Functional analysis—-spaces of holomorphic functions and their duality

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Contents

1 Some results on topology

Definition 1 *A topological space is a set X together with a collection τ of subsets which contains the X and the empty set, is closed under arbitrary unions and finite intersections.*

Remark. The sets of τ are referred to as **open sets**. The complements of open sets are called closed sets.

Examples.

1. The most frequently occuring topological spaces are those which arise from a metric d on X. We then define τ or, more precisely τ_d to be the family of open sets in the sense of metric spaces (recall that a subset of X is open if for each x in X there is a positive ϵ so that the open ball $U(x, \epsilon)$ lies in U (in other words, open sets are those which are unions of open balls). Two important special cases are the real line $\mathbf{R}$ with the usual metric $(x, y) \mapsto |x - y|$ and the discrete metric (which can be defined on any set), where the distance of any two non-coincident points is defined to be one.

2. The indicrete topology. This is the smallest topology on a given set X and thus consists precisely of the family $\{X, \emptyset\}$.

3. The Sierpinski topology. This is a topologigy on a two-point set, say $\{0, 1\}$. The only open set apart from the two canonical ones is $\{0\}$.

4. The cofinite topology and the cocountable topology. These can be defined on any infinite (resp. uncountable set). The open sets (again in addition to the two canonical ones) are those with finite (resp. countable) complements.

Definition 2 *The above examples show that although the family of open sets can be vast (think of the topology on $\mathbf{R}^n$ defined by the euclidean metric), the topology can often be specified by a simpler clas sof open sets (think of the opne balls). The appropriate concepts are those of a* **Basis***—that is, a subfamily $\mathcal{B}$ of τ so that each open set is a union of sets from $\mathcal{B}$ resp. a* **Subbasis***—that is a subfamily $\mathcal{B}$ of τ so that the family of all finite intersections of sets of $\mathcal{B}$ forms a basis.*

Examples. The family of open intervals forms a basis for the natural topology of $\mathbf{R}$, the intervals of the form $]a, \infty[$ resp. $]-\infty, b[$ form a subbasis. The open balls in a metric space form a basis.

Definition 3 *we can use the open sets to extend the concepts of convergnece (sequential convergnece or convergence for nets resp. filters).*

EXAMPLES. Convergence in general topological spaces can have rather non-intuitive properties. For example, the reader should work out as an exercise which sequences convergence to which limits in the Sierpinski space, discrete and indiscrete spaces resp. in the cofinite and cocountable topologies.

Definition 4 *Using open sets we can extend the definition of continuity for a function between topological spaces in the natural. This allows us to use the concept of a homeomorphism and to talk of homeomorphic spaces.*

Definition 5 *In order to avoid certain difficulties concerning the uniqueness of limits it is convenient to restrict to general class of topological spaces. We shall find the folloing definition most useful: A topological space is* **Hausdorff** *or T_2 if for each distinct pair $xy \in X$ there are disjoint open sets U and V with $x \in U$, $y \in V$.*

Definition 6 *A subset A of a topological space X is* **connected** *if whenever U and V are open, disjoint subsets of X with $A \subset U \cup V$, then either $A \subset U$ or $A \subset V$.*

EXAMPLES. The space $\mathbf{Q}$ is not connected whereas $\mathbf{R}$ is.

Some simple properties of connectedness are contained in the following

Proposition 1

1. *If A is connected, then so is each B with $A \subset B \subset \bar{A}$.*
2. *If $f : X \to Y$ is continuous and surjective, then Y is connected, whenever X is.*
3. *If (A_α) is a family of connected subsets with non-empty intersection, then the union is connected.*
4. *If $X_1, \dots, X_n$ are connected then so is their cartesian product.*
5. *A subset of the real line is connected if and only if it is an interval.*

EXAMPLES. Show that S^1 and $0, 1]$ are not homeomoprhic. The same for $\mathbf{R}$ and $\mathbf{R}^n$ $(n > 10)$.

EXAMPLES. Generalise one of the above results by showing that if there is a connected B so that each A_α intersects B, then the union of the A_α is connected. Use this fact to deduce the connectedness of $\mathbf{R}^n$ from that of $\mathbf{R}$. More generally show that the product of connected spaces is connected.

Show that if A is a finite (or even countable) subset of the plane then its complement is connected.

Definition 7 *Using the open sets we can define in the natural way the interior of a set (the union of the open sets which it contains), the closure* $\bar{A}$ *of s set* A *(this is the complement of the interior of its complement. More intuitively, a point* x *lies in* $\bar{A}$ *if each open set contining* x *meets. A set is an open neighbourhood of* x *if it is open and contains* x*. It is a neighbourhood if it contains an open neighbourhood. We write* $\mathcal{N}(\S)$ *to denote the family of neighbourhoods of* x*. It is a filter and so is often called the neighbourhood filter. The boundary* ∂A *of* A *is the intersection of the closure of* A *and that of its complement. In other words,* $x \in \partial A$ *if and only if each neighbourhood of* x *meets both* A *and its complement.*

Definition 8 X *is compact if it is* T_2 *and each open covering of* X *has a finite subcovering.*

For metric spaces there is a more intuitive restatement of this definition:

Proposition 2 *If* X *is a metric space then* X *is compact if and only if it is sequentially compect (i.e. each sequence has a convergent subsequence) resp.* σ*-compact (i.e. each sequence has a cluster point). Either of these conditions is equivalent to the fact that* X *is complete and totally bounded.*

A simple consequence of this is the fact that a subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded

We mention briefly some properties of compact spaces:

Proposition 3

1. *A subset of a compact space is compact if and only if it is closed;*
2. *A continuous, surjective image of a compact space is compact provided that it is* T_2*;*
3. *Products of compact spaces are compact;*
4. *If* $f : X \to Y$ *is a continuous bijection from a compact space onto a* T_2 *space, then* f *is a homeomorphism.*

Definition 9 *A Baire space is a topological space which is of second category in itself. (Recall that a subset* A *of* X *is* **meagre** *if the interior of its closure is empty.* A *is of first category in* X *if it is expressible as a countable union of meagre sets, otherwise it is of second category. The fact that each complete, non empty metric space is a Baire space is a major result of topology. The same is true for compact or even locally compact spaces (A space is locally compact if it is* T_2 *and each point has a compact neighbourhood).*

Definition 10 *Further definitions which can easily be extended from the context of metric spaces to that of general topological spaces are those of continuous curves in X, arcwise-connectedness, dense subsets, separable space.*

EXAMPLES. Show that X is a Baire space if and only if for each sequence (U_n) of dense open subsets, the intersection $\bigcap U_n$ is also dense.

2 Holomorphic functions and the space $H(U)$

In this section we shall collect without proof those result on holomorphic functions which we shall refer to. Most of these are proved in the Course on analytic functions. Some will be proved using functional analytic methods below. Notation: U is a domain i.e. an open, connected subset of $\mathbf{C}$ (or $\hat{\mathbf{C}}$):

We define: $C(U)$ to be the space of continuous, complex-valued functions on U;

$H(U)$ to be the subspace of holomorphic functions. Of course $C(U)$ is a vector space and $H(U)$ is a vector subspace.

We recall some basic properties of analytic functions:

EXAMPLES. The conformal property If $f : U \to \mathbf{C}$ is holomorphic and $f'(z_0) \neq 0$, then f is conformal near z_0. This means that that the images of two smooth curves which meet in z_0 at an angle θ are again smooth (near z_0) and meet at the same angle at $f(z_0)$.

Definition 11 *The length of a curve If $c : [a, b] \to \mathbf{C}$ is a smooth curve, then its length is given by the integral:*

$$L(c) = \int_a^b |c'(t)|\, dt.$$

Similarly, the curvilinear integral of a function $f : U \to \mathbf{C}$ (where U is a domain containing the image of c) is

$$\int_v f(z)\, dz = \int_a^b f(c(t))c'(t)\, dt.$$

As is easy to see, if f has a primitive i.e. a holomorphic function F on U whose derivative is f, then the above integral is equal to $F(c(b)) - F(c(a))$. In particular, the integral is path independent and the integral around a closed curve is zero. Curvilinear integrals.

For our treatment, we require the following version of Cauchy's theorem for rectangles:

Proposition 4 *Let f he holomorphic in a neighbourhood of the rectangle $[a, b] \times [c, d]$. Then the integral of f around the boundary of the rectangle is zero.*

As usual, this implies the following version of Cauchy's integral theorem.

Proposition 5 Cauhy's integral formula for rectangles *With the above hypotheses we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for each z in the interior of R.

EXAMPLES. Use Green's theorem to show that

$$\frac{1}{2\pi i} \int_{\partial R} \frac{1}{\zeta - z} \, d\zeta = 1.$$

Using this result it is elementary to prove that each function $f : D \to \mathbf{C}$ where D is a disc has a primitive. This easily allows us to prove the Cauchy theorem and integral formula for functions on discs where the corresponding integrals are now taken around circles. From this we can deduce in the usual way the existence of Taylor series resp. Laurent series for functions holomorphic on discs resp. annuli. More precisely

Proposition 6 *1) Let f be holomorphic on the disc $D(z_0, R)$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z0)^n$$

where $a_n = \frac{1}{2\pi i} \int \frac{f(\zeta}{(\zeta - z_0)^{n+1}}$, the integral being taken around any circle with centre z_0 which lies in the disc;

a) Let f be holomorphic on the annulus $)0 < r < |z - z_0| < R$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z0)^n$$

where $a_n = \frac{1}{2\pi i} \int \frac{f(\zeta}{(\zeta - z_0)^{n+1}}$, the integral being taken around any circle with centre z_0 which lies in the annulus.

From these we can deduce the **Cauchy inequalities**: If f is holomorphic on a neighbourhood of the closed disc with radius R, then $|a_n| \leq \frac{1}{R^n} max_{|z|=R} |f(z)|$ where a_n is the corresponding Taylor coefficinet of f.

If f is holomorphic in a punctured neighbourhood of z_0 then the latter is

a removable singularity if f is bounded near z_0. (This is equivalent to the fact that the Laurent series of f is actually a Taylor series so f has a holomorphic extension to the non-punctured neighbourhood);

a pole if $|f|$ goes to infinity as $z \to z_0$. This is equivalent to the fact that the Laurent series has the form

$$\sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

where k is positive resp. that $f(z) = (z - z_0)^{-k} g(z)$ where g is holomorphic near z_0 and $g(z) \neq 0$. In this case, we say that f has a pole of order k at z_0;

An essential singularity if f is unbounded near z_0 but does not go to infinity in absolute value. This means that the for any k there is $n > k$ with $a_{-n} \neq 0$. A typicl example of such a function is $e^{1/z}$ (near 0).

We remark at this point the theorem of Weiserstraß-Casaroti. If z_0 is an essential singularity of f then fow each $w \in \mathbf{C}$ and each positive 4 and δ, there is a point $z \in U z_0, \delta)$ so that $|f(z) - w| < \epsilon$.

This is proved by noting that if there is a w which violates the condition then the function $g(z) = \dfrac{1}{f(z) - w}$ is holomorphic and bounded in a punctured neighbourhood of z_0. Hence z_0 is a removable singularity of g. But it is easy to see that this is not compatible with the fact that f has an essential singularity there.

A similar reasoning shows that if $f(z_0) = 0$, then there is a $k \in \mathbf{N}$ so that f has the form $(z - z_0)^k g(z)$ where $g(z_0) \neq 0$. f then has a zero of order k at z_0. From this it follows easily that there is a neighbourhood of z_0 in which f has no further zeros. This means that the zero set of f is discrete i.e. it is a sequence whose only cluster points are on the boundary of U. Simple consequence are the following:

The identity theorem: If f is holomorphic on a domain U and f vanishes on a non-trivial open subset then f is the zero function.

A sharper version is: if f is as above and the set of zeros of f has a cluster point in U, then f is the zero function.

We can then deduce as corollaries the Identity principle and the residuum theorem (for rectangles).

Proposition 7 *Let f be holomorphic on a neighbourhood of a rectangle R and suppose that f has zeroes on the boundary ∂R of R. Then the integral*

$$\frac{1}{2\pi i} \int_{\partial r} \frac{f'(z)}{f(z)}\, dz$$

is a whole number and is equl to the number of zeroes of f in R (counted according to multiplicities).

Definition 12 *If X and Y are topological spaces, then $f : X \to Y$ is an open mapping if the images of open sets are open. If X and Y are metric spaces we have the following ϵ-δ version of this definition: for each $x \in X$ and each positive δ there is a positive ϵ so that*

$$f(U(x, \delta) \supset U(f(x), \epsilon).$$

Proposition 8 *if $f : U \to \mathbf{C}$ is holomorphic and non constant, then it is open.*

As a consequence, we have the maximum and minimum modulus theorems.

Suppose that $f : U \to \mathbf{C}$ is holomorphic and its modulus attains its maximum. Then f is constant.

Under the same conditions, it f has no zeroes and its modulus attains its minimum, then once again f is constant.

As a final comment in this section we remark briefly on the topic of holomorphic functions in two complex variables. For our purposes, the following definition will suffice: a function $(z, w) \mapsto f(z, w)$ on an open subset of $\mathbf{C}^2$ is holomorphic if it is continuous and the partial mappings

$$z \mapsto f(z, w_0) \quad w \mapsto f(z_0, w)$$

are holomorphic for fixed z_0 and w_0. (In fact the continuity condition is redundant, due to the Theorem of Hartogs).

It then follows that if f is defined on a neighbourhood of a rectangle i.e. a set of the form $R_1 \times R_2$ where The R's are rectangles in the above sense, then we have

$$f(z, w) = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial R_1} \int_{\partial R_2} \frac{f(z', w')}{(z - z')(w - w')} \, dz' dw'$$

where (z, w) is from the interior of $R_1 \times R_2$.

This can be derived by iterating the one dimensional case. Note that since f is continuous, the above exists as a (four-dimensional) Riemann integral. (In fact, the formula is valid without the continuity condition it is regarded as an iterated Riemann integral. This is useful in the proof of the Hartogs result mentioned above).

From this formula one can deduce as in the one dimensional case, that a holomorphic function of two dimensions is infinitely differentiable (in the sense that all partial derivatives exists) and even analytic i.e. representable locally by a power seris.

3 $C(G)$ and $H(G)$ as locally convex spaces

We regard $C(G)$ (and hence its subspace $H(G)$) as locally convex spaces with the topology of compact convergence. Both are complete and, since the topology can be defined by a sequence of seminorms they are metric spaces (i.e. Fréchet spaces). This metric is defined as follows:

Let (K_n) be a squence of compact subsets so that for each n K_n is contained in the interior of K_{n+1} and so that $G = \bigcup K_n$. Then the metric is defined to be

$$d(f,g) = \sum \frac{1}{2^n} \frac{p_n(f-g)}{1+p_n(f-g)}$$

where p_n is the seminorm $\sup\{|f(z)|; z \in K_n\}$. This is the topology corresponding to the locally convex structure induced by the family of seminorms (p_n) (cf. the Lecture "Functional analysis").

Of course this can also be regarded as a locally convex structure on $C(G)$.

Examples. In the two cases whcih will interest us most, namely $G = \mathbf{C}$ resp. $G = D$, then the natural choices for (K_n) are

$K_n = \bar{U}(0,n)$ resp. $K_n = U(0, 1 - \frac{1}{n})$.

The details of the above facts are contained in the following exercise:

Examples. Let (X_n, d_n) be a sequence of metric space. Then the mapping

$$d(x,y) = \sum \frac{1}{2^n} \frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}$$

is a metric on the Cartesian product $X = \prod X_n$ (where a typical member of the latter is $x = (x_n)$. Show that this metric induces the natural topology and notion of convergence on X and that the product is complete if and only if each X_n (provided that the X_n are non-trivial.

(Particularly important examples of spaces which can be constructed natural in this way are

$2^{\mathbf{N}}$—the Cantor set;

$\mathbf{N}^{\mathbf{N}}$—the irrational numbers;

$[0,1]^{\mathbf{N}}$—the Hilbert cube.

We note the following property of sequential convergence in $H(G)$ the principle of whose proof will frequently be useful. A sequence (f_n) converges in $H(U)$ if and only if it converges locally (i.e. each point $z_0 \in G$ has a neighbourhood on which the sequence converges uniformly).

From this and the Cauchy integral formula (for the derivative) it is easy to see that the operation D of differentiation is continuous (as a linear operator on $H(U)$).

We remark that it follows from the Weierstraß theorem (on the denseness of polynomials) that $C(G)$ is separable. Hence $H(G)$ enjoys the same property (as a subspace of a separable metric space).

It follows from another theorem of Weierstraß (on the limits of convergent sequences of analytic functions) that $H(G)$ is a closed subspace of $C(G)$ (and so complete).

4 Properties of $C(G)$ and $H(G)$

An important property of $H(G)$ is that it is a so-called Montel space i.e. the theorem of Bolzano-Weierstraß is valid. (Note that the corresponding fact for a Banach space is always false, except for the trivial case of finite dimensional spaces).

Definition 13 *A subset B of a locally convex space (E, S) is* **bounded** *if and only if for each $p \in S$, $\sup_{x \in B} p(x) < \infty$.*

Hence in $H(G)$, a set is bounded if and only if it is uniformly bounded on compact subsets of G.

In a general locally convex space, each compact set is clearly bounded. A crucial property of $H(G)$ is that the converse holds:

Definition 14 *A subset of $H(U)$ is relatively compact if and only it is bounded.*

We remark that the classical formulation of this result is the famous theorem of Montel: if (f_n) is a sequence in $H(G)$ which is uniformly bounded on compact subsets of G, then we can extract from (f_n) a subsequence which converges uniformly on compacta. Hence, the concept of bounded in $H(G)$ almost corresponds to the classical concept of a normal family (only almost since in the classical case one allows for convergence to infinity. This can be accomodated in our abstract formulation if one considers functions with values in the extended complex plane, something which we shall not do here but see the course "Geometric function theory").

This is proved using several steps:

Firstly we quote the theorem of Ascoli characterising relatively compact subsets of $C(G)$. A subset of $C(G)$ is relatively compact if and only if it is bounded and equicontinuous.

We also use the following localisation properties:

A subset A of $H(G)$ is bounded if and only if it is locally bounded i.e. for each $z_0 \in G$ there is a positive ϵ so that A is uniformly bounded on $U(z_0, \epsilon)$.

A subset of $H(G)$ is relatively compact if and only if it is locally relatively compact i.e. for each $z_0 \in G$ there is a positive ϵ so that the restrictions of A to $U(z_0, \epsilon)$ are compect for the supremum norm (i.e. have a subsequence which converges uniformly there).

The result now follows form piecing together the above information and using the simple fact that it follows fom the Cauchy integral formula for the derivative that if a family is bounded on a disc then it is equicontinuous and so relatively compact on each smaller disc with the same centre.

EXAMPLES. Let A be a relatively compact subset of $H(G)$ (G connected). Show that if $f \in H(G)$ and (f_n) is a sequence in A, then $f_n \to f$ in $H(G)$ if one of the following conditions holds:

a) $f_n \to f$ pointwise on a non-trivial subset of G (Stieltjes' theorem);

or

b) $f_n \to f$ pointwise on a subset of G which has a cluster point.

5 Duality

Definition 15 *The dual E' of a locally convex space (E, S) is defined to be the space of continuous linear functionals on E. There are various ways of regarding this as a locally convex space (or convex bornological space) (see the course "Functional analysis") but we will skip the details in this course, while at some points calling on results from the general theory of duality for locally convexc spaces.*

We shall be interested in giving a concrete description of the dual of $H(G)$. We begin with some simple remarks on special elements of the dual. If $z \in G$, then $\delta_z : f \mapsto f(z)$ is an element of the dual. In fact, we have the following:

EXAMPLES. Let $L : H(G) \to \mathbf{C}$ be linear, multiplicative and non-zero. Then L has the form δ_z for some $z \in U$.

(Note that we do not require continuity in the above result).

Proposition 9 *The dual space of $H(G)$ is the space $H_0(G \setminus \hat{\mathbf{C}})$, the space of germs of functions on the complement of U which vanish at ∞.*

We remark here that if K is a closed subset of the extended complex plane, then we define the space of germs of analytic functions on K as follows. The family $H(U)$ of spaces indexed by the set of open neighbourhoods of K forms an inductive family (see the course "Functional analysis") and we define $H(K)$ to be its inductive limit. More concretely, an element of $H(K)$ is an equivalence class of analytic functions, each defined on an open neighbourhood of K, whereby two such "germs" are identified if they agree on an open neighbourhood of K. We remark that we can substitute $H(U)$ by the Banach space $H^\infty(U)$ of bounded analytic functions on U (with the supremum norm). This shows that $H(K)$ has a very special structure, that of a so-called Silva space (i.e. an inductive limit of a sequence of Banach spaces with compact linking maps) and this means that there is an abstract theory available which simplifies many of the follwoing considerations.

The concrete form of the above duality will be described below.

Before proving the general result, we begin by considering the case of the unit disc U where the proof is particularly transparent.

In this case we can identify a function f in $H(U)$ with the sequence (a_n) of its Taylor coefficients i.e. we have a vector space isomorphism

$$(a_n) \mapsto \sum_{n=0}^{\infty} a_n z^n$$

with corresponding inverse

$$f \mapsto \left(\frac{f^(n)(0)}{n!} \right)$$

between $H(U)$ and the sequence space E_1 consisting of those sequences (a_n) for which $\limsup |a_n|^{1/n} \leq 1$.

Examples. Show that the so-called α-dual of E_1 is

$$E_2 = \{(b_n) : \limsup |b_n|^{1/n} < 1\}$$

where the α-dual of a sequence space E is defined to be

$$E^\alpha = \{(b_n) : \sum a_n b_n \text{ converges for each } (a_n) \in E\}.$$

Now it follows from the Banach-Steinhaus theorem for Fréchet spaces (see below) that the dual of E is E^α whenever E is a suitable sequence space (cf. the proof that the dual of ℓ^p is ℓ^q from Analysis III). This means that E_1 and E_2 are in duality via the canonical bilinear mapping

$$(a, b) \mapsto \sum a_n b_n$$

as in the case of ℓ^p and ℓ^q.

Now we can identify E_2 again with the space of germs of analytic functions on the complement of the open unit disc which vanish at infinity via the correspondence

$$(b_n) \mapsto \sum_{n=0}^{\infty} b_n \left(\frac{1}{z}\right)^(n+1).$$

Hence if we combine these facts, we see that the above result holds for the case where G is the open unit disc. We remark in this case that the bilnear form which defines the duality can be described directly. For if $f(z) = \sum a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n \left(\frac{1}{z}\right)^(n+1)$ then

$$\sum a_n b_n = \frac{1}{2\pi i} \int_{c_r} f(z) g(z)\, dz$$

where c_r is any cirle in the joint domains of definition of f and a representant of g. The reader can also verify that if L is an element of the dual of $H(U)$, then the corresponding element g of $H(\hat{\mathbf{C}} \setminus U)$ is given by the formula

$$g(z) = L\left(\frac{1}{z-w}\right).$$

In order to prove the above result we use the following Lemma:

Lemma 1 *The topology of $H(U)$ can be generated by the family of seminorms $p_r(f) = \sum |a_n| r^n$ resp. $q_r(f) = \max |a_n| r^n$ with $0 < r < 1$.*

These are proved using simple estimates (involving the Cauchy inequalities) and show two equivalent ways of defining an intrinsic topology on E_1 so that the above correspondence is an isomorphism.

Using this result, we can use the theorem of Hahn-Banach to prove a famous approximation theorem of Runge.

Proposition 10 First version: *Let G be simply connected (i.e. so that $\hat{\mathbf{C}} \setminus G$ is connected. Then the polynomials are dense in $H(G)$.*

Proposition 11 Second version: *Let G be a domain. Then the family of rational functions whose poles are in the complement of G is dense in $H(G)$.*

We note that these results can be obtained in the above generality provided we use the duality theory in the corresponding generality. We note however that with the above simple version it is easy to prove the theorem of Runge for regions which are disjoint unions of discs.

In fact the proof provides a much finer version of this theorem, the statement of which is left as an exercise.

As examples of applications we consider the

Theorem 1 of Weierstraß *Let (z_n) be a sequence in $\mathbf{C}$ without cluster points in G. Then there is a function f in $H(G)$ with zero set $Z(f)$ equal to (z_n).*

We remark that we regard sequences with multiplicity, with the corresponding natural interpretation of the result and concept of a cluster point.

Theorem 2 Interpolation theorem: *Let (z_n) be a sequence as above (in this case without multiplicities), let (m_n) be a sequence of positive integers and (a_n^i) $(n \in \mathbf{N}, 0 \leq i \leq m_n$ a doule sequence of complex numbers. Then there exists an $f \in H(G)$ with $f^{(i)}(z_n) = a_n^i$.*

Examples. In order to prepare for the proof in the general case, we invite the reader to prove the following: Let $((z_n)$ be a sequence in $\mathbf{C}$ which converges in absolute value to infinity in such a way that $\sum \frac{1}{|z_n|} < \infty$. Then the infinite product

$$f(z) = \prod \left(1 - \frac{z}{z_n}\right)$$

converges absolutely and uniformly on compact sets to a solution of the interpolation problem.

The general solution can also be obtained by means of an infinite product. For completeness we recall the method: if (z_n) is a sequence as above then there there is a sequence of polonomials p_n so that the infinite product

$$f(z) = \prod \left(1 - \frac{z}{z_n}\right) d^{p_n(z)}$$

converges as above and solves the interpolation problem.

For the next result, we remark that $f : G \to \mathbf{C}$ is meromorphic if it is analytic on $G \setminus \{z_1, z_2, \dots\}$ where (z_n) is a sequence as above and each z_i is a pole of f. Typical examples of such functions are quotients of analytic functions.

Theorem 3 Mitteg-Leffler's theorem: *Let (z_n) be as above and let (p_n) be a sequence of polynomials. Then there exists a mermorphic function f on G so that the sequence $P(f)$ of poles of f is precisely (z_n) and for each n the principal part of f at z_n is $p\left(\frac{1}{z-z_n}\right)$.*

Corollar 1 *If f is meromorphic on G, then there exist g and h in $H(G)$ so that $f = \frac{g}{h}$.*

We shall use these to prove a less classical result:

Proposition 12 *Each closed ideal of $H(G)$ is a principal ideal.*

PROOF. **Sketch of Proof** To each f in the ideal I we assoociate the sequence (z_n, m_n) of zeros with multiplicity. Then we define the "intersection" of these sequences in the natural way (i.e. it consists of the sequence of elements which occur in all such zero sets, whereby the corresponding multiplicity is the infimum of all its mulitplicity). Then by the above we can find a function g which has precisely this sequence with multiplicities as its zero set. Then one can show that $I = gH(U)$.

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EXAMPLES. Show that if f and g are in $H(G)$ and such that $f|g$ resp. $g|f$ then there is a function $h \in H(G)$ so that $f = ge^h$. (This shows that the g in the above result is unique up to a facto of a function of the form e^h). (Hint show that if $f \in H(G)$ has no zeros, then $f = ce^g$ where g is a primitive of $\frac{f'}{f}$).

PROOF. Proof that the theorem of Mittag-Leffler follows from the interpolation theorem We retrieve the required meromorphis function as a quotient $\frac{g}{h}$ where g and h are the solutions of suitable interpolation problems. For at a zero we have an equation of the form

$$\frac{g(z)}{h(z)} = \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \frac{c_l}{z^l} + \phi(z)$$

where ϕ is analytic.

If we suppose that $h(z) = z^l(1 + k(z))$ where k is analytic and $k(0) = 0$, say $k(z) = z^m\psi(z)$, then we get:

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PROOF. **of Weierstraß' theorem from Mittag-Leffler and the interpolation theorem** Firstly it follows from the interpolation theorem that we can find an f with the required sequence as zeroes (with suitable multiplicity). The problem is that it may have more zeroes. We consider the the reciprocal function $\frac{1}{g}$. We can construct a mermorphic function h which has the same principal parts at the superfluous zeros. Then the reciprocal of $\frac{1}{g} - h$ is the required function.

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Hence it follows that if we can produce a simple functional analytic proof of the interpolation theorem then the results of Mittag-Leffler and Weierstraß will be relatively simple consequences.

In order to prove these theorems (the duality theorem and the interpolation theorem) we use versions of the theorems of Banach which are valid for certain types of locally convex spaces:

The Hahn-Banach theorem: Let (E, S) be a locally convex space, F a subspace and f a member of the dual of F. Then there is an element g of the dual of E whose restriction to F is f.

As in the case of Banach spaces, this has the Corollary that a subspace A of a locally convex space E is dense if and only if each element of the dual which vanishes on A is zero.

The theorem of Banach Steinhaus: Let (E, S) and (F, S_1) be Fréchet spaces, (T_n) a sequence of continuous linear operators from E into F which is pointwise convergent. Then the pointwise limit is also continuous.

In the case where the image space is one-dimensional i.e. the T_n are from the dual, then we can also deduce that the sequence is equicontinuous i.e. the sequence is dominated by a single continuous seminorm.

The closed graph theorem: Let (E, S) and (F, S_1) be Fréchet spaces. Then a linear operator $T : E \to F$ is continuous if and only if its graph is closed.

The isomorphism theorem: Let E and F be as above. Then a continuous linear bijection from E onto F is an isomorphism.

The open mapping theorem. Let $T : E \to F$ be a continuous linear mapping where E and F are as above. Then if T is surjective, it is open.

As a corollary, we have that $T : E \to F$ is onto if and only if its adjoint T' has the following property: if B is a bounded subset of F', then $T^{-1}(B)$ is bounded in E'.

We now remark briefly on how to prove the duality theorem in the general case. We remark that the non-trivial part is to define an analytic function associated to a form $L \in H(G)'$. We remark that formally we would like to define

$$g(z) = L\left(\frac{1}{z-w}\right).$$

However, this only works for z in the complement of U. In order to exend it to within U we requre the following more delicate argument. L is an element of the dual of $H(U)$ which we regard as a subspace of $C(U)$. By the Hahn-Banach theorem we can extend L to an element of the dual of $C(U)$ which, by an abuse of notation we continue to denote by L. By the Riesz representation theorem there is a Radon measrue μ with support in a compact subset of U so that $L(f) = \int_K f\, d\mu$. Hence we can now define $g(z) = \int_K \frac{1}{z-w}\, dmu(w)$ and this works for $z \in \hat{\mathbf{C}} \setminus K$. The rest of the details we will omit.

At this point, we remark that there are three natural topologies on the dual of a locally convex space, in particular on $H(U)'$, the weak topology, the strong topology and the Mackey topology. In this case the latter two agree and also coincide with the Silva topology defined on the space $H_0(\hat{\mathbf{C}} \setminus U)$

mentioned above. Also the sequential convergence induced from any of the four structures coincides. This all follows form what we have proved and from the general theory of locally convex spaces, in particular, Silva spaces. For our purposes we spell out the following consequences:

I. The duality between $H(U)$ and $H_0(\hat{\mathbf{C}} \setminus U$ is completely symmetrical i.e. each of the spaces is, as a locally convex space, the strong dual of the other via the blinear form generated by the Cauchy integral formula.

II. Both $H(U)$ and it's dual $H_0(\hat{\mathbf{C}} \setminus U$ are complete.

III. If K is a closed subset of $\bar{\mathbf{C}}$ then a sequence f_n converges to f in $H(K)$ if and only if there is an open neighbourhood U of K so that each of f_n and f can be extended analytically fo U and the resulting functions converge uniformly there.

Proof. **of the interpolation theorem:** We now sketch the proof of the interpolation theorem. In order to simplify the notation, we confine attention to the case where G is the unit disc D and the interpolation has the simple form $f(a_n) = b_n$ i.e. where we do not interpolate derivatives.

We consider the linear mapping

$$L : f \mapsto (f(a_n))$$

from $H(D)$ into $\mathbf{C}^n$. It suffices to show that this mapping is surjective and so, by the open mapping theorem, that the adjoint mapping has the property that the pre-images of bounded sets are bounded. This is not difficult.

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Examples. Give an alternative proo of the above result as follows. Consider the linear space in the dual of $H(D)$ spanned by the functions of the form $\frac{1}{a_n - w}$. Define a functional L on this space by putting

$$L\left(\frac{1}{a_n - w}\right) = b_n.$$

Show that L is continuous by showing that its kernel is closed (this is the non-trivial part). Now use the Hahn-Banach theorem.

We close this course by indicating how the injection of some functional analysis can lead to a streamlined approach to the standard results of complex function theory. We remark that using only the Cauchy integral theorem for rectangles we succed in obaining the general duality theorem. From this we can deduce the Runge theorem in its general form. From the Runge theorem we can dedcue the general form of the Cauchy integral theorem. This then allows us to deduce in the usual way the theorem on maximal modulus, Rouché's theorem, Hurwitz' theorem etc. We then showed how to deduce the interpolation theorem using the epimorhism theorem or directly from the duality theorem. This then can be used to deduce the Weierstraß theorem and Mittag-Leffler's theorem.

In order to state the general form of Runge's theorem we introduce some notation. We have a domain G. A set with multiplicity is a pair (E, m) where E is a subset of $\hat{\mathbf{C}}$ and m is a function from E into $\mathbf{N}_0 \cup \{\infty\}$. Then $\mathcal{F}(\mathcal{E})$ is the set of functions from the following list:

if $z_0 \in E \cap \mathbf{C}$ and $m(z_0) < \infty$, then we include the function $\dfrac{1}{z - z_0}$;

if $z_0 \in E \cap \mathbf{C}$ and $m(z_0) = \infty$, then we include the functions $\dfrac{1}{z - z_0}, \quad \dfrac{1}{(z - z_0)^2} \dots$;

if $z_0 \in E$, $z_0 = \infty$, then we include the functions $z, z^2, z^3, \dots$.

Theorem 4 *Runge's theorem Let (E, m) be a set with multiplicity as above and suppose that for each component G of $\hat{\mathbf{C}} \setminus U$ the following holds:*

either there is a $z_0 d \in E$ with $m(z_0) = \infty$ in G

or there is a sequence (z_n) of elements with finite multiplicity in E in G with cluster point in G.

Then the linear hull of $\mathcal{F}(\mathcal{E}) \cup \{\infty\}$ is dense in $H(U)$.

PROOF. The proof is a simple application of the theorem of Hahn-Banach. It suffices to show that each element of the dual which vanishes on the above family of functions vanishes. But if we regard such an element as a holomorphic function on the complement of G, then the above conditions are sufficient to ensure that the function vanishes (by the identity theorem) on each component of the latter set.

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From the above it is easy to deduce any of the classical versions of the theorem of Runge. In particular one sees that if G is a simply connected open subset of $\mathbf{C}$ (i.e. such that its complement in $\hat{\mathbf{C}}$ is connected), then the polynomials are dense in $H(G)$. We now round off the course by showing how we can use this result to get one of the most general version of the Cauchy integral theorem:

Cauchy-integral for simply connected domains. Let G be simply connected. Then $\int_c f(z)\, dz = 0$ for any $f \in H(G)$ and any closed curve in G.

PROOF. Note that the theorem holds trivially for polynomials (since they have primitives). Hence by an approximation argument, it holds for any function in $H(G)$.

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