# Riesz spaces

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## 1 Introduction

## 2 Banach lattices

In this final chapter we discuss Banach lattices. These are Banach spaces with an ordering related to the norm structure. Of course, the classical function spaces C(K),  $L^p(\mu)$ ,  $\ell^p$  which we have studied up till now have such a structure (with the pointwise, resp. coordinatewise ordering).

In the first section we bring a concise introduction to the basic theory of ordered sets with very brief proofs. In section 2 we discuss the theory of partially ordered vector space and Riesz spaces (i.e. partially ordered vector spaces which are lattices with repect to their ordering). Finally, in section 3 we introduce and study Banach lattices.

### 2.1 Ordered sets

**1.1. Definition** Recall that a **partial ordering** on a set X is a bineary relation  $\leq$  on X so that

- $x \le x$   $(x \in X);$
- if  $x \leq y$  and  $y \leq z$  then  $x \leq z$   $(x, y, z \in X)$ ;
- if  $x \leq y$  and  $y \leq x$ , then x = y  $(x, y \in X)$ .

A partially ordered set is a pair  $(X, \leq)$  where  $\leq$  is a partial ordering on X. We use the abbreviation "poset" for a partially ordered set. A poset  $(X, \leq)$  is totally ordered if for each pair x, y in X either  $x \leq y$  or  $y \leq x$ . It is directed if for each pair x, y in X there is a  $z \in X$  with  $x \leq z$  and  $y \leq z$ . The following are some simple examples of posets.

A. The relation of equality on a set X is a partial ordering.

B. If P(X) denotes the family of all subsets of a given set X, then the relation of inclusion is a partial ordering on P(X).

C. The space  $\mathbf{R}^X$  of real-valued functions on a given set X has a natural paral ordering defined by

$$x \le y \iff x(t) \le y(t) \quad (t \in X)$$

(this is called the pointwise ordering).

Note that every subset of a given poset is automatically a poset. Thus the set of all open subsets of a topological space, the set of measurable subsets of a measure space etc. are in a natural way posets. Similarly, the spaces C(K),  $\mathcal{S}(\mu)$  resp.  $\mathcal{L}^{\checkmark}$  are posets.

**1.2.** Exercises A. Let X be a set and R a binary relation on X which satisfies conditions 1) and 2) of definition 1.1 (such a relation is called a **pre-order**). Show that the the relation

$$x \sim y \iff xRy \text{ and } yRx$$

is an equivalence. relation. Show that the relation  $\hat{R}$  defined by

 $A\tilde{R}B \iff$  there exist  $x \in A, y \in B$  with xRy

is a partial ordering on the family  $X_{\sim}$  of equivalence classes.

(This is a convenient way of dealing with situations where condition 3) fails. Elements which cannot be distinguished by the ordering are identified with each other).

B. Let A be a ring with identity. Show that the relation

$$xRy \iff$$
 there is a  $z \in A$  with  $x = yz$ 

is a pre-oder on R. What is the equivalence relation defined in A. above?

If X and Y are posets, a mapping between X and Y is **isotone** if  $f(x) \leq f(y)$  whenever  $x \leq y$ . Such an f is a (poset) **isomorphism** if it is a bijection and its inverse  $f^{-1}$  is also isotone.

**1.3. Exercise** Let  $(X, \leq)$  be a poset. If  $x \in X$ , define

$$L_x = \{ y \in X : y \le x \}.$$

Show that  $x \leq y$  if and only if  $L_x \subset L_y$ . Deduce that X is isomorphic to a subset of P(X) (with the natural ordering of inclusion).

**1.4.** Definition Let  $(X, \leq)$  be a poset.  $a \in X$  is maximal if for each  $x \in X, x \geq a$  implies x = a. Similarly, a is maximal for a subset A of X if a is maximal for A with the induced ordering.  $a \in X$  is an upper bound for A if for each  $x \in A, x \leq a$ . If, in addition,  $a \in A$ , then a is called a greatest element for A. a is called the supremum of A in X (written  $\sup_X A$  or simply  $\sup A$ ) if a is a least element of the set of all upper bounds of A.

Lower bounds, least elements and infima are defined in the obvious way.

The following construction is often useful: let X be a poset without a greatest element. Then we can embed X in a natural way into a poset with a greatest element as follows: Let  $X_{\infty} = X \cup \{\infty\}$  (i.e. we are adding an ideal "element at infinity" to X). On  $X_{\infty}$  we define a partial ordering as follows:

if x, y are in X then the order relation between x and y remains unchanged. All elements of X are smaller than  $\infty$ .  $X_{\infty}$  is the **poset obtained from** X **by adding a greatest element**. Similarly, if X has no smallest element, we can embed it into a poset  $X_{-\infty}$  with smallest element.

In general we write  $\tilde{X}$  for the poset obtained by adding a largest and smallest elements (of course, if X already has one or both of these, we refrain from adding it or them).

**1.5.** Exercises A. In this exercise we show how to define products of posets. If  $\{X_{\alpha}\}_{\alpha \in A}$  is a family of posets we define an ordering on the product space as follows

$$(x_{\alpha}) \leq (y_{\alpha})$$
 if and only if  $x_{\alpha} \leq y_{\alpha}$  for all  $\alpha$ .

Show that this is a partial ordering. When is it a total ordering resp. directed?

B. Show that a set can have at most one greatest elements (so that we can talk of *the* greatest element) and hence at most one supremum.

C. A poset X satisfies the **ascending chain condition** (abbreviated ACC) if for each sequence  $(x_n)$  in X with  $x_n \leq x_{n+1}$  for each n, there is an  $N \in \mathbb{N}$  so that  $x_n = x_N$  for  $n \geq N$ . Show that this implies that X has a maximal element.

D. The following exercise shows that if Y is a subset of a poset X and  $A \subset Y$  then it can happen that both  $\sup_Y A$  and  $\sup_X A$  exist but are distinct (so that the subscript X is not completely superfluous). X is the poset C([0, 1]) with the pointwise ordering and Y is the subset of X consisting of the affine functions i.e. those of the form  $t \mapsto at + b$   $(a, b \in \mathbf{R})$ . Then if A is the set consisting of the functions t resp. 1 - t, the suprema of A in Y and X both exist but are distinct.

E. A **chain** in a poset X is a subset which is totally ordered in the induced ordering. Note that Zorn's Lemma can be stated in the following form. If every chain in a poset X has an upper bound, then X has a maximal element. Show that every chain in a poset is contained in a maximal chain i.e. a chain which is not properly contained in a larger chain.

F. Let  $\{x_{(\alpha,\beta)}\}$  be a set of elements in a poset X indexed by the set  $A \times B$ . Suppose that for each  $\alpha inA \ y_{\alpha} = \sup\{x_{(\alpha,\beta)} : \beta \in B\}$  exists. Show that  $\{x_{(\alpha,\beta)}\}$  has a supremum if and only if  $\{y_{\alpha}\}$  has and that in this case

$$\sup_{(\alpha,\beta)\in A\times B} \{x_{(\alpha,\beta)}\} = \sup_{\alpha\in A\}\{y_{\alpha}\}.$$

G. Let  $(X_n)$  be a sequence of posets. On  $X = \prod_{n \in \mathbb{N}} X_n$  we define a partial ordering, distinct from that of B above, called the **lexicogrphic ordering** 

for obvious reasons:  $(x_n) \leq (y_n)$  if and only if they are equal or  $x_k < y_k$ where k is the first integer for which  $x_k \neq y_k$ .

Show that X is totally ordered if each  $X_n$  is.

We come to the important concept of a lattice i.e. a poset with finite suprema and infima:

**1.6. Definition** A **lattice** is a poset X in which every set with two elements has a supremum and an infimum. Examples of lattices are  $\mathbf{R}^X$  (X a set), C(K) (K compact), P(X) (X a set).

 $x \lor (y \land z) \le (x \lor y) \land (x \lor z) \quad (x, y, z \in X).$ 

**PROOF.** For the first statement we have the inequalities  $x \wedge y \leq x$  and  $x \wedge y \leq y \leq y \vee z$ . Hence  $x \wedge y \leq x \wedge (y \vee z)$ . Similarly  $x \wedge z \leq x \wedge (y \vee z)$  and the result follows. The second part is proved similarly.

**1.12. Definition** A lattice  $(X, \leq)$  is **distributive** if equality holds in the above expressions, that is, if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad (x, y, z \in X)$$

resp.

$$x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad (x, y, z \in X).$$

If S is a set, the lattices P(S) and  $\mathbf{R}^{S}$  are distributive. As an example of a non-distributive lattice we consider the poset with five elements, a greatest element 1 and a smallest element 0 together with three further elements x, y, z which are not related to each other by the order but all lie between 0 and 1.

**1.13.** Exercise A. Show that a totally ordered lattice is distributive. B. Show that if  $(X, \leq)$  is a distributive lattice then the following "cancellation law" holds:

$$(z \land x = z \lor y)$$
 and  $(z \lor x = z \lor y)$  implies  $x = y \quad (x, y, z \in X).$ 

**Proposition 1** 1.14. Proposition Let  $(X, \leq)$  be a distributive lattic with 0 and 1. Then each  $x \in X$  possesses at most one complement.

**PROOF.** Suppose that y and  $y_1$  are elements of X such that

$$x \lor y = 1, x \land y = 0, x \lor y_1 = 1, x \land y_1 = 0.$$

Then

$$y_1 = y_1 \land 1 = y_1 \land (x \lor y) = (y_1 \land x) \lor (y_1 \land y) = 0 \lor (y_1 \land y) = y_1 \land y$$

and so  $y = y_1$  by symmetry.

1.15. Definition A lattice X is complete (resp.  $\sigma$ - complete) if each subset (resp. each countable subset) has a supremum and infimum. X is **Dedekind complete** if each subset with an upper bound (resp. lower bound) has a supremum and an infimum. Dedekind  $\sigma$ -completeness is defined in the obvious way. For example, if S is a set, P(s) is complete and  $\mathbf{R}^{S}$  is Dedekind complete but not complete. Top (S) is complete. The space C([0, 1]) is not Dedekind complete.

#### **1.16. Exercise** Show that

- any complete lattice has a 0 and 1;
- if X is a lattice, X the lattice obtained by adding a zero and 1, then X is Dedekind complete if and only if X is complete;
- a lattice is complete if and only if it is Dedekind complete and has a 0 and 1.

In fact, in the definition of completness (resp. Dedekind completeness) it suffices to demans that suprema exist. For example if X is a lattice in which every nonempty set with an upper bound has a supremum, then X is Dedekind complete. For if B is a subset with a lower bound. Denote by  $\tilde{B}$  the set of lower bounds for B. Then  $x = \sup \tilde{B}$  exists and is an infimum for B by definition.

Many concrete lattices appear in the form of lattices of suitable subsets of a given set. Most of the useful cases can be subsumed in the following general scheme:

**1.18. Definition** Let S be a set. A Moore family on S is a subfamily  $\mathcal{F}$  of P(S) which satisfies the following conditions:

•  $S \in \mathcal{F}, \emptyset \in \mathcal{F};$ 

• if  $\mathcal{A}$  is a subfamily of  $\mathcal{F}$ , then  $\bigcap \mathcal{A} \in \mathcal{F}$ .

We regard  $\mathcal{F}$  as a poset with the inclusion ordering. Examples of Moore families are:

- Cl(S)—the family of closed subsets of a topological space S;
- SS(V)—the family of subspaces of a vector space V;
- ClSS (E)—the family of closed subspaces of a Banach space E;
- SubG (G)—the family of subgroups of a group G;
- Conv (V)—the family of convex subsets of a vector space V.

We now claim that a Moore famaily  $\mathcal{F}$  is a lattice i.e. that  $A \vee B$  and  $A \wedge B$  exist for  $A, B \in \mathcal{F}$ . (Note that we de not claim that it is a sublattice of P(S)—the above examples show that this is not the case in general).

The existence of  $A \wedge B$  is obvious—we simply take the intersection  $A \cap B$  of A and B. For the supremum, we define

$$A \lor B = \cap \{ C \in \mathcal{F} : \mathcal{A} \cup \mathcal{B} \subset \mathcal{C} \}.$$

Then one can check that  $A \lor B$  is in fact a supremum for  $\{A, B\}$ .

Exactly the same reasoning provides infima and suprema for arbitrary subfamilies of  $\mathcal{F}$  and hence  $\mathcal{F}$  is a complete lattice.

**1.19.** Exercise Calculate the suprema of two elements A and B in the Moore families listed above.

Note that since the supremum in a Moore family need not be the set theoretical union, such a lattice may fail to be distributive as the next exercise shows.

**1.20.** Exercise Show that the lattice of subgroups of an abelian group need not be distributive (consider the subgroups  $G_1$ ,  $G_2$ ,  $G_3$  of  $G = \mathbb{Z} \times \mathbb{Z}$  where  $G_1$  is generated by (1, 0),  $G_2$  by (0, 1) and  $G_3$  by (1, 1).

We now consider the possibility of embedding a poset in a complete one. Some of the preliminaries are taken care of in the next exercise: **1.21.** Exercise Let X be a poset with greatest element. If  $A \subset X$ , define L(A) to be the set of lower bounds for A, U(A) to be the set of upper bounds. A is said to be saturated if A = L(U(A)). Show that

- if  $A \subset X$ , then L(U(A)) is saturated;
- if  $x \in X$ , then  $L_x$  is saturated;
- the family of saturated sets is a Moore family.

**Proposition 2** 1.22. Proposition Let X be a poset. Then X can be embedded into a complete lattice  $\hat{X}$  so that

- if  $A \subset X$  is such that  $\sup_X A$  exists, then  $\sup_X A = \sup_{\hat{X}} A$ ;
- the analogous condition for infima holds;
- $x = \sup_{\hat{X}} \{ y \in X; y \le x \}$   $(x \in \hat{X}).$

PROOF. We can assume that X has a greatest element. Then  $\hat{X}$ , the family of saturated subsets of X, being a Moore family, is complete and the mapping  $x \mapsto L_x$  embeds X into  $\hat{X}$ . It remains to show that the above three properties hold.

Suppose firstly that  $A \subset X$  and  $x_0 = \sup_X A$  exists. We must show that  $L(U(\bigcup_{x \in A} L_x)) = L_{x_0}$ . Clearly,  $U(A) = U(\bigcup_{x \in A} L_x)$  and so  $x_0$  is the smallest element of  $U(\bigcup_{x \in A} L_x)$ . Hence  $L(U(\bigcup_{x \in A} L_x)) = \{y : y \leq x_0\} = L_{x_0}$ .

For the second part suppose that  $A \subset X$  and  $x_0 = \inf_X A$  exists. We must show that  $\bigcap_{x \in A} L_x = L_{x_0}$  and this is obvious.

For the third part suppose that A is saturated. We show that  $A = \sup_{\hat{X}} \{L_x : x \in A\}$  i.e.  $A = L(U(\bigcup_{x \in A} L_x))$ . But this follows immediately from the equation A = L(U(A)) and  $U(A) = U(\bigcup_{x \in A} L_x)$ .

Using the order structure of a poset, we can define a notion of convergence of nets:

**1.23.** Definition A net  $(y_{\alpha})_{\alpha \in A}$  in a poset X is increasing (resp. decreasing) if whenever  $\alpha \leq \beta$ , then  $y_{\alpha} \leq y_{\beta}$  (resp.  $y_{\alpha} \geq y_{\beta}$ ). A net  $(y_{\alpha})$  increases to y if  $(y_{\alpha})$  is increasing and y is its supremum.  $y_{\alpha}$  decreases to y is defined similarly. A net  $(x_{\alpha})_{\alpha \in A}$  is said to be order convergent to x (written  $x_{\alpha} \to x$ ) if there are nets  $(y_{\beta})_{\beta \in B}$  and  $(z_{\gamma})_{\gamma \in \gamma}$  so that

•  $(y_{\beta})$  increases to x;

- $(z_{\gamma})$  decreases to x;
- for each  $\beta \in B$ ,  $\gamma \in \gamma$ , there is an  $\alpha_0 \in A$  so that  $z_{\gamma} \leq x_{\alpha} \leq y_{\beta}$  for  $\alpha \geq \alpha_0$ .

**Proposition 3** • *if*  $(x_{\alpha})$  *is increasing then*  $x_{\alpha} \to x$  *if and only if* x *is the supremum of the net;* 

- if  $(x_{\alpha})_{\alpha \in A}$ ,  $(x'_{\beta})_{\beta \in B}$  are nets with order limits x and x' resp. and if for each  $\alpha \in A$  there is a  $\beta_0 \in B$  so that  $x'_{\beta} \ge x_{\alpha}$  for  $\beta \ge \beta_0$ , then  $x \le x'$ ;
- A net  $(x_{\alpha})$  can have at most one order limit;
- if  $(x_{\alpha})$  is order convergent to x then so is each subnet.

Proof.

(i) Suppose that  $(y_{\beta})$  and  $(z_{\gamma})$  are as in the definition. For  $\beta \in B$ , there is an  $\alpha_0$  in A so that  $x_{\alpha} \leq y_{\beta}$  if  $\alpha \geq \alpha_0$ . Hence if  $\alpha \in A$  and  $\alpha' \in A$  are such that  $\alpha' \leq \alpha$  and  $\alpha' \geq \alpha_0$ , then  $x_{\alpha'} \leq x_{\alpha} \leq y_{\beta}$  and so  $x_{\alpha} \leq y_{\beta}$  for each  $\alpha$  and  $\beta$ . Thus  $x_{\alpha} \leq \inf\{y_{\beta}\} = x$  for each  $\alpha$ . On the other hand, if z is an upper bound for  $\{x_{\alpha}\}$  then z is also one for  $\{z_{\gamma}\}$  and so  $z \geq \sup\{z_{\gamma}\} = x$ .

We leave the remainder of the proof as an exercise.

**1.25.** Exercise Let X be a complete lattice and define, for a net  $(x_{\alpha})_{\alpha \in A}$ 

$$\liminf \{x_{\alpha}\} = \sup_{\beta \in A} \inf \{x_{\alpha} : \alpha \ge \beta\}\}$$
$$\limsup \{x_{\alpha}\} = \inf_{\alpha = A} \sup \{x_{\alpha} : \alpha \ge \beta\}\}.$$

Show that  $x_{\alpha} \to x$  if and only if  $= \liminf(x_{\alpha}) = \limsup(x_{\alpha})$ . Deduce that in an arbitrary lattice,  $x_{\alpha} \to x$  if and only the same condition holds, the lim inf and lim sup being taken in the completion  $\hat{X}$ .

**1.26.** Exercise A. Let  $(X, \leq)$  be a lattice. A subset  $A \subset X$  is defined to be **closed** if it contains the limit of each order convergent net in A. Show that this defines a topology on X (i.e. the set of all subset of X whose complements are closed in this sense is a topology). Prove that this topology is  $T_1$ . Show that if  $x_{\alpha} \to x$  in X for the order, then  $x_{\alpha} tox$  for the topology (the converse is not true in general).

B. Consider the sequence  $(t^n)$  in C([0, 1]). Show that it decreases to zero in this lattice, but does not converge pointwise to zero.

C. If  $(X, \leq)$  is a lattice with zero elements 0, a non-zero  $x \in X$  is an **atom** if each  $y \in X$  with  $0 \leq y \leq x$  is either 0 or x. Identify the atoms in the lattices P(S),  $\mathbf{R}^S$ , SS (V), ClSS (E) from above. A lattice is said to be **atomic** if each  $x \in X$  is the supremum of the atoms which are smaller than x. Which of the above lattices are atomic?

D. Show that if S and T are  $T_1$  topological spaces so that  $\operatorname{Cl}(S)$  and  $\operatorname{Cl}(T)$  are lattice isomorphic, then S and T are homeomorphic. (Let  $\Phi$  be a lattice isomorphism from  $\operatorname{Cl}(S)$  onto  $\operatorname{Cl}(T)$ . If  $s \in S$ , then the corresponding singleton is an atom and hence so is its range. This defines a mapping from S into T. Show that this is the required homeomorphism).

E. Let  $\{X_{\alpha}\}_{\alpha \in A}$  be a family of lattices. Show that their product is complete if and only if this holds for each  $X_{\alpha}$ . Prove the same result for the properties: Dedekind complete, distributive, complemented. F. Let X be a lattice with ACC and DCC (i.e. each descending sequence is eventually stationary). Show that X is complete. Show that if it has one of these properties, then it is Dedekind complete. Show that if X is a lattice with ACC and a is the supremum of a subset A of X, then there is a finite subset of A whose supremum is a.

G. Let f be an isotone mapping from a complete lattice into itself. Then f has at least one fixed point (consider the supremum of the set of those x for which  $x \leq f(x)$ .

We now turn to one of the most important topics in the theory of lattices that of Boolean algebras. These are lattices which possess those properties of P(S) which are relevant to the most elementary level of set theory.

**1.27. Definition** A Boolean algebra is a distributive lattice  $(X, \leq)$  with a zero and a unit in which each element has a complement. Then by 1.14 the complement is uniquely determined and so we can introduce the notation x' for the complement.

If S is a set, P(S) is a Boolean algebra. Similarly, if S is a topological space, Clopen (S), the set of subsets of S which are simultaneously open and closed, is a Boolean algebra under the partial ordering induced from P(S).

**Proposition 4** If x and y are elements of the Boolean algebra X, then

- (x')' = x;
- $x \wedge y = 0 \iff y \le x';$
- $x \le y \iff x' \ge y'$ .

Proof.

We prove the second statement. Suppose that  $x \wedge y = 0$ . Then

$$y = y \land 1 = y \land (x \lor x') = (y \land x) \lor (y \land x') = 0 \lor (y \land x') = y \land x'$$

and so  $y \leq x'$ .

On the other hand, if  $y \leq x'$ , then  $x \wedge y \leq x \wedge x' = 0$ . For the third part, it follows from the second one that

$$x \le y \iff x \le (y')' \iff x \land y' = 0 \iff y' \land x = 0 \iff y' \le x'.$$

Corollar 1 •  $(x \lor y)' = x' \land y'$  and  $(x \land y)' = x' \lor y';$ 

• if  $A \subset X$  has a supremum, then  $(\sup A)' = \inf\{y' : y \in A\}$ .

**Proposition 5** (the infinite distributive law) If  $x \in X$  and  $A \subset X$  has a supremum, then  $x \land (\sup A) = \sup(x \land A)$ .

**PROOF.** For each  $y \in A$ ,  $x \wedge y \leq x \wedge \sup A$  and so  $x \wedge \sup A$  is an upper bound for  $x \wedge A$ . Suppose that z is also an upper bound. Then for each  $y \in A$ ,

$$z \lor x' \ge (x \land y) \lor x' = (x \lor x') \land (y \lor x') = y \lor x' \ge y$$

and so  $z \lor x' \ge \sup A$ . Hence

$$z = z \lor 0 = z \lor (x \land x') = (z \lor x) \land (z \lor x') \ge (z \lor x) \land \sup A \ge x \land \sup A$$

and so  $z \ge x \land \sup A$ . Hence  $x \land \sup A$  is the smallest upper bound for  $x \land A$ .

**Corollar 2**  $x \lor (\inf A) = \inf(x \lor A).$ 

**1.32.** Exercises A. Let S be a topological space. A subset A of S is regularly closed if it is the closure of an open set. Show that the family of such sets is a Boolean algebra under the natural ordering but that infimum and complements of elements do not, in general, coincide with the set theoretical intersection or complement.

B. A **Boolean ring** is a ring (R, +, ., e) with unit e so that  $x \cdot x = x$   $(x \in R)$ . Show that x + x = 0  $(x \in R)$  and that R is commutative. Show that if  $(X, \leq)$  is a Boolean algebra, then x with the operations

$$: (x, y) \mapsto (x \land y') \lor (y \land x') \tag{1}$$

$$: (x,y) \mapsto x \wedge y \tag{2}$$

is a Boolean ring. On the other hand, if (R, +, ., e) is a Boolean ring, then R under the operations

$$\lor : (x, y) \mapsto x + y - x.y \tag{3}$$

$$\wedge : (x, y) \mapsto x.y \tag{4}$$

is a Boolean algebra. Thus the concepts of a Boolean algebra and a Boolean ring are essentially equivalent.

Our main purpose in the remaining part of this chapter will be to prove a famous result of Stone, namely that every Boolean algebra X is representable in the form Clopen (S) for some topological space S. To do this we introduce some new concepts:

**1.33.** Ideals Let X be a Boolean algebra. A subset A of X is normal if for each  $y \in A$ ,  $x \leq y$  implies  $x \in A$   $(x \in X)$ . It is an ideal if it is normal and closed under  $\lor$ . In particular, a normal subset (and so an ideal) is clased under  $\land$ . As usual, the intersection of a family of ideals is an ideal and so there is a smallest ideal I(A) containing any subset A of X (it is called the ideal generated by A). An ideal is maximal if is proper (i.e.  $\neq X$ ) and not properly contained in another ideal other than X itself. Note that an ideal is proper if and only if it does not contain the unit.

An example of an ideal is  $[0, x] = \{y \in X : 0 \le y \le x\}$  for  $x \in X$ .

The following characterisation of ideals generated by normal subsets will be useful later:

**Proposition 6** 1.34. Proposition Let A be a normal subset of X. Then I(A), the ideal generated by A, consists of the set  $\{\sup F : J \in \mathcal{J}(A)\}$  of those elements which are the suprema of finite subsets of A.

**PROOF.** The set on the right hand side is clearly contained in any ideal containing A and so is contained in I(A). Conversely, since A is normal, it is an ideal and so contains I(A).

**Corollar 3** Let A be a normal subset of a Boolean algebra. Then I(A) is proper if and only I(J) is for each  $J \in \mathcal{J}(\mathcal{A})$ .

**Proposition 7** Every proper ideal is contained in a maximal ideal.

The proof is a typical application of Zorn's Lemma.

**Corollar 4** Let x be an element of the Boolean algebra X with  $x \neq 1$ . Then x is contained in a maximal ideal.

PROOF. Apply 1.36 to [0, x].

**Proposition 8** Let I be a maximal ideal in the Boolean algebra X. Then if  $x \in X, x \in I$  or  $x' \in I$ .

**PROOF.** Suppose that  $x \notin I$ ,  $x' \notin I$ . Then let

$$I_1 = [0, x], \quad I_2 = \{ y \lor y_1 : y \in I, y_1 \in I_1 \}.$$

 $I_2$  is an ideal containing I. However,  $I_2$  is proper for if 1 were an element of  $I_2$ , then we could write  $1 = y \lor y_1$  with  $y \in I$  and  $y_1 \le x$  and so  $y \ge x'$  i.e.  $x' \in I$ . This contradicts the maximality of I.

**Corollar 5** If I is a maximal ideal and  $I_1$  and  $I_2$  are ideals whose intersection lies in I, then either  $I_1 \subset I$  or  $I_2 \subset I$ .

PROOF. Suppose that there exist  $x_1 \in I_1 \setminus I$  and  $x_2 \in I_2 \setminus I$ . Then  $x'_1 \in I$ ,  $x'_2 \in I$  and  $x_1 \wedge x_2 \in I_1 \cap I_2 \subset I$ . Then  $1 = x'_1 \vee x'_2 \vee (x_1 \wedge x_2)$  is an element of I and this is a contradiction.

**1.40.** Exercise let I be a subset of the power set P(S) of a set S. Then I is a proper ideal if and only the family of complementary sets  $I^c = \{S \setminus A : A \in I\}$  is a filter on S. I is a maximal ideal if and only if  $I^c$  is an ultrafilter.

Let X and  $X_1$  be Boolean algebras. A (Boolean algebra) morphism from X into  $X_1$  is a lattice morphism f which preserves 0 and 1 i.e. f is a ring morphism for the associated Boolean rings.

For example if  $\phi$  is a mapping from a set S into a set  $S_1$ , then the mapping  $f: A \mapsto \phi^{-1}(A)$  is a morphism from  $P(S_1)$  into P(S).

We now discuss briefly subalgebras and quotient algebras of Boolelan algebras.

**Subalgebras** Let X be a Boolean algebra. A non-zero subset  $X_1$  of X is a subalgebra if it is a sublattice and is closed under complementation. Then  $X_1$  contains the zero and one of X and so is a Boolean algebra under the induced ordering. If S is a topological space, then Clopen (S) is a subalgebra of P(S). On the other the family of regularly closed subsets is not in general a subalgebra.

**Quotient algebras** Let I be an ideal in a Boolean algebra X. Then I is an ideal in the associated ring and so the quotient space X/I is also a ring. In fact, it is easy to see that it is a Boolean ring i.e. satisfies the conditions of 1.32. B.

Hence the quotient of a Boolean algebra by an ideal is also a Boolean algebra—called the **quotient algebra** of X with respect to I. The natural mapping form X onto X/I is, of course, a Boolean algebra morphism. For example, if  $S_1$  is a subset of S, then  $I = \{A \in P(S) : A \subset S_1\}$  is an ideal in P(S) and the quotient algebra P(S)/I is naturally isomorphic to  $P(S \setminus S_1)$  as the reader can check for himself.

From this it follows that if X is a Boolean algebra, a subset I is an ideal if and only if there is a Boolean algebra morphism f from X into a Boolean algebra Y so that I is the kernel of f i.e. the set  $\{x \in X : f(x) = 0\}$ . The special case where I is a maximal ideal is important enough to be stated as a proposition.

**Proposition 9** 1.41. Proposition A subset I of X is a maximal ideal if and only if it is the kernel of a non-trivial morphism f from X into the Boolean  $\mathbf{Z}_2$  (i.e. the unique Boolean algebra with two elements—0 and 1).

**1.42.** Exercise Prove 1.41 and deduce that if x is a non zero element of a Boolean algebra, then there is a morphis from X into  $\mathbb{Z}_2$  so that f(x) = 1.

Let S be a set. A collection  $\mathcal{F}$  of subsets of S is a **field** if it is a Boolean subalgebra of P(S), i.e. if it is closed under finite unions, finite intersections and complementation. If  $\mathcal{F}$  is a field of sets and  $x_0 \in S$ , then  $I_{x_0} = \{A \in \mathcal{F} : \S, \notin \mathcal{A}\}$  is a maaximal ideal of the Boolean algebra  $\mathcal{F}$ . It is called the maximal ideal determined by  $x_0$ . A field  $\mathcal{F}$  of sets in S **separates** S if whenever  $x \neq y$  in S there is an  $A \in \mathcal{F}$  so that  $x \in A$  and  $y \notin A$ .  $\mathcal{F}$  is **perfect** if each maximal ideal in  $\mathcal{F}$  is determined by a point of S.

For example if S is a set, the P(S) separates S but is not perfect unless S is finite (this follows from the existence of ultrafilters on an infinite set S which are not **fixed** i.e. of the form

$$\mathcal{U}_{\S} = \{\mathcal{A} \subset \mathcal{S} : \S \in \mathcal{A}\}$$

for some  $x \in S$ . This in turn is a consequence of the axiom of choice consider an ultrafilter containing the filter of cofinite subsets of S i.e. those subsets whose complements are finite).

If S is a Hausdorff topological space, then Clopen (S) does not separate S in general. A necessary condition for this to be the case is that S be **totally-disconnected** i.e. the connected component of each  $x \in S$  is the singleton

x (this is the case, for example, when S is discrete or S =). It is also easy to see that if S is compact, then Clopen (S) is perfect. For if  $\mathcal{A}$  is any maixmal ideal of this algebra, then by the finite intersection property the intersection of the complements of sets of  $\mathcal{A}$  is non-empty. It is then clear that  $\mathcal{A}$  is the maximal ideal determined by a point of this set.

The next result shows that if a field is both separating and perfect then it must be of the form Clopen(S) for a suitable topology on S.

**Proposition 10** 1.43. Proposition Let  $\mathcal{F}$  be a perfect field of subsets of a set S which separates S. Then there is a compact, totally disconnected topology  $\tau$  on S for which  $\mathcal{F}$  is the algebra of clopen subsets.

PROOF. Let  $\tau$  be the topology on S which has  $\mathcal{F}$  as a basis i.e. a subset U of S is open if and only it it is the union of sets of  $\mathcal{F}$ . Suppose that  $\mathcal{U}$  is an open covering of S. We show that  $\mathcal{U}$  has a finite subcovering. We can assume that each  $U \in \mathcal{U}$  is in  $\mathcal{F}$  and that  $\mathcal{U}$  is normal as a subset of the Boolean algebra  $\mathcal{F}$ . If there is no  $J \in \mathcal{J}(\mathcal{U})$  which covers S then the ideal  $I(\mathcal{U})$  generated by  $\mathcal{U}$  is proper in  $\mathcal{F}$  and so is contained in a maximal ideal. But this maximal ideal is determined by a point  $x_0 \in S$  and so  $x_0 \notin \bigcup \mathcal{U}$  and this is a contradiction.

We now show that  $\mathcal{F}$  consists of the clopen subsets of  $\tau$ . If  $A \subset S$  is clopen, then A is the union of sets in  $\mathcal{F}$ . Since A is closed in S and S is compact, it is the union of a finite number of those sets and so is in  $\mathcal{F}$ .  $(S, \tau)$  is totally disconnected since  $\mathcal{F}$  is perfect.

We now come to our main result on Boolean algebras:

**Proposition 11** Let X be a Boolean algebra. Then there exists a totally disconnected space S so that X is isomorphic to the Boolean algebra of clopen subsets of S.

PROOF. We let S denote the set of maximal ideals of X and put  $U(x) = \{I \in S : x \notin I\}$  for  $x \in X$ . Then  $\mathcal{F} = \{\mathcal{U}(\S) : \S \in \mathcal{X}\}$  is a field of subsets of S and the mapping  $x \mapsto U(x)$  is an isomorphism from X into  $\mathcal{F}$ . For if  $x, y \in X$ , then

$$U: (x \lor y) = \{I \in S : x \lor y \notin I\}$$

$$(5)$$

$$= \{I \in S : (x \notin I) \text{ or } (y \notin I)\}$$

$$(6)$$

$$= U(x) \cup U(y). \tag{7}$$

Similarly,

$$U: (x \wedge y) = \{I \in S : x \wedge y \notin I\}$$
(8)

$$= \{I \in S : (x \notin I) \text{ and } (y \notin I)\}$$

$$\tag{9}$$

$$= U(x) \cap U(y). \tag{10}$$

(for  $x \land y \in I \iff (x \in I)$  or  $(y \in I)$  since if  $x \land y \in I$ ,  $[0, x] \cap [0, y] \subset I$ and so  $[0, x] \subset I$  or  $[0, y] \subset I$ .

It is clear that  $U(x') = S \setminus U(x)$ .

 $x \mapsto U(x)$  is injective since if  $x \neq 0$ , there is a maximal ideal I so that x', the complement of x, is in I (1.36) and so  $x \notin I$  (1.38) i.e.  $U(x) \neq \emptyset = U(0)$ .

 $\mathcal{F}$  is perfect: let I be a maximal ideal in  $\mathcal{F}$ . Then its pre-image  $I_0$  in X is also a maximal ideal. For each  $x \in X$ ,  $U(x) \in \mathcal{F}$  if and only if  $x \in I_0$  i.e. if and only if  $I_0 \notin U(x)$  and so I is the ideal determined by  $I_0$ .

 $\mathcal{F}$  separates S: for if  $I \neq J$ , there is an x = inX so that  $x \in I \setminus J$  say. Then  $J \in U(x)$  and  $I \notin U(x)$ . The result follows now by applying 1.43 to  $\mathcal{F}$ .

In the light of this result, we discuss briefly conditions on the topological space S which ensure that Clopen (S) is complete (resp.  $\sigma$ -complete).

1.44. Definition A topological space S is extremally disconnected (resp.  $\sigma$ -extremally disconnected) if the closure of every open set (resp. the closure of every  $F_{\sigma}$ -open set) is open. (Recall that an  $F_{\sigma}$ -set is, by definition, a countable union of closed sets). S is a **Stonian space space** (resp.  $\sigma$ -Stonian) if it is compact and extremally disconnected (resp. compact and  $\sigma$ -extremally disconnected). Spaces with either of the above properties are automatically totally disconnected.

**Proposition 12** Let S be a totally disconnected compact space. Then Clopen(S) is

- $\sigma$ -complete if and only if S is  $\sigma$ -Stonian;
- Dedekind complete if and only if S is Stonian.

PROOF. We prove the second claim. Suppose that S is Stonian. Let  $\mathcal{A}$  be a subset of Clopen (S). Then if U is its union, this is an open set in S and so its closure is in Clopen (S). This is the supremum of  $\mathcal{A}$  in Clopen (S).

Conversely, suppose that the latter algebra is complete. Let U be open in S. Since S is totally disconnected, there is a family  $\mathcal{A}$  of clopen sets in S so that U is their union. Let  $U_1$  be the supremum of the family  $\mathcal{A}$  in the algebra. Then we claim that  $U_1$  is the closure of U and so the latter set is open. (It is clear that  $\overline{U} \subset U_1$ . If  $U_1 \setminus \overline{U}$  were non-empty then it would contain a non-empty clopen set V. But then  $U_1 \setminus V$  would be an upper bound for  $\mathcal{A}$  which is strictly smaller than  $U_1$ ).

**Corollar 6** If X is a complete Boolean algebra (resp.  $a \sigma$ -Dedekind complete Boolean algebra), then the Stone space of X is Stonian (resp.  $\sigma$ -Stonian).

**1.47.** Exercises A. Show that a product of Boolean algebras is also a Boolean algebra.

B. Identity this product as a field of subsets in the disjoint union of the  $S_{\alpha}$  when each Boolean algebra is a field  $\mathcal{F}_{\alpha}$  of subsets of a set  $S_{\alpha}$ .

C. Show that any Boolean algebra is isomorphic to a subalgebra of a product of Boolean algebras, all of which are isomorphic to  $\mathbb{Z}_2$ .

D. Let S be compact and totally disconnected. Show that the mapping

$$s \mapsto I_s = \{U \in \operatorname{Clopen}(S) : s \notin U\}$$

is a bijection from S onto the set of maximum ideals of Clopen (S).

E. If S is a set, the Stone space of P(S) is  $\beta S$ , the Stone- $\lor Cechcompactification of Swith the discree F. Classify all finite Boolean algebras.$ 

#### 2.2 Riesz spaces

We now turn to the topic of vector spaces which are also provided with an order structure. Of course we assume that some compatibility condition between the two structures holds. The classical function spaces and sequence spaces which we have considered all have natural orderings and these supply a motivation for the theory.

**Definition A partially ordered vector space** (abbreviated (POVS)) is a poset  $(E, \leq)$  where E is a real vector space such that the following conditions are satisfied: if  $x, y, x \in E$  and  $\lambda > 0$  then  $x + z \leq y + z$  and  $\lambda x \leq \lambda y$  whenever  $x \leq y$ .

E is a **Riesz space** if it is a POVS and, at the same time, a lattice under its ordering.

The following vector spaces are Riesz spaces under their natural ordering:  $\mathbf{R}^{S}, C(K), \mathcal{S}(\mu), L^{p}(\mu).$ 

We list some properties of Riesz spaces which are simple consequences of the definition:

- $x + \sup A \sup(x + A)$   $(x \in E, A \subset E);$
- $\sup(\lambda A) = \lambda \sup A \quad (\lambda > 0, A \subset E);$
- $\sup(\lambda A) = \lambda \inf(A)$   $(\lambda < 0, A \subset E)$ . In particular  $x \lor y = -((-x)(-y))$   $(x, y \in E)$ .
- $x + y = xy + x \lor y$   $(x, y \in E)$ .

(For we have:  $((x - y) \lor 0) + y = x \lor y$  resp.  $(x - y_{\lor} 0 = x + ((-y) \lor (-x)) = x - (x \land y)$  and so  $x \lor y - y = x - (x \land y)$ ).

If x is an element of a Riesz space, we put

$$x^+ := x \lor 0; \tag{11}$$

$$x^{-} := x^{+} + x^{-}. \tag{12}$$

Then

 $x=x^{+}-x^{-}$  (for  $x+0=s \lor 0+x \land 0=x^{+}-x^{-}$ ).

If x and y are elements of E we say that x and y are **disjoint** if  $|x| \wedge |y| = 0$  (written  $x \perp y$ ).

The decomposition  $x = x^+ - x^-$  expresses x as the difference of two positive elements. In general, such decompositions ate not unique. However, we do have the following:

if x = y - z where  $y, z \ge 0$  then  $x^+ \le y, x^- \le z$ ;

if x = y - z where  $y, z \ge 0$  and yz are disjoint, then  $y = x^+, z = x^-$  (note that  $x^+$  and  $x^-$  are always disjoint.)

**Definition** For elements x, y of Riesz space  $E, \lambda \in \mathbf{R}$ , the following relationship hold:

- $(x+y)^+ \le x^+ + y^+;$
- $(\lambda x)^+ = \lambda x^+ (\lambda > 0);$
- $|x+y| \le |x|+|y|;$

• 
$$x \wedge y = (x + y - |x - y|)/2, \quad x \vee y = (x + y + |x - y|)/2;$$

• |x| = 0 if and only if x = 0.

Note that for the Riesz function spaces listed above, the symbols  $x^+, x^-, |x|$  habe their usual meaning (positive part, negative part resp. absolute value of a function).

In constrast to the situation for general lattices, the distributive law always holds in a Riesz space: **Proposition 13** If X is an element of a Riesz space E, A a subset of E, then

$$x \wedge \sup(A) = \sup(x \wedge A).$$

**PROOF.** We always have the inequality:  $x \wedge \sup S \ge \sup(x \wedge A)$ . Suppose that  $x \ge wy$  for each  $y \in A$ . Then

$$z \ge (x - y) \land 0 + y \ge (x - \sup A) \land 0 + y.$$

Since this holds for each  $y \in A$ , we have

$$z \ge (x - \sup A) \land 0 + \sup A = x \land \sup A.$$

The following is an important property of Riesz spaces:

**Proposition 14** (Riesz decomposition property) Let x, y, z be positive elements of a Riesz space so that x = y + z. Then for every representation  $x = x_1 + \cdots + x_n$  of x as the sum of n positive elements, there exist positive  $\{y_1, \ldots, y_n\}$  resp.  $\{z_1, \ldots, z_n\}$  so that

$$y = \sum_{i=1}^{n} y + i, z = \sum_{i=1}^{n} z_i \text{ and } x_i = y_i + z_i$$

for each i.

PROOF. We give the proof for n = 2. The general case follows by induction. We put  $y_2 := x_2 \wedge y$ . Then  $y_2 \leq y$  and  $y_2 \leq x_2 \leq x$ . Hence if  $y_1 := y - y_2$ , then  $0 \leq y_1 \leq x_1$  since  $x_1 = x - x_2 \geq (y - x_2) \vee 0 = y + [(-x_2) \vee (-y)] \geq y - y_2 = y_2$ . Thus if we define  $z_1 := x_1 - y_1$  and  $z_2 := x_2 - y_2$  we have the required decomposition.

**Exercises** A. Show that P([0,1]), the space of real polynomials of degree n on [0,1] with the pointwise ordering, does not have the above property; B. Show that  $C^k([0,1])$ , the space of k-times continuously differentiable functions in C([0,1]) (k < 1) with the pointwise ordering is a POVS which is not a Riesz space but does have the property above.

C. E be a POVS. Show that each of the following properties is equivalent to the Riesz decomposition property:

• [0, x] + [0, y] = [0, x + y] (x > 0, y > 0);

• for all pairs A, B of finite subsets of E,  $A \leq B$  (i.e.  $x \leq y$  for each  $x \in A$  and each  $y \in B$ ) implies the existence of a  $x \in E$  with  $A \leq z \leq B$ .

**Lemma 1** If x, y, z are positive elements of a Riesz space, then

$$x \wedge (y+z) \le x \wedge y + y \wedge z.$$

PROOF. Let  $u = x \land (y + z)$ . Applying the Riesz decomposition property to the decomposition,

$$y + z = u + (y + z - u)$$

we obtain elements  $u_1$  and  $u_2$  so that  $0 \le u_1 \le y, 0 \le u_2 \le z$  and  $u = u_1 + u_2$ . But then  $u_1 \le x \land y$  and  $u_2 \le x \land z$  and so  $u \le x \land y + x \land z$ . q.e.d.

**Proposition 15** If x, y, z are elements of a Riesz space E and  $l \in \mathbf{R}$ , then x and (ly + z) are disjoint whenever x and y and x and z are.

PROOF. It clearly suffices to prove the two special cases l - 1 and z = 0. a) the case l = 1:

$$0 \le |x| \land |y+z| \le |x| \land (|y|+|z|) \le |x| \land |y|+|x||w|z| = 0.$$

b) the case z = 0:

$$0 \le |x| \land |ly| \le \sup(1, |l|) \quad (|x| \land |y|) = 0.$$

In the next result we use the following notation: a subset A of a Riesz space E is **disjoint** if each distinct pair x, y of elements from A is disjoint.

**Proposition 16** Let E be a Riesz space.

- if  $x, y \in E$ ,  $x \perp y$ , then  $(x + y)^+ = x^+ + y^+, (x + y)^- = x^- + y^-;$
- if  $x_1, \ldots, x_n$  are disjoint elements of E then

$$x_1 + \dots + x_n = x_1 \vee \dots \vee x_n;$$

- if  $x_1, \ldots, x_n$  are disjoint elements of E shose sum  $\sum_{i=1}^n x_i$  is non-negative, then each  $x_i$  is non-negative;
- a disjoint set of (non-zero) elements is linearly independent.

PROOF. 1. We have  $x^+ \wedge y^- = x^- \wedge y^+ = x^+ \wedge x^- = y^+ \wedge y^- = 0$ and so  $(x^+ + y^+) \perp (x^- + y^-)$ . The result follows now from the fact that  $(x^+ + y^+) - (x^- + y^-)$  is a decomposition of x + y into a difference of disjoint positive elements.

2. For n = 2 the result follows from the relation  $x_1 + x_2 = x_1 \lor x_2 + x_1 \land x_2$ and the fact that

$$0 \le |x_1 \land x_2| \le |x_1| \land |x_2| = 0$$

and so  $x_1 \wedge x_2 = 0$ . The general case follows by a simple induciton argument. 3. It follows from 1. that

$$\left(\sum_{i=1}^n x_i\right)^- = x_1^- + \dots + x_n^-.$$

Hence if for some *i* we have  $x_i > 0$ , then  $(\sum_{i=1}^n x_i)^- > 0$ . 4. Suppose that there are scalars  $l_1, \ldots, l_n$  so the  $\sum_{i=1}^n l_i x_i = 0$ . Then by 3.  $l_i x_i \ge 0$  and  $l_i x_i \le 0$  for each *i* and so  $l_i = 0$ .

**Exercise** Show that if x and y are disjoint elements of a Riesz space, then

$$|x + y| = |x - y| = |x| + |y| = |x| \lor |y|.$$

As the above results show, elements of a Riesz space tend to satisfy the simple identities which hold in  $\mathbf{R}$  (and so in spaces of the type  $\mathbf{R}^{S}$ ). There is one aspect, however, where care is required. Riesz spaces can contain "infinitely small" elements i.e. non-zero elements x so that  $\{nx : n \in \mathbf{N}\}$  is order bounded. A simple example is  $\mathbf{R}^{2}$  with the lexicographics ordering where the element (0, 1) satisfies this condition. For this reason we introduce the following definition:

**Definition** A Riesz space E satisfies the axiom of Archimedes (or is Archimedean) if for each  $x \ge 0$  the fact that  $\{nx : x \in \mathbf{N}\}$  is bounded implies that x = 0.

The examples  $\mathbf{R}^{S}$ , C(K),  $\mathcal{S}(\mu)$ ,  $\mathcal{L}_{\sqrt{\mu}}$  are all Archimedean. Another important concept is that of a unit:

**Definition** An element e of a Riesz space E is a **unit** for E if  $x \land e < 0$  for each x > 0. It is a **strong unit** if for each  $x \in E$  there is an  $n \in \mathbb{N}$  with  $|x| \le ne$ .

For example, any strictly positive function is a unit for  $\mathbf{R}^{S}$  resp. a strong unit for C(K) (in particular, the constant function 1).  $S(\mu)$  has a unit but

not a strong unit in general. If S is an infinite set and  $_{C}(S)$  is the set of functions in  $\mathbb{R}^{S}$  with finite support then  $_{C}(S)$  with the induced order is a Riesz space which does not have a unit.

**Exercise** If E is a Riesz space with strong unit e, then the mapping

$$|| ||_e : y \to \inf\{\lambda > 0 : |y| \le \lambda e\}$$

is a seminorm on E and it is a norm if E satisfies the Archimedes axiom.  $|| ||_e$  is the Minkowski functional of the unit interval  $\{x \in E; |x| \leq e\}$  and, for  $y \in E$ ,  $||y||_e = \max(||y^+||_e, ||y^-||_e)$ .

Except in the trivial case of a zero-dimensional space, a Riesz space will never be complete (as a lattice). However, many of the classical spaces are Dedekind complete (resp. Dedekind  $\sigma$ -complete). We note some simple properties of such spaces:

- For a Riesz space to be Dedkind complete, it suffices that every bounded subset of *E* have a supremum;
- A Dedekind  $\sigma$ -complete spaces is automatically Archimedean (for if  $x \ge 0$  is such that  $\{nx : n \in \mathbb{N}\}$  is bounded and  $y = \sup\{nx : x \in \mathbb{N}\}$ , then clearly y + x = x and so x = 0).

**Exercise** Show that if E is a Dedekind  $\sigma$ -complete Riesz space with strong unit e then  $(E, || ||_e)$  is a Banach space.

We now turn to the standard types of constructions on Riesz spaces. As usual these are characterised by so-called universal properties with respect to suitable classes of mappings. For Riesz spaces, there are several possible choises of these classes: if  $T : E \to F$  is a linear operator between RIesz spaces, then we say that

T is **isotone** if  $x \ge 0$  implies that  $Tx \ge 0$  (this coincides with the definition after 1.2 for lattices); T is a **Riesz morphism** if  $T(x \land y) = Tx \lor Ty$  (i.e. T is a lattice morphism); T is **order continuous** if for each net  $(x_{\alpha}), x_{\alpha} \downarrow 0$  implies that  $Tx_{\alpha} \downarrow 0$ ; T is  $\sigma$ -order continuous if for each sequence  $(x_n) n_n \downarrow 0$  implies that  $Tx_n \downarrow 0$ .

**Exercise** Let T be a linear mapping from a Riesz space E into a Riesz space F. Show that T is a Riesz morphism if the following condition holds:

$$(Tx)^+ = T(x^+) \quad (x \in E).$$

If T is a Riesz morphism show that it is order continuous if and only if the following condition holds: for each subset A of E for which  $\sup A$  exists,  $\sup T(A)$  exists and  $\sup(T(A)) = T(\sup A)$ .

If E is a Riesz space, then each vector subspace F which is closed under the lattice operations is also a Riesz space – it is called a **Riesz subspace** of E. For example, C(K) is a Riesz subspace of  $\mathbf{R}^{K}$  and each  $\mathcal{L}_{\sqrt{\mu}}$  is a Riesz subspace of  $\mathcal{S}(\mu)$ .

We now discuss quotient spaces. Firstly, let E be a POVS, I a subspace and denote by  $\pi_I$  the natural projection from E onto E/I we define a relation  $\leq$  as follows:

$$x \leq y$$
 if and only if $(y - x) = \pi_i(z)$  for some  $z \geq 0$ .

This relation is a pre-ordering on E/I but not always an ordering. In fact it is an ordering if and only if I satisfies the following condition:

if 
$$x \ge 0$$
 and  $x \le y$  for some  $y \in I$ , then  $x \in I$ .

**Exercise** Show that if I satisfies this condition, then E/I is a POVS. Show that if T is an isotone linear mapping between POVS's then KerT satisfies this condition (so that it is a necessary condition for E/I to be a POVS).

We now turn to quotients of Riesz space. For this we introduce the following concepts. A subset A of a Riesz space E is **solid** if whenever  $y \in A$  and  $x \in E$  is such  $|x| \leq |y|$ , then  $x \in A$ . An **ideal** in E is a solid Riesz subspace.

• **Proposition 17** Let I be an ideal in Riesz space E. Then the quotient space E/I is also a Riesz space.

PROOF. We first show that in this situation an element  $\pi_I(x)$  is positive if and only if  $x^- \in I$ . The sufficiency of this latter condition is clear since  $\pi_I(x) = \pi_I(x^+) - \pi_I(x^-) = \pi_I(x^+) \ge 0$ . On the other hand, if  $\pi_I(x) \ge 0$  then there is a  $y \ge 0$  in E so that  $\pi_I(x) = \pi_I(y)$  i.e.  $y - x \in I$ . Now  $-x \le y - x$ and so

$$0 \le x^{-} = (-x) \lor - \le (y - x) \lor 0 = (y - x)^{+} \le |y - x|.$$

Hence  $x^- \in I$  since  $y - x \in I$  and I is an ideal. We know that E/I is POVS. We now show that for each  $x \in E$ ,  $\pi_I(x^+)$  is the supremum of the pair  $\{\pi_I(x), 0\}$ . It is clear that  $\pi_I(x^+)$  is greater that both elements. Suppose that  $y \in E$  is such that  $\pi_I(y) \ge \pi_I(x)$  and  $\pi_I(y) \ge 0$ . Then  $y^- \in I$  and  $(y - x)^- \in I$ . Hence

$$(y-x)^- = ((x \lor 0) - y) \lor 0 = (x-y) \lor (-y) \lor 0 = (y-x)^- \lor y^- \in I$$

and so  $\pi_I(y) \geq \pi_I(x^+)$ . It now follows easily that E/I is a Riesz space.

The space E/I is called the **quotient Riesz space** of E by I. Typical examples are as follows:

if  $S_1$  is a subset of S and  $I_{S_1}$  denotes the set of those functions in  $\mathbb{R}^S$ which vanish on  $S_1$ , then  $I_{S_1}$  is an ideal in  $\mathbb{R}^S$ . The corresponding quotient space is naturally isomorphic to  $\mathbb{R}^{SS_1}$ . If  $(\Omega, \sum, \mu)$  is a measure space and

$$I := \{ x \in S(\mu) : x(t) = 0 \text{ almost everywhere} \}$$

then I is an ideal in  $S(\mu)$  (and so in each  $L^p(\mu)$ ). The corresponding quotient spaces are just  $S(\mu)$  resp.  $L^p(\mu)$ .

**Exercises** A. Suppose that I is an ideal in the Riesz space E. If E is Dedekind complete, does it follow that E/I also is? If e is a strong unit for E, need  $\pi_I(e)$  be a strong unit for E/I?

B. The space  $\ell^{\infty}$  of bounded sequences is an ideal in  $\mathbf{R}^{\mathbf{N}}$ . Show that the quotient space  $\mathbf{R}^{\mathbf{N}}/\ell^{\infty}$  is non-Archimedean.

C. Let  $\{E_{\alpha}\}$  be a family of POVS's. Show that the product space  $E = \prod_{\alpha \in A} E_{\alpha}$ , provided with the product ordering, is also a POVS and that it is a Riesz space if each  $E_{\alpha}$  is. Show that if  $\{S_{\alpha}\}$  is a family of sets and S is their disjoint union, then  $\mathbf{R}^{S}$  is naturally (Riesz) isomorphic to the product  $\prod_{\alpha \in A} \mathbf{R}^{S_{\alpha}}$ .

D. With the notation of C show that each  $E_{\alpha}$  is an ideal in  $\prod_{\alpha \in A} E_{\alpha}$  and so isomorphic to a quotient of E. Show that E is Dedekind complete if and only if each  $E_{\alpha}$  is.

We consider the question of embedding a given Riesz space in a Dedekind complete one. Since every such space is Archimedean and the latter is obviously a hereditary property it is clear that a necessary condition for this to be possible is that the spece be Archimedean. In fact his condition is also sufficient as the next result shows:

**Proposition 18** Let E be an Archimedean Riesz space. Then there excists a Dedekind complete Riesz space  $\hat{E}$  with the following properties:

- $E \subset \hat{E};$
- for each subset  $A \subset E$  such that  $\sup_{E}(A)$  exists, we have  $\sup_{E}(A) = \sup_{\hat{E}}(A)$ ;
- if  $x \in \hat{E}$ ,  $x = \sup\{y \in ELy \le x\} = \inf\{z \in E : z \ge x\}$ ;
- every order continuous Riesz morphism from E into a Dedekind complete Riesz space F factors in a unique way through  $\hat{E}$ .

PROOF. We use the notation of 1.22 with the exception that E denotes the family of all saturated subsets of E with the exception of and E. We remark here that a nonempty saturated subset A is in E if and only if there is a  $b \in E$  so that  $A \leq b$ .

(Clearly, if this condition is satisfied, then A|neE. If it is not satisfied, then A has no upper bounds i.e. U(A) =. Hence A = L(U(A)) = L() = E).  $\hat{E}$  is a Dedekind complete lattice and (1) - (2)) are satisfied (where we understand (1) in the sense that the inclusion  $x \to L_x$  is a lattice isomorphism from E onto a sublattice of  $\hat{E}$ ).

On = hatE we define a linear structure as follows:

$$A \xrightarrow{\cdot} + := L(U(A+B)); \tag{13}$$

$$\lambda \cdot A := \lambda A(\lambda > 0); \tag{14}$$

$$0 \cdot A := L_0; \tag{15}$$

$$\lambda \cdot A := -U(|\lambda|A) \quad (\lambda < 0). \tag{16}$$

(We note that all the sets on the right hand sides are saturated. The only non-trivial case is  $-U(|\lambda|A)$ . Then we have

$$L(U\{-U(|\lambda|A)\}) = L(-U(|\lambda|A))) = L(-|\lambda|A) = -U(|\lambda|A).$$

By applying the above criterion, it can be checked that they are in  $\hat{E}$ .) In the verification that  $\hat{E}$ , with these operations, is a vector space we check the following points. (Those remaining can be proved with similar methods.)

- (a) addition  $\rightarrow$  + is associative;
- (b)  $L_0$  is a zero for addition;
- (c)  $A \rightarrow +(-1)A = L_0$   $(A \in \hat{E});$ (d)  $\lambda(A \rightarrow +B) = \lambda A \rightarrow +\lambda B$   $(\lambda \in \mathbf{R}, A, B \in \hat{E}).$

(a) We show that  $(A + \rightarrow +B) \rightarrow +C = L(U(A + B + C))$  and the result follows by symmetry. Since  $A + B \subset A \rightarrow +B$ , then  $(A + B) + C \subset (\rightarrow +B) \rightarrow +C$  and so

$$L(U(A+B+C)) \subset (A \to +B) \to +C.$$

Suppose that  $x \in (A + B) + C$  – we show that  $x \leq u$  if u is an upper bound for (A+B+C) i.e. if  $u \geq a+b+c$  for each  $a \in A, b \in B, c \in C$ . Then  $a+b \leq u-c$  and so  $z \leq u-c$  for each  $z \in A+B$ . Hence for any  $x \in A \rightarrow +B$ ,  $c \in C$ , then  $z + c \leq u$  and so u is an upper bound for  $(A \rightarrow +B) + C$ . Thus  $x \leq u$  as required.

(b)  $A + L_0 = \{x + z : x \in A, z \leq 0\} = A$  since A is saturated and so  $A \rightarrow +L_0 = A$ .

(c)  $x \in (-1)A$  if and only if x = -y where  $y \in U(A)$ . Hence if  $a \in A$ ,  $a + x = a - y \leq 0$ . Thus  $A + (-1)A \subset L_0$  and so  $A \xrightarrow{\cdot} + (-1)A \subset L_0$ . If this inclusion is strict, there is a  $z \in U(A + (-1)A)$  with  $z^- > 0$ .

Then  $-z^- = z \wedge 0 \in U(A + (-1)A)$  and so  $-z^- \ge a + a'$  for each  $a \in A$ ,  $a'_1 - U(A)$ . Repeating this process, we obtain  $\{a + nz : n \in \mathbb{N}\} \subset A$  and this, together with the fact that  $U(A) \neq \emptyset$ , gives a contradiction since E satisfies the axiom of Archimedes.

(d) This is clear if  $\lambda \geq 0$ . So we suppose that  $\lambda < 0$ . Then

$$\lambda(\dot{\rightarrow} + B) = (-|\lambda|)(A \dot{\rightarrow} + B) = (-1)|\lambda|(A \dot{\rightarrow} + B) = (-1)(|\lambda|A \dot{\rightarrow} + |\lambda|B)$$
(17)

$$= (-|\lambda|A \xrightarrow{\cdot} + (-|\lambda|)B = \lambda A \xrightarrow{\cdot} + \lambda B.$$
(18)

**Exercise** Complete the proof by showing

- that  $x \to L_x$  embeds E as a Riesz subspace of  $\hat{E}$ ,
- that 4) holds (for  $x \in \hat{E}$  define Tx to be

$$\sup\{Ty: y \in E, y \le x\}).$$

A useful property of ideals in certain Riesz space is the fact that they split the space up into two parts in a way reminiscent of the splitting of Hilbert space into the sum of orthogonal subspaces. For example, if A is a measurable subspace of a measure space  $\Omega$  and

$$I_1 = \{ x \in L^p(\mu) : x|_{\Omega A} = 0 \}$$
(19)

$$I_2 = \{ x \in L^p(\mu) : x|_A = \Omega \}$$
(20)

then  $I_1$  and  $I_2$  are ideals in  $L^p(\mu)$  and this space is the algebraic direct sum  $I_1 \oplus I_2$ . Note that  $I_2$  is just  $I_1^{\perp}$ , the set of elements disjoint form  $I_1$ . We will now discuss an abstract version of this decomposition. Note that the corresponding decomposition fails for the Riesz space C([0, 1]). We shall see that the decisive factor is the Dedekind completeness of  $L^p(\mu)$  in contrast to C([0, 1]).

First we introduce the term **band** to denote an ideal I in a Riesz space E wihich is **sup-closed** i.e. such that if A is a subset of E whose supremum sup A exists, then sup  $A \in I$ . Examples of bands are the ideals  $I_1, I_2$  described above. More generally, if A is a subset of a Riesz space and  $A^{\perp}$  is defined to be the set  $\{y \in E : y \perp x \text{ for } x \in A\}$  of elements which are disjoint from A then  $A^{\perp}$  is clearly a band. (This follows from the simple fact that if A is a subset of Riesz space and  $x \perp A$  then  $x \perp \sup A$ -provided the latter exists).

If A is a subset of a Riesz space, then there exists a smallest band containing A – the intersection of all bands containing A – we denote it by B(A). Since, by the above remark,  $A^{\perp\perp}(=(A^{\perp})^{\perp})$  is a band containing A, it is clear that  $B(A) \subset A^{\perp\perp}$ . We shall discuss below the question of when equality holds. Firstly we give a characterisation of those elements of the band generated by an ideal.

**Proposition 19** Let I be an ideal in a Riesz space E. Then a positive element x in E is in B(I) if and only if it is representable as the supremum of some subset of I.

**PROOF.** Let  $I := \{x \ge 0 : x = \sup A_1 \text{ for some } A_1 \subset I\}$ . We show that

- (i) if  $x \ge 0$  is bounded above by a  $y \in \tilde{I}$ , then  $x \in \tilde{I}$ ;
- (ii)  $x + \lambda y \in \tilde{I}$   $(x, y \in I, \lambda \ge 0);$
- (iii)  $\sup_E A_1 \subset \tilde{I}$  for each  $A_1 \subset \tilde{I}$  so that  $\sup_E A_1$  exists. From this it follows easily that  $\tilde{I} \tilde{I}$  is a band. Since  $I \subset \tilde{I} \tilde{I} \subset B(I)$ , the result follows.

(i) there is a set  $A \subset I$  so that  $y = \sup A$ . Then  $x = \sup(x \land A)$  and  $x \land A \subset I$ .

(ii) we can assume that  $\lambda = 1$ . Let  $A_1 := \{z \in I, 0 \le z \le x + y\}$ . x + y is an upper bound for  $A_1$ . On the other hand, if z' is an upper bound, then  $z' \ge x' + y'$  for each  $x', y' \in I$  with  $0 \le x' \le x, 0 \le y' \le y$  and so  $z' \ge x + y$  i.e.  $x + y = \sup A_1$ . (iii) follows from 1.5.F.

We now show that  $B(A) = A^{\perp \perp}$ , provided that the Riesz sspace is Archimedean. The proof uses the following Exercise:

**Exercise** Show that if I is an ideal in a Riesz space E, then for  $x > 0, x \in I^{\perp \perp}$  there is a y > 0 in I with  $y \leq x$ .

**Proposition 20** Let E be an Archimedean Riesz space. Then for each  $A \subset E, B(A) = (A^{\perp})^{\perp}$ .

PROOF. We can suppose that A is an ideal. Suppose that there is an  $x \in A^{\perp \perp}B(A)$ . We can assume that  $x \ge 0$ . Then if  $A' := \{y' \in A : 0 \le y' \le x\}$ , we have  $x \ge y := \sup A'$ . Thus  $0 < x - y \in (A^{\perp})^{\perp}$  and so there is a  $z \in A$  with 0 < z < x - y. Hence  $z + A' \subset A'$  and so  $\{nz : n \in \mathbf{N}\} \subset A'$ . But this contradicts the fact that E is Archimedean.

**Exercise** Show that the converse holds i.e. that a Riesz space E with the property that  $B(A) = A^{\perp \perp}$  for each subset A is automatically Archimedean.

Our main aim is to obtain a decomposition of the form

$$E = E_1 \oplus E_1^{\perp}$$

for suitable bands  $E_1$  in a Riesz space  $E_{\dot{L}}$  We begin by defining the projection of an element  $x \in E$  onto  $E_1$ . Here it is useful to bear in mind the splitting of  $L^p(\mu)$  discussed above.

Suppose first that x is positive element of E. We say that the projection of x into  $E_1$  exists if

$$\{\sup\{y \in E_1 : 0 \le y \le x\}\$$

exists and in this case we denote it by  $\pi_{E_1}(x)$ . For general x in E we say that  $\pi_{E_1}(x)$  exists if  $\pi_{E_1}(x^+)$  and  $\pi_{E_1}(x^-)$  exists and we define  $\pi_{E_1}(x)$  to be their difference.

Of course, if E is Dedekind complete, then  $\pi_{E_1}(x)$  exists for each  $x \in E$ and each band  $E_1$  in E. On the other hand, if  $E_1$  is the band

$$\{x \in C([0,1]) : x = 0 \text{ on } ]1/2, 1$$

in C([0,1]), then  $\pi_{E_1}(1)$  does not exist.

If A is a measurable subset of  $\Omega$  and  $E_1$  is the band of functions which vanish outside of A then  $\pi_{E_1}(x)$  is the function  $\chi_A x$  ( $\chi_A$  is the characteristic function of A).

The connection between projections on ideals and decompositions of the space is made explicit in teh following result:

**Proposition 21** Let x be an element of a Riesz space E,  $E_1$  a band in E. Then  $\pi_{E_1}(x)$  exists if and only if x has a representation of the form x = y + z where  $y \in E_1$ ,  $z \in E_1^{\perp}$ . This representation is unique and, in fact,  $y = \pi_{E_1}(x), z = \pi_{E_1^{\perp}}(x)$ .

**PROOF.** Suppose first that  $x \ge 0$  has such a representation, x = y + z $(y \in E_1, z \in E_1)$ . Then  $y \ge 0$  and  $z \ge 0$  since y and z are disjoint. Suppose that  $u \in E_1$  is such that  $0 \le u \le x$ . We show that  $u \le y$  which implies that  $y = \pi_{E_1}(x)$  by the definition of the latter.

For  $u = u \wedge x \leq u \wedge y + u \wedge z = u \wedge y$ . If a general x has a representation x = y + z, then  $x^+ = y^+ + z^+$ ,  $x^- = y^- + z^-$  by 2.8 and so we can apply the above case.

Now assume that  $y = \pi_{E_1}(x)$  exists and put z = x - y (where  $x \ge 0$ ). It suffices to show that  $z \in E_1^{\perp}$ . Hence consider  $x_1 \in E_1$  we shall show that  $z \wedge |x_1| = 0$ .

Now  $0 \le z \land |x_1|$  is in  $E_1$  (since  $E_1$  is a band) and  $y + z \land |x_1| \le x$  and so  $y + z \land |x_1| \le y$  i.e.  $z \land |x_1| = 0$ .

Using this and the fact that projections always exist in Dedekind complete spaces we get:

**Proposition 22** Let  $E_1$  be a band in a Dedekind complete Riesz space E. Then E is the algebraic direct sum

 $E_1 \oplus E_1^{\perp}$ 

and the vector space isomorphism  $x \to (\pi_{E_1}(x), \pi_{E_1^{\perp}}(x))$  from E onto  $E_1 \oplus E_1^{\perp}$ is also a Riesz space isomorphism. In particular,  $\pi_{E_1}$  is a Riesz morphism from E onto  $E_1$  (which is also a vector space projection).

**Definition** Let *E* be a Dedekind complete Riesz space. A family  $\{E_{\alpha}\}$  of bands in *E* decomposes *E* if

- they are disjoint i.e.  $E_{\alpha} \perp E_{\beta} \ (\alpha \neq \beta);$
- if  $x \in E$  is such that  $x \perp E_{\alpha}$  for each  $\alpha$  then x = 0.

For example if  $\{x_{\alpha}\}$  is a collection of disjoint elements so that if  $x \in E$  is such that  $x \perp x_{\alpha}$  for each  $\alpha \in A$ , then x = 0, the bands  $B(x_{\alpha})$  decomposes E. (The typical example of this is the canonical basis  $(e_n)$  in any of the Riesz spaces  $\omega, \ell_{c_{\alpha}}^p$ .)

The word decomposes in the above definition is motivated by the next result, where the sum of two elements in 2.25 is replaced by an infinite sum. Note however that since we have topology available, the notion of an infinite sum is replaced by the suprema of finite sums of disjoint elements.

**Proposition 23** If  $\{E_{\alpha}\}$  decomposes the Dedekind complete Riesz space E, then for  $x \in E$ 

$$x = \sup_{\alpha \in A} \pi_{E_{\alpha}} x^{+} - \sup_{\alpha \in A} \pi_{E_{\alpha}} x^{-}.$$

PROOF. It suffices to show that this holds for  $x \ge 0$ . By the Dedekind completeness  $y := \sup_{\alpha \in A} \pi_{E_{\alpha}} x$  exists and in less than x. We claim that  $x - y \perp E_{\alpha}$  for each  $\alpha$  and so x - y = 0. For if this were not the case, there would be a  $\beta \in A$  and a  $z_{\beta} \in E$  so that  $(x - y) \wedge z_{\beta} > 0$ . But  $u = (x - y) \wedge z_{\beta} \in E_{\beta}$  and  $0 \le \pi_{E_{\beta}}(x) + u \le y + u \le x$  which contradicts the definition of  $\pi_{E_{\beta}}(x)$ .

**Proposition 24** Corollary If the Dedekind complete Riesz space is decomposed by the bands  $\{E_{\alpha}\}$ , then E is Riesz isomorphic to an ideal in  $\prod_{\alpha \in A} E_{\alpha}$ .

**PROOF.** We map x onto the element  $(\pi_{E_{\alpha}}(x))_{\alpha \in A}$  of the product.

The example  $E = \ell^1$ ,  $E_n = B(e_n)$  shows that the image of E in the product space can be (and usually is) a proper ideal. Using this decomposition we can now prove a result which can often be used for reducing proofs to the case where a given Riesz space has a unit. First we introduce the adjective **principal** to describe an band which is generated by one element i.e. is of the form B(u) ( $u \in E$ ). We note the following simple facts:

- if E is Archimedean, then |u| is a unit for B(u);
- if E is Dedekind  $\sigma$ -complete, then the projection of an element  $x \in E$  onto B(u) (which we denote by  $\pi_u(x)$ ) always exists and is given by the formula:

$$\pi_u(x) = \sup\{x \land n|u| : n \in \mathbf{N}\}.$$

The converse also holds i.e. if  $E_1$  is a band in the Archimedean space E and u is a unit for  $E_1$ , then  $E_1$  is generated by u (for, by 2.25, it suffices to show that  $(u)^{\perp} \subset E_1^{\perp}$ . Suppose that  $x \perp u$  and  $y \in E_1$  with  $|x| \wedge |y| > 0$  (.e. x is not i  $E_1$ ). Then  $|x| \wedge |y| \wedge |u| > 0$  and  $|x| \wedge |u| > 0$  -contradiction.

**Proposition 25** Let *E* be a Dedekind complete Riesz space. Then there exist Dedekind complete Riesz spaces  $\{E_{\alpha}\}$  with units so that *E* is an ideal in  $\prod_{\alpha \in A} E_{\alpha}$ .

PROOF. By 2.28 and the above remarks it suffices to produce a family  $\{E_{\alpha}\}$  of principal bands which decomposes E. This is equivalent to producing a family  $\{x_{\alpha}\}$  of disjoint elements which is maximal i.e. cannot be extended to a properly greater one. The existence of such a family is established by applying Zorn's Lemma to the family S of disjoint subsets of E.

In our proof of the spectral theorem for self-adjoint operators, the order structure of the linear operators played an important role. If we extract

the order theoretical kernel of the proof we can prove an abstract spectral theorem for Riesz space, originally due to Freundenthal. While reading the following results, it is helpful to keep in mind the example  $L^{\infty}(\mu)$  of bounded, measurable functions on a measure space  $\Omega$ . The theorem that we prove is an abstract form of the fact that such functions can be approximated uniformly by step functions. To do this, we begin by indtroducing an abstract version of the concept of a characteristic function.

**Definition** Let E be an Archimedean Riesz space with unit 1.  $e \in E$  is called a **component** if  $e \wedge (1 - e) = 0$ . We write C(E) for the set of components of E. Then 0 and 1 are components and  $C(E) \subset [0, 1]$  (the set of elements x in E with  $0 \leq x \leq 1$ ).

In C([0, 1]) 0 and 1 are the only components. For a general compact space K the components of C(K) are the characteristic functions  $\chi_U$  where U is a open subset of K i.e. a subset which is simultaneously closed and open (note that this condition means precisely that  $\chi_U$  is continuous). In  $S(\mu)$  the components are the characteristic functions of easurable subsets – the same holds for the space  $L^p(\mu)$ . In  $S(\mu)$  resp.  $L^p(\mu)$  they are the corresponding equivalence classes.

An important fact for us is that C(E) is a Boolean algebra.

**Proposition 26**  $C(\mathcal{E})$ , with the ordered induced from E, is a Boolenalgebra and, for each  $e \in C(\mathcal{E})$ , e', the complement of e in the sence of 1.27, is 1 - e.  $C(\mathcal{E})$  is complete if E is Dedekind complete.

PROOF. (i) We show that  $\mathcal{C}(\mathcal{E})$  is closed under suprema in E. If  $A \subset \mathcal{C}(\mathcal{E})$  and  $e := \sup_{E} A$  exists, then

 $0 \le e \le 1$  and  $0 \le 1 - e \le 1 - e$  for each  $e_1 \in A$ .

Hence  $0 \le e_1 \land (1-e) \le e_1 \land (1-e) = 0$  for each  $e_1 \in A$  and so

$$e \wedge (10e) = (\sup A) \wedge (1-e) = \sup_{e_1 \in A} e_1 \wedge (1-e) = 0$$

i.e.  $e \in \mathcal{C}(\mathcal{E})$ .

(ii) Since  $e \in \mathcal{C}(\mathcal{E})$  if and only if  $(1 - e) \in \mathcal{C}(\mathcal{E})$  it follows that  $\mathcal{C}(\mathcal{E})$  is closed under infima in E.

Hence  $\mathcal{C}(\mathcal{E})$  is a lattice with 0 and 1. It is distributive as a sublattice of the distributive lattice E and it is complete if E is Dedekind complete.

(iii) We show that  $e \in \mathcal{C}(\mathcal{E})$  has a complement, namely (1-e). By definition of  $\mathcal{C}(\mathcal{E})$ ,  $e \wedge (1-e) = 0$ . On the other hand

$$e \lor (1-e) = e + (1-e) = 1.$$

Corollar 7 (i)  $e_1 \perp e_1 e_1 + e_2 \in C(\mathcal{E})$   $(e_1, e_2 \in C(\mathcal{E}));$ (ii) if  $e_1 \leq e_2, e_2 - e_1 \in C(\mathcal{E});$ (iii)  $e_1 - (e_1 \wedge e_2) \perp e_2(e_1, e_2 \in C(\mathcal{E})).$ 

Proof.

(i)  $e_1 + e_2 = e_1 \lor e_2$ (ii)  $(1 - e_2) \perp e_2$  and so  $(1 - e_2) \perp e_1$ . Hence  $1 - e_2 + e_1 \in C(\mathcal{E})$  and so  $e_2 - e_1 \in C(\mathcal{E})$ .

(iii)  $e_1 - (e_1 \lor e_2) = e_1 + [(-e_1) \lor (-e_2)] = 0 \lor (e_1 - e_2) \le 1 - e_2.$ 

For concrete Riesz spaces we can identify this Boolean algebra se follows: if  $E = \mathbf{R}^S$  then  $\mathcal{C}(\mathcal{E})$  is naturally isomorphic to P(S). Similarly, if K is compact, then  $\mathcal{C}(\mathcal{C}(\mathcal{K}))$  is the algebra Clopen (S) of clopen subsets of K. If E is one of the spaces  $S(\mu)$ , of  $L^p(\mu)$ , then  $\mathcal{C}(\mathcal{E})$  is the albegra of measurable (resp. *p*-integrable) subsets of  $\Omega$  (a measurable set is *p*-integrable if its characteristic function is in  $\mathcal{L}^{\checkmark}$ ).

**Exercise** Use 1.44 to show that if X is a Boolean algebra then there is a Riesz space E with  $C(\mathcal{E}) = \mathcal{X}$ .

The following simple result shows how to construct components in certain Riesz spaces: if E is an Archimedean Riesz space with unit 1 and  $E_1$  is a band in E then  $\pi_{E_1}(1)$  (if it exists) is a componen in E. For 1 has the decomposition e + (1 - e) in  $E_1 \oplus E_1^{\perp}$  and so  $e \perp (1 - e)$ .

Another useful observation is given in the following exercise:

**Exercise** Let E be a Dedekind -complete Riesz space with unit. Then the mapping  $e \mapsto B(e)$  is a bijection from  $\mathcal{C}(\mathcal{E})$  onto the set of principal bands in E. If now x is an element of a Dedekind -complete Riesz space E with unit 1 we write

$$e(x) := \pi_x(1) = \sup\{1 \land n | x | : n \in \mathbf{N}\}.$$

This is a lattice theoretical version of the support of a function. For example, if  $x \in S(\mu)$ , then  $e(x) = \chi_N$  where N is the support of x.

We list some simple properties of e(x).

- if  $x \leq y$ , then  $e(x) \leq e(y)$  (for  $x \leq y$  implies that  $B(x) \subset B(y)$  and this in turn means that  $e(x) \leq e(y)$ );
- if  $x \perp y$  then  $e(x) \perp e(y)$  (for  $x \perp yB(x) \perp B(y)e(x) \perp e(y)$ );
- if A is a subset of E whose supremum exists, then  $e(\sup A) = \sup\{e(x) : x \in A\}$  (for if  $y = \sup A$  then

$$e(y) = \sup_{n \in \mathbf{N}} \{1 \land |ny| : n \in \mathbf{N}\} = \sup_{n \in \mathbf{N}} \sup_{x \in A} (1 \land |nx|)$$
(21)

$$= \sup_{x \in A} \sup_{n \in \mathbf{N}} (1 \wedge |nx|) = \sup_{x \in A} e(x).$$
(22)

(We are tacitly assuming that

Ais closed under finite supremaind in fima.)

**Exercise** Show that  $e(x) = e(x^+) + e(x^-)$ .

Now suppose that  $x \in E$  with E Dedekind -complete, with unit 1. Then we define:

$$e_{\lambda}(x) := e((\lambda 1 - x)^{+} \quad (\lambda \in \mathbf{R});$$
(23)

$$B_{\lambda} := B((\lambda 1 - x)^+), \tag{24}$$

$$\pi_{\lambda} := \pi_{B_{\lambda}} \quad (\lambda \in \mathbf{R}). \tag{25}$$

In  $S(\mu)$  for example,  $e_{\lambda}(x)$  is the characteristic function of  $N_{\lambda}$  where  $N_{\lambda}$  is the set  $\{x < \lambda\}$ . The reader is invited to compare the following result with III.1.5.

**Proposition 27** Lemma If  $x \in E$  and  $\lambda \leq \mu$  in **R**, then

(i) 
$$e_{\lambda}(x) \leq e_{\mu}(x);$$
  
(ii)  $\lambda(1 - e_{\lambda}(x)) \leq x - \pi_{\lambda}x;$   
(iii)  $\lambda(e_{\mu}(x) - e_{\lambda}(x)) \leq (\pi_{\mu} - \pi_{\mu})(x) \leq \mu(e_{\mu}(x) - e_{\lambda}(x));$   
(iv)  $e_{\mu}(x) = \sup\{e_{\lambda}(x) : \lambda < \mu\};$   
(v)  $\sup_{\lambda \in \mathbf{R}} \{e_{\lambda}(x)\} = 1, \inf\{e_{\lambda}(x)\} = 0;$   
(vi)  $(e_{\mu_{2}}(x) - e_{\mu_{1}}(x) \perp (e_{\lambda_{2}}(x) - e_{\lambda_{1}}(x)) \text{ where } \lambda_{1} \leq \lambda_{2} \leq \mu_{1} \leq \mu_{2} \text{ in } \mathbf{R}.$ 

Proof.

(i)  $\lambda \leq \mu \lambda 1 - x \leq \mu 1 - x(\lambda 1 - x)^+ \leq (\mu 1 - x)^+ e_{\lambda}(x) \leq e_{\mu}(x)$ (ii) if  $y \in E$ , then  $y - \pi_{B(y^+)}(y) = y - t^+ \leq 0$ . Applying this with  $y = (\lambda 1 - x)$  we get

$$(\lambda 1 - x) - \pi_{\lambda}(\lambda 1 - x) \le 0$$
 i.e.  $\lambda(1 - e_{\lambda}(x)) \le \chi - \pi_{\lambda}(x)$ .

(iii) if  $\lambda \leq \mu$ , then  $\pi_{\lambda} \circ \pi_{\mu} = \pi_{\mu} \circ \pi_{\lambda} = \pi_{\lambda}$  and so, applying  $\pi_{\mu}$  to (*ii*), we get

$$\lambda(e_{\mu}(x) - e_{\lambda}(x)) \le \pi_{\mu}(x) - \pi_{\lambda}(x).$$

On the other hand

$$(\pi_{\mu} - \pi_{\lambda}) = (\pi_{\mu} - \pi_{\lambda})\pi_{\mu}(x) \le (\pi_{\mu} - \pi_{\lambda})(\mu e_{\mu}(x))$$
(26)

$$=\mu(e_{\mu}(x) - e_{\lambda}(x)). \tag{27}$$

(iv) since E is Archimedean

$$\mu 1 - x = \sup\{\lambda 1 - x : \lambda < \mu\}$$

and so

$$(\mu 1 - x)^{+} = \sup\{(\lambda 1 - x)^{+} : \lambda < \mu\}.$$

Hence  $e_{\mu} = \sup\{e_{\lambda}(x) : \lambda < \mu\}$ noindent (v) let  $e := \sup\{e_{\lambda}(x)\}$ . Then for each  $\lambda > 0$ 

$$0 \le \lambda(1-e) \le \lambda(1-e_{\lambda}(x)) \le x - \pi_{\lambda}x \le |x|$$

ando so 1 - e = 0 e.e. e = 1.

The second part can be proved similarly. (vi)  $e_{\mu_2}(x) - e_{\mu_1}(x) \leq 1 - e_{\lambda_2}(x)$  and  $e_{\lambda_2}(x) - e_{\lambda_1}(x) \leq e_{\lambda_2}(x)$  and so

$$e_{\mu_2}(x) - e_{\mu_1}(x) \perp e_{\lambda_2}(x) - e_{\lambda_1}(x).$$

Motivated by this result we define a **spectral system** in an Archimedean Riesz space E with unit 1 to be a mapping :  $\mathbf{R} \to \mathcal{C}(\mathcal{E})$  so that

- $(\lambda) \leq (\mu)$  if  $\lambda \leq \mu$ ;
- $(\mu) = \sup\{(\lambda) : \lambda < \mu\};$
- $\inf\{(\lambda) : \lambda \in \mathbf{R}\} = 0$ ,  $\sup\{(\lambda) : \lambda \in \mathbf{R}\} = 1$ .

In particular,  $\lambda \mapsto e_{\lambda}(x)$  is a spectral system.

Using a spectral system we can define a functional calculus in the following way (recall the definition of the Riemann integral): into **R**. Then if  $p = [t_0, \ldots, t_n]$  is a partition of [a, b] (i.e.  $a = t_0 < \cdots < t_n = b$ ) we define

$$s_P^u = \sum_{i=0}^{n-1} \sup\{x(\lambda) : \lambda \in [t_i t_{i+1}]\}((\lambda_{i+1}) - (\lambda_i))$$
$$s_P^l = \sum_{i=0}^{n-1} \inf\{x(\lambda) : \lambda \in ]t_i, t_{i+1}[\}((\lambda_{i+1}) - (\lambda_i)).$$

The set of all such partitions is a directed set (under the ordering " $P_1$  refines P") and so we can regard  $\{s_P^u : P \in\}$  and  $\{s_P^l : P \in\}$  as nets.

**Proposition 28** • The net  $s_P^u$  (resp.  $s_P^l$  is decreasing resp. increasing;

- $\inf_{P \in S_P^u} = \sup_{P \in S_P^l} s_P^l;$
- both nets converge in the sense of the norm induced by the unit element to a common value.

The proof is entirely analogous to the proof of the existence of the Reimann integral for continuous function and is left to the reader. By virtue of this result we can write  $\int xd$  for the common value of these limits. We can now state our order theoretical spectral theorem:

**Proposition 29 (Freundenthal spectral theorem:** If E is a Dedekind -complete Riesz space with strong unit 1 and  $x \in E$  with  $|x| \leq K1$  (K > 0) then

$$x = \int_{-K}^{K} de_{\lambda}(x).$$

**PROOF.** We need only remark that  $s_P \leq x \leq s_P^u$  for each P (where we are using the above notation applied to the integral of the identity function with respect to the spectral system  $(e_{\lambda})$ .

**Exercises** A. A Riesz space E has the **principal projection property** if for every principal band  $E_1$  in E the projection  $\pi_{E_1}$  exists. Prove

- that this property implies that E is Archimedean;
- that 2.38 holds also for spaces with the principal projection property (in place of Dedekind -complete ness).

B. (an order theoretical version of the spectral theorem for unbounded operators). Let E be a Dedekind -complete space with unit 1. Show that for each  $x \in E$ 

$$\int_{-N}^{N} \lambda de_{\lambda}(x) @>(o) >> x \text{ as } N \to \infty.$$

(Assume that  $x \ge 0$  and put  $x_N := x \land N1$ . Apply 2.38 to  $x_N$  and show that  $X_N @>o>> x$ .)

C. (this exercise yields yet another proof of the Radon-Nikodym theorem).

Let  $\Omega$  be a set, a Boolean algebra of subset. Denote by  $M_f()$  the set of finitely additive measures on so that

$$\sup\{|\mu(A)|:A\in\}<\infty.$$

(i) Show that  $M_f()$ , with the natural ordering is a Dedekind complete Riesz space;

(ii) if is a -algebra and  $M_{()}$  denotes the countable additive measures in  $M_{f}()$ , show that  $M_{()}$  is a band in  $M_{f}()$ ;

(iii) if  $\mu, \nu \in M_{()}$ , then  $\mu \perp \nu$  if and only if  $A = A \cup B$  where  $A, B \in A \cap B =$ and  $|\nu|(A) = |\mu|(B) = 0$ ;

(iv) if  $\mu, \nu \in M_{()}$ ,  $\nu$  is in the band generated by  $\mu$  if and only if  $|\nu|$  is absolutely continuous with respect to  $|\mu|$ ;

(v)  $\nu$  is a component of  $B(\mu)$  if and only if there is an  $E \in$  so that

$$\nu: A \to \int_A \chi_E d\mu;$$

(vi) (the Radon-Nikodym theorem) let  $\mu$  be a positive measure in  $M_{()}$  and let  $\nu$  be a positive  $\mu$ -absolutely continuous measure. Then there is an  $x \in L^{1}(\mu)$  so that

$$\nu: A \to \int_A x d\mu \qquad (A \in).$$

One of the most important questions on Riesz spaces is that of their representation as funciton spaces and we now bring a simple result of this type whic will be used as the basis for more important ones later. We prove namely that every Dedekind complete Riesz space with strong unit is isomorphic to a space C(K) for some compact, extremally disconnected K (cf. the representation theorem for commutative  $B^*$ -algebras). To do this we first gibe a lattice theoretical definition of simple functions or step functions.

**Definition** Let E be a Riesz space with unit 1.  $x \in E$  is said to be simple if it has a representation  $\sum_{i=1}^{n} \lambda_i e_i$  where the  $\lambda_i$  are in **R** and the  $e_i$  are elements of the Boolean algebra  $\mathcal{C}(\mathcal{E})$  of components. We denote the set of simple elements by  $E_s$  then it clearly has a representation of the form  $x = \sum_{i=1}^{n} \lambda_i e_i$  where  $\sum e_i = 1$  and  $e_i \perp e_j$  for  $i \neq j$ . We tacitly assume below that our representations have this property. Note that if x, y are in  $E_s$ they have such representations

$$x = \sum_{i=1}^{n} \lambda_i e_i$$
$$y = \sum_{i=1}^{n} \mu_i e_i$$

with the same  $e_i$  (for if  $x = \sum_{j=1}^l \lambda_j e'_j$  and  $y = \sum_{k=1}^m \mu_k e''_k$  then x, for example, has the representation

$$\sum_{j=1}^{l} \sum_{k=1}^{m} \lambda_j e'_j \wedge e''_K.$$

For the Riesz spaces C(K),  $\mathcal{S}(\mu)$ ,  $\mathcal{L}_{\sqrt{\mu}}$  etc., the simple elements are the appropriate step functions (continuous or measurable).

- **Lemma 2** if  $x = \sum_{i=1}^{m} \lambda_i e_i = \sum_{j=1}^{n} \mu_j e'_j$  then  $\lambda_k = \mu_1$  whenever  $e_k \wedge e'_l > 0$ ;
  - if  $\sum_{i=1}^{n} \lambda_i e_i \neq \sum_{j=1}^{n} \mu_j e'_j$  there exist k and l with  $e_k \wedge e'_l > 0$  and  $\lambda_k \neq \mu_l$ ;
  - if  $x = \sum_{i=1}^{n} \lambda_i e_i$ ,  $y = \sum_{i=1}^{n} \mu_i e_i$  then  $x \leq y$  implies that  $\lambda_k \leq \mu_k$  whenever  $e_k > 0$ ;
  - if  $x = \sum_{i=1}^{n} \lambda_i e_i$  and  $|x| \le 1$  then  $\lambda_k \le$  whenever  $e_k > 0$ .

Proof.

- $\pi_{e_k \wedge e'_l}(x) = \sum_{i=1}^n \lambda_i \pi_{e_k \wedge e'_l}(e_i) = \lambda_k e_k \wedge e'_l;$
- follows from the fact that  $x = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i e_i \wedge e'_k$ ;
- $\lambda_k e_k = \pi_{e_k}(x) \le \pi_{e_k}(y) = \mu_k e_k;$
- follows from 3).

Now let E be a Riesz space with unit 1. Then we know that there is a compact space S and a Boolean algebra isomorphism  $e \to S_e$  from  $\mathcal{C}(\mathcal{E})$  into Clopen (S). If  $e \in \mathcal{C}(\mathcal{E})$  we write  $\hat{e}$  for the characteristic function  $\chi_{S_e}$  of  $S_e$  – of course, this is an element of C(S).

If  $x = \sum_{i=1}^{n} \lambda_i e_i$  is a simple element of E we write  $\hat{x}$  for the function  $\sum_{i=1}^{n} \lambda_i \hat{e}_i$  in C(S).

**Proposition 30** Lemma The mapping  $x \to \hat{x}$  is a Riesz space isomorphism from  $E_s$  onto the space of simple function in C(S). It is an isometry from  $E_{s'}$  provided with the norm  $|| ||_1$  defined by the unit of E, into C(S) (with the supremum norm).

**PROOF.** It follows from 2.41 1) and 2) that the mapping is well defined and injective. It is clearly surjective. By 3) it is posivite and so a Riesz isomorphism. It is a norm-isometry by 2.41.4.

**Exercise** Show that if Sis a -stonian space then the continuous characteristic functions separate S. Deduce that the simple functions in C(S) are dense in C(S).

**Proposition 31** Let E be a Dedekind -comlete Riesz space with strong unt 1. Then  $E_s$  is norm dense in E.

**PROOF.** This is a consequence of the Freudenthal spectral theorem 2.38 since the partial sums  $s_P^u$  with converge to x are in  $E_s$ .

If we combine the facts contained in 2.42, 2.43 and 2.44 then we see that the isometry  $x \to \hat{x}$  form  $E_s$  into the space of simple functions in C(S) can be extended in a unique way to a linear isometry from E onto C(S). We denote this extension also by  $x \to \hat{x}$ . Then we have the following representation theorem: **Proposition 32** Let E be Dedekind -complete Riesz space with strong unti 1. Then there is a unit-preserving Riesz isomorphism and norm isometry from E onto a space C(S) where S is a compact -Stonian space.

We conclude this section with some remarks on elementary properties of spaces of operators between Riesz spaces, in particular, on duality for Riesz spaces. Firstly we note the simple fact that if E is a Riesz space and  $P = \{x \in E : x \ge 0\}$ , then any operator T from P into a linear space F which satisfies the conditions

$$T(x+y) = T(x) + T(y) \quad x, y \in P$$

and

$$T(\lambda x) = \lambda T(x) \quad (\lambda \ge 0, x \in P)$$

can be extended (in a unique manner) to a linear operator from E into F. The required extension is the operator

$$T: x \mapsto T(x^+) - T(x^-).$$

**Definition** A linear operator  $T : E \to F$  between Riesz spaces is **regular** if it can be expressed as the difference  $T_1 - T_2$  of positive linear operators. We write  $L_r(E, F)$  for the space of regular oparators. These operators can be conveniently charcterised as follows:

**Proposition 33** Consider the following conditions on a linear operator  $T \in L(E, F)$  between Riesz spaces:

• ther is a positive linear operator S so that

$$Tx \leq Sx \quad (x \geq 0);$$

- T is regular;
- for each  $x \ge 0$  in E, T([0, x]) is bounded from above in F.

Then 1) and 2) are equivalent and each imply 3). If F is Dedekind complete, then all three are equivalent.

**PROOF.** 2)  $\Rightarrow$  1): If  $T = T_2 - T_1$  where  $T_1$  and  $T_2$  are positive, then

$$T(x) = T_2(x) - T_1(x) \le T_2(x) \quad (x \ge 0).$$

1)  $\Rightarrow$  2): T = S - (S - t) and S - T is positive by condition 1). 1)  $\Rightarrow$  3): S(x) is an upper bound for T([0, x]). 3)  $\Rightarrow$  1): (for F Dedekind complete): For  $x \ge 0$  we define

$$s(x) = \sup\{T(y) : y \in [0, x]\}.$$

Then the mapping  $x \to Sx$  satisfies the conditions stated befor 2.46 and so can be extended to a positive linear operator from E into F which majorises T (we verify the additive condition on S – the positive homogenity is trivial. If  $x_1 \ge 0, x_2 \ge 0$  in E, we have

$$S(x_1 + x_2) = \sup T([0, x_1 + x_2])$$
(28)

$$= \sup T([0, x_1] + [0, x_2])$$
 by (29)

$$= \sup T([0, x_1]) + \sup T([0, x_2])$$
(30)

$$= \sup S(x_1) + S(x_2).$$
(31)

Exercises A. Let T be a linear operator between the Riesz spaces E and F. Show that the following are equivalent:

- T is positive;
- if  $x \le y$ , then  $T(x) \le T(y)$ ;
- $|Tx| \le T(|x|) \quad (x \in E).$

B. Show that if T is an additive operator from E into F where E is a Riesz space and F is an Archimedean RIesz space so that  $Tx \leq Ty$  whenever  $x \leq y$ , then T is homogeneous.

C. Let T be a linear operator from a Riesz space E into a Dedekind complete Riesz space F. Show that T is regular if and only if for each  $u \ge 0$  in E, there is a  $v \ge 0$  in F so that T is continuous from  $(E_{u'}|| ||_u)$  into  $(F_{v'}|| ||_v)$ .

Note that the space  $L_r(E, F)$  is a POVS under the natural ordering (i.e.  $S \leq TT - s$  is positive).

**Proposition 34** If F is Dedekind complete, then  $L_r(E, F)$  is a Dedekind complete Riesz space.

PROOF. To show that  $L_r(E, F)$  is a Riesz space, it suffices to show that for each  $T \in L_r(E, F)$ ,  $\sup(T, )$ ) exists. But the operator A constructed in the proof of 2.47, 3)  $\Rightarrow$  1) is clearly  $\sup(T, 0)$ .

To show that  $L_r(E, F)$  is Dedekind complete, it suffices to construct a supremum for a set M of positive operators which is closed under finite suprema and is bounded from above, say by  $T_0$ . We use the operator

$$S: x \mapsto \sup\{T(x): T \in M\}$$

which is defined on  $\{x \in E : x \ge 0\}$  and can be extended to E in the standard way. (Note that the above supremum exists since the appropriate set is bounded from above by  $T_0x$ ).

**Exercises** A. Show that if  $T, T_1, T_2 \in L_r(E, F)$ , and  $x \ge 0$  in E, then

$$|T|(x) = \sup\{T(y) : |y| \le x\};$$
$$(T_1 \lor T_2)(x) = \sup\{T_1x_1 + T_2x_2 : x_1, x_2 \ge 0, x_1 + x_2 = x\}.$$

Find a corresponding formula for the supremum of an arbitrary bounded family of operators.

B. Show that

• if  $x \in E$  and  $T \in L_r(E, F)$ , then

$$|Tx| \le |T| \, |x|;$$

- if  $T_{\alpha}$  @>(0)>>T in  $L_r(E,F)$ , then  $T_{\alpha}x$  @>(0)>>Tx for each  $x \in E;$
- if  $(T_{\alpha})$  is increasing in  $L_r(E, F)$ , then  $T_{\alpha} @>(0) >> T$  if and only if  $T_{\alpha} @>(0) >> Tx$  for each  $x \in E$ .

**Definition** Let E and F be a Riesz spaces. Recall that a positive linear operator  $Tx_{\alpha} \downarrow 0$ . A linear operator  $T \in L(E, F)$  is defined to be **order continuous** if it is expressible in the form  $T_2-T_1$  where  $T_1$  and  $T_2$  are positive and order continuous. We write  $L_{\tau}(E, F)$  for the space of such mappings. Similarly T is -order continuous if it is expressible as the difference of two positive, -order continuous operators (defined after 2.13). L(E, F) denotes the space of -order continuous linear operators from E into F.

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**Exercise** Let T be a positive linear operator from the Riesz space E into the Riesz space F and let  $P = \{x \in E : x \ge 0\}$ . Show that the following are equivalent:

- T is order continuous;
- T preserves suprema i.e. for every bounded subset A of P with supremum

$$\sup T(A) = T(\sup A);$$

• for every net  $(x_{\alpha})$  in P,

$$x_{\alpha} \uparrow x \Rightarrow T(x_{\alpha}) \uparrow T(x);$$

• for every net  $(x_{\alpha})$  in E,

$$x@>(0)>>x \Rightarrow T(x_{\alpha})@>(0)>>T(x).$$

State and prove the corresponding result for  $\sigma$ -order continuous operators.

**Proposition 35** Let E be a Riesz space, F a Dedekind complete Riesz space. Then  $L_{\tau}(E, F)$  and  $L_{\sigma}(E, F)$  are bands in the Riesz space  $L_r(E, F)$ .

PROOF. We prove this for  $L_{\tau}(E, F)$ . First note that it is clear that the sum of two positive, order continuous linear operators is order continuous and hence  $L_{\tau}(E, F)$  is a linear subspace of  $L_r(E, F)$ .

It is also clear that if  $0 \leq T \in L_{\tau}(E, F)$  and if  $S \in L_{r}(E, F)$  with  $0 \leq S \leq T$ , then  $S \in L_{\tau}(E, F)$ . Hence  $L_{\tau}(E, F)$  is an ideal in  $L_{r}(E, F)$ . For if  $S \in L_{r}(E, F)$  and  $S \leq |T_{2} - T_{1}|$  where  $T_{1}$  and  $T_{2}$  are positive and order continuous, then

$$0 \le S^+ \le |S| \le |T_1| + |T_2| = T_1 + T_2$$

and so  $S^+$  is order continuous. Similarly,  $S^-$  is order continuous and hence so is  $S = S^+ - S^-$ .

Finally, let M be a subset of  $L_{\tau}(E, F)$  whose supremum T exists in  $L_r(E, F)$ . We show that T is order continuous. Without loss of generality, we can suppose that each  $S \in M$  is positive and that M is sup-closed. Then if  $x_{\alpha} \uparrow x$  in E,

$$T(x) = \sup_{S \in M} \{S(x)\} = \sup_{S \in M} \sup_{\alpha \in A} \{S(x_{\alpha})\}$$
(32)

$$= \sup_{\alpha \in A} \sup_{S \in M} \{ S(x_{\alpha}) \} = \sup_{\alpha \in A} \{ T(x_{\alpha}) \}.$$
(33)

**Exercise** Let F be an ideal in a Riesz space E. Show that the natural mapping

$$\pi_F: E \to E/F$$

is order-continuous if and only if F is a band. State and prove a corresponding result for  $\sigma$ -order continuity.

For the next result, we say that a subset  $E_1$  of a Riesz space E is **order dense** in E if for each  $x \in E$  there is an  $A \subset E_1$  with  $x \rightarrow = \sup A$ . If A can always be chosen to be countable, then  $E_1$  is  $\sigma$ -order dense.

**Proposition 36** If E is an Archimedean Riesz space, then E is order dense in its Dedekind completion. If E is a Dedekind  $\sigma$ -compelete Riesz space with unit, then  $E_S$  is  $\sigma$ -order dense in E.

**PROOF.** The first statement in just 2.13.3), while the second follows immediately from the Freundenthal spectral theorem.

Just as for metric completions, we have the standard type of result on the extension of order continuous mappings on dense subsets:

**Proposition 37** Let  $E_1$  be a  $(\sigma)$ -order dense Riesz subspace of a Dedekind  $(\sigma)$ -complete Riesz space E, T a  $(\sigma)$ -complete Riesz space. Then T has a unique extension to a  $(\sigma)$ -order continuous, linear operator  $\tilde{T}$  from E into F.

The proof is standard.

We now turn to duality for Riesz spaces. If E is such a space, we have the following dual spaces:

 $E^*$  – the algebraic dual;

 $E^r$  – the space of bounded forms on E (i.e.  $L_r(E, \mathbf{R})$ ;

 $E_{\sigma}$  – the space of  $\sigma$ -continuous forms (i.e.  $L_{\sigma}(E, \mathbf{R})$ ;

 $E_{\tau}$  – the space of  $\tau$ -continous forms (i.e.  $L_{\tau}(E, \mathbf{R})$ .

It follows immediately from 2.53 that  $E^r$  is a Dedekind complete Riesz space and that  $E_{\sigma}$  and  $E_{\tau}$  are bands therein.

We close this section with a discussion of the lattice theoretical properties of the canonical emedding of a space in its bidual.

**Lemma 3** Let f be a non-negative linear form on a Riesz space E and let  $x \in E$  be non-negative. Then there exists a  $g \in E^r$  with  $0 \leq g \leq f$ , g(x) = f(x) and g(y) = 0 if  $x \wedge y = 0$ .

PROOF. If z > 0 in E we define

$$g(z) = \sup\{f(z \wedge nx) : m \ge 0\}.$$

It is clear that g is additive and positive-homogeneous. We extend it to a linear form on E and this has the required properties.

Now if G is an ideal in the dual  $E^r$  of a Riesz space, there is a natural linear mapping  $x \to \hat{x}$  from E into  $G^*$  where  $\hat{x} : f \mapsto f(x)$ .

**Proposition 38** If G is an ideal in  $E^r$ , then the mapping  $x \mapsto \hat{x}$  is a Riesz space morphism from E into  $G^{\tau}$ .

PROOF. Firstly  $\hat{x} \in G^{\tau}$  since if  $f_{\alpha} \downarrow 0$  in G, then  $f_{\alpha}(x) @>(0) >> 0$  and so  $(f_{\alpha}) @>(0) >> 0$ . To complete the proof, it suffices to show that if  $x \land y = 0$ , then  $\hat{x} \land \hat{y} = 0$ . Suppose that  $f \in G$  with  $f \ge 0$ . There is a  $g \in E^r$  with  $0 \le g \le f$ , g(x) = f(x) and g(y) = 0 by the above Lemma. Then  $g \in G$  and

$$(\hat{x} \wedge \hat{y})(f) \le \hat{x}(f-g) + \hat{y}(g) = 0.$$

**Proposition 39** Corollary Let E be a Riesz space,  $f \in E^r$  non-negative. Then

$$f(|x|) = \sup\{g(x) : g \in E^r, |g| \le f\}$$
(34)

$$f(x \lor y) = \sup\{g(x) + (f - g)(y) : 0 \le g \le f\}.$$
(35)

We remark that one can actually show that the image of

Ein G(Eand Gasin 2.58) isorder dense.

### 2.3 Banach lattices

We now consider vextor spaces with the double structure of an ordering and a norm. As usual we impose a suitable compatibility condition:

**Definition** A normed lattice is a vector space with a norm and an ordering under which it is a Riesz space and the following condition holds: if  $|x| \leq |y|$ , then  $||x|| \leq ||y||$ . If E is complete with respect to the norm, it is called a **Banach lattice**.

Examples of Banach lattices are the classical spaces  $L^p(\mu)$   $(1 \leq p \leq \infty)$  and C(K) with the usual structures. Another instructive example is

the following: let E be a Banach space with unconditional basis  $(x_n)$  with constant one (i.e. we have

$$\left\|\sum_{n} l_n x_{\pi(n)}\right\| \le \left\|\sum l_n x_n\right\|$$

for any permutation  $\pi$  of **N** and any choise (i) of signs). Then we can define on E an ordering by putting

$$x \le y f_n(x) \le f_n(y)$$

for each n where  $(f_n)$  is the series biorthogonal to  $(x_n)$ .

**Exercise** Show that if x and y are elements of a normed lattice then

$$||x|| = ||x||, ||x + y|| = ||x - y||$$
 if  $|x| \land |y| = 0$ 

and

$$\max(||x^+||, ||x^-||) \le ||x|| \le ||x^+|| + ||x^-||.$$

It follows easily from the definitions that the operations

$$(x,y) \mapsto x \wedge y$$

and

$$(x,y) \mapsto x \wedge y$$

on a normed lattice E are uniformly continuous. Hence the positive cone  $P = \{x \in E : \ge 0\}$  is closed as are intervals i.e. sets of the form  $\{x \in E : a \le x \le b\}$ .

Using the first statement, one can extend the lattice operations to the norm completion of a normed lattice and it is not difficult to see that this completion is a Banach lattice.

We now turn to the dual of a normed lattice. Of course there are two natural candidates – namely  $E^r$  and E'. Fortunately, they coincide as we now shall see (for Banach lattices):

**Proposition 40** If E is a Banach lattice, then  $E^r$  and E' coincide.

PROOF. Firstly, it is clear that each order interval of the form [0, x] is norm bounded and so every element of E' is bounded on such intervals and so is in  $E^r$ . To prove the converse, suppose that  $f \in E^r$  is not norm bounded. By considering  $f^+$  nad  $f^-$  separately, we can assume that  $f \ge 0$ . Then we can find a sequence  $(x_n)$  in  $B_E$  so that  $|f(x_n)| \ge 2^n$ . Then if  $x = \sum_n \frac{1}{2^n} |x_n|$  one checks that  $f(x) \ge f\left(\sum_{k=1}^n \frac{1}{2^k} |x_k|\right) \ge n$  for each n which is a contradiction.

This result has the convenient consequence that the dual of a Banach lattice is itself a lattice, in fact a Dedekind complete Riesz space. This follows from the above and the simple observation that if  $f \ge 0$  in E', then

$$||f|| = \sup\{f(x) : x \in B_E, x \ge 0\}.$$

It follows immediately from 2.58 that the natural injection  $J_E$  from E into E'' is a Riesz space isomorphism from E onto sublattice of E''.

We now turn to concrete representations of suitable Banach lattices. Recall that the norm on a Banach lattice satisfies the inequalities

$$\max(||x^+||, ||x^-||) \le ||x|| \le ||x^+|| + ||x^-||.$$

We now examine the two extreme cases:

**Definition** A Banach lattice E is an **abstract** L-space if the norm satisfies the condition

$$||x + y|| = ||x|| + ||y||$$

for  $x, y \ge 0$  in E. It is an abstract M-space if

$$||x + y|| = \sup(||x||, ||y||)$$

if  $x \ge 0$  and  $y \ge 0$  are disjoint elements of E.

Of course, the typical examples are  $L^1$ -spaces and C(K) spaces respectively. Just how typical these are is the object of the next result.

Lemma 4 Let E be an abstract L-space. Then E is Dedekind complete.

**PROOF.** Suppose that  $(x_{\alpha})_A$  is an increasing net which is bounded above by x. It will suffice to show that it has a supremum. We claim firstly that each increasing sequence of elements of the net is Cauchy. For if

$$x_1 \le x_2 \le \dots \le$$

is such a sequence, then by the additivity of the norm

$$||x_2 - x_1|| + ||x_3 - x_1|| + \dots + ||x_{n+1} - x_n|| = ||x_{n+1} - x_1|| \le ||x - x_1||$$

and so  $\sum_{n} ||x_{n+1} - x_{n}|| < \infty$  which implies the result. Now if the net  $(x_{\alpha})$  where not itself Cauchy, one could construct an increasing sequence of elements from it which is also non-Chauchy. Hence  $(x_{\alpha})$  is Cauchy and so convergent. It is clear that its limit is a supremum.

We now come to our main representation theorem for L-spaces:

**Proposition 41** Let E be an L-space with order unit 1. Then there is a compact space K and positive Radon measure on K so that E is isometrically isomorphic to  $L^1(\mu)$  under an mapping which preserves the lattice structures.

PROOF. By 2.45, applied to the band generated by 1, we know that the corresponding space  $E_s$  of simple functions is lattice isomorphic to the subspace of simple functions of a C(K)-space where K is Stonian. We define a measure on K by putting

$$\mu(U) = ||e||$$

where  $e \in \mathcal{C}(\mathcal{E})$  is such that  $\hat{e} = \chi_U$ .

This measure is -additive on the clopen subset of K (since if a union of clopen sets is clopen, then it is essentially a finite union by compactness). By the results of I.5 it can be extended to a Radon measure on K. Now we show that E and  $L^1(K)$  are isomorphic in the sense of the above statement. First note that the mapping  $x \mapsto \hat{x}$  from  $E_s$  into the space of simple function of C(K) is a norm isometry for the  $L^1(\mu)$  norm on the latter. For the norm of a component is unchanged by the very definition of |mu| and so, by the additivity property, the norm of an element of  $E_s$  is also preserved. Now the simple function  $\sigma$  in C(K) are dense in  $L^1(\mu)$ . In addition,  $E_s$  is densein E. This follows from the Freudenthal spectral theorem. For if say  $x \ge 0$ , then it follows from the latter that x is the supremum of an incrasing sequence  $(x_n)$ is  $E_s$ . It follows as in the proff of 3.5 that  $||x_n - x|| \to )$ .

**Exercise** A. Generalise 3.6 as follows. Show that if E is an L-space, not necessarily with unit, then there is a locally compact space S and a Radon measure  $\mu$  on S so that E and  $L^1(\mu)$  are isomorphic in the sence of the Proposition.

B. If  $1 , then E is an abstract <math>L^p$ -space if

$$||x + y||^{p} = ||x||^{p} + ||y||^{p}$$

whenever  $x, y \ge 0$ . Show that if E is such a space and has an order unit then there is a compact space K and a Radon measure  $\mu$  on K so that E is isomorphic to  $L^p(\mu)$ . What happens in the case where E fails to have a unit.

We now turn to representations of M-spaces. Our method will be to use duality theory to reduce to the case of L-spaces.

**Lemma 5** Let E be an M-space. Show that its dual space E' is an abstract L-space.

Using this fact, we can immediately obtain the following representation for M-spaces:

**Proposition 42** Let E be an abstract M-space. Then E is isometrically and lattice isomorphic to a sublattice of a C(K)-spaces.

PROOF. By the above Lemma, E' is an abstract L-space and so isomorphic to an  $L^1(\mu)$  space. Hence E'' is isomorphic to a C(K). The result now follows from the fact hat  $J_E$  displays E as a sublattice of E''.

In view of this result the sublattices of C(K)-spaces are of some interst. One way of construction such sublattices is as follows: Let S be a subset of  $K \times K \times \mathbb{R}^+$ . If  $(s, t, \lambda)$  is a triple in S, then

$$\{x \in C(K) : x(s) = \lambda x(t)\}\$$

is clearly a sublattice of C(K). Hence so is the intersection

$$E = \{ x \in C(K) : \text{ for each triple } (s, t, \lambda) \in S \\ x(s) = \lambda x(t) \}.$$

The interest of this example lies in the fact that these are the only closed sublattices, a fact that we sahll not prove here (its proff uses the idea of the proof of the Stone-Weierstraß theorem). However, we remark that once we have this result, the following theorem can immediately be deduced from 3.9:

**Proposition 43** Let E is an abstract M-space with strong unit, then E is isometrically ordere isomorphic to a C(K)-space.