A \cup B$ of $\mathbf{Q}$, whereby

1. A and B are disjoint and $A \neq \emptyset$, $B \neq \emptyset$;
2. if $x \in A$ and $y \in B$, then $x < y$;
3. A has no greatest element.

We define $\mathbf{R}$ to be the family of all Dedekind cuts of $\mathbf{Q}$. One can then show that the set $\mathbf{R}$ defined in this way has the required properties.

Cauchy sequences: (motivation: each real number is the limit of a Cauchy sequence of rational numbers). We consider the family $\mathcal{Q}$, whose elements are Cauchy sequences in $\mathbf{Q}$. On this family we define an equivalence relationship $\sim$ as follows:

$$(r_n) \sim (s_n) \iff r_n - s_n \to 0.$$

$\mathbf{R}$ is the corresponding family of equivalence classes i.e. $\mathcal{Q}|_{\sim}$.

We remark that the technicalities of proving that the real numbers have the desired properties can be slightly reduced by applying this construction firstly to$\mathbf{Q}_+$ (and thus constructing $\mathbf{R}_+$). The extension from $\mathbf{R}$ to $\mathbf{R}_+$ can then be carried out as for the extension from $\mathbf{N} \to \mathbf{Z}$.

We conclude this section with Cantor's proof that $\mathbf{R}$ is uncountable. We use the fact that each real number x between 0 and 1 has a decimal expansion $0, a_1 a_2 \dots$. The proof is by contradiction. We suppose that the numbers of $]0,1[$ can be numerated. Thus we can list them as follows:

$$\begin{aligned} r_1 &= 0, a_1^1 a_2^1 a_2^1 \dots \\ r_2 &= 0, a_1^2 a_2^2 a_2^2 \dots \\ r_3 &= 0, a_1^3 a_2^3 a_2^3 \dots \\ &\vdots \end{aligned}$$

(in order to avoid ambiguities we avoid expansions which end with $99999\ldots$). One can then easily construct a number $0, b_1^1 b_2^2 b_3^3 \ldots$ which is not on the list, simply by going along the diagonal $a_1^1 a_2^2 a_3^3 \ldots$ and changing each number in a corresponding manner.