Habilitation thesis

Martingale Properties of Spline Sequences

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ABSTRACT. The underlying thesis consists of a collection of published journal papers and each of the Chapers 2–10 corresponds to one article. In Chapter 1 we summarize the results of those papers and point out their common underlying theme and the interrelations among all of them. The Bibliography chapter starting on page 13 lists all the references needed in Chapter 1 and the references corresponding to the included articles can be found at their respective end.

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CHAPTER 1

Introduction

In this work we investigate how well known and established results for martingales carry over to certain polynomial spline sequences, where the degree of the splines is allowed to be arbitrary. Here, by splines we mean functions that are piecewise algebraic polynomials. In particular, we are especially interested in properties that are true for *any* underlying filtration, where we will see below what we mean in detail by filtration in the context of splines.

Martingales are a central notion in probability theory and their applications reach far beyond probability theory to many branches of mathematics and physics. We begin by defining the notion of a martingale. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)$ an increasing sequence of σ-algebras (i.e., a filtration) contained in the σ-algebra $\mathcal{F}$. We assume here for simplicity that $\mathcal{F}$ is generated by $\cup_n \mathcal{F}_n$. A sequence of real-valued integrable random variables (X_n) on Ω is called a martingale if each X_n is measurable with respect to $\mathcal{F}_n$ and, moreover, the consecutive random variables are connected via conditional expectations as follows

$$X_n = \mathbb{E}_n X_{n+1}, \tag{1}$$

where $\mathbb{E}_n f := \mathbb{E}(f|\mathcal{F}_n)$ denotes the conditional expectation of the function $f \in L^1$ with respect to the σ-algebra $\mathcal{F}_n$. It is the unique (up to equality almost surely) $\mathcal{F}_n$-measurable function g so that

$$\int_A g \,\mathrm{d}\mathbb{P} = \int_A f \,\mathrm{d}\mathbb{P}, \qquad A \in \mathcal{F}_n. \tag{2}$$

The operator $\mathbb{E}_n$ can be seen as local averaging and in the case that $\mathcal{F}_n$ is generated by the partition $(A_j)_{j=1}^m$ of Ω into sets of positive probability, it is given by

$$\mathbb{E}_n f = \sum_{j=1}^m \frac{\int_{A_j} f \,\mathrm{d}\mathbb{P}}{\mathbb{P}(A_j)} \mathbb{1}_{A_j}, \tag{3}$$

where $\mathbb{1}_{A_j}$ denotes the characteristic function of the set A_j. A standard example for a filtration is the dyadic one $(\mathcal{G}_n)$ on $[0,1)$, and its corresponding orthogonal function system, the classical Haar system, which both can be constructed as follows: we set $\mathcal{G}_0 = \{\emptyset, [0,1)\}$ and $h_0 = \mathbb{1}_{[0,1)}$. By induction, $\mathcal{G}_{n+1}$ is constructed out of $\mathcal{G}_n$ by dividing the leftmost atom in $\mathcal{G}_n$ of maximal length into two intervals I and J of equal length (left-closed and right-open) and defining $\mathcal{G}_{n+1}$ as the σ-algebra that is generated by $\mathcal{G}_n \cup \{I, J\}$, so for instance $\mathcal{G}_1 = \sigma(\{[0,1/2),[1/2,1)\})$, $\mathcal{G}_2 = \sigma(\{[0,1/4),[1/4,1/2),[1/2,1)\})$ and so on. We additionally set $h_{n+1} := \mathbb{1}_I - \mathbb{1}_J$. The conditional expectation $\mathbb{E}(f|\mathcal{G}_n)$ is now the projection of f onto the linear span of the first $n+1$ Haar functions $(h_j)_{j=0}^n$. The fact that taking conditional expectations is tantamount to taking orthogonal projections is true not only for the dyadic filtration $(\mathcal{G}_n)$, but for general σ-algebras. In fact the conditional expectation operator $\mathbb{E}_n$ is characterized by the property that it is the orthogonal projection with respect to the canonical L^2-inner product onto the space of $\mathcal{F}_n$-measurable L^2-functions. In the setting of filtrations generated by a finite partition of Ω into intervals, as can be seen by (3), the operator $\mathbb{E}_n$ is an orthogonal projection operator onto a space of piecewise constant functions. Also note that $\mathbb{E}_n$ respects positivity, i.e., it maps non-negative functions to non-negative functions.

An extension of this idea is to consider, on $\Omega = [0,1]$ equipped with Lebesgue measure $|\cdot|$, orthogonal projection operators onto spaces of piecewise polynomial functions. This setting can be described as follows: Let $(\mathcal{F}_n)$ be an interval filtration, i.e. a sequence of increasing sub-σ-algebras of the Borel σ-algebra $\mathcal{B}$ on $[0,1]$ so that each $\mathcal{F}_n$ is generated by a partition of $[0,1]$ into a finite number of intervals having positive length. For convenience, we also assume that $(\mathcal{F}_n)$ has the properties

(4)
- $\mathcal{F}_0 = \{\emptyset, [0,1]\}$,
- $\mathcal{B}$ is generated by $\cup_n \mathcal{F}_n$,
- for each $n, \mathcal{F}_{n+1}$ is generated by $\mathcal{F}_n$ and the subdivision of one atom of $\mathcal{F}_n$ into two subintervals.

For any positive integer k, the spline space $S_k(\mathcal{F}_n)$ of order k (or degree $k-1$) is defined by

$$S_k(\mathcal{F}_n) = \{f \in C^{k-2}[0,1] : f \text{ is an algebraic polynomial of order } k \text{ on each atom of } \mathcal{F}_n\}$$

and its corresponding orthogonal projection by

$$P_n^{(k)} = \text{orthogonal projection operator onto } S_k(\mathcal{F}_n) \text{ w.r.t. } L^2(\mathcal{B}).$$

If $k = 1$, $S_k(\mathcal{F}_n)$ consists of piecewise constant functions without any smoothness conditions (interpreting $C^{-1}[0,1]$ as the space of all real-valued functions on $[0,1]$). Note that the requirement $f \in C^{k-2}[0,1]$ only means continuity of $k-2$ derivatives at the boundary points of each atom of $\mathcal{F}_n$, since polynomials are infinitely differentiable.

The above restriction to the space $\Omega = [0,1]$ arises naturally, as we want to be able to talk about polynomials on Ω and smoothness properties of functions. Additionally, as we will see in more detail later, in order for a property of the sequence $(P_n^{(k)})$ of projection operators to be true for arbitrary filtrations $(\mathcal{F}_n)$, we use results that depend on the fact that Ω is totally ordered in a way that the ordering admits a crucial relationship with a special local basis of $S_k(\mathcal{F}_n)$ that we introduce now. The main difficulty in analyzing properties of the operators $(P_n^{(k)})$ lies in the fact that for $k \geq 2$, the space $S_k(\mathcal{F}_n)$ does not have a basis consisting of disjointly supported functions as opposed to the basis $(\mathbb{1}_{A_j})_{j=1}^m$ in which $\mathbb{E}_n f$ is expanded in equation (3). The substitute for this sharply localized basis in $S_k(\mathcal{F}_n)$ is called the B-spline basis $(N_{j,k})$, where each function $N_{j,k}$ has the property that it is non-negative, its support consists of exactly k neighbouring atoms of $\mathcal{F}_n$ and it forms a partition of unity, i.e., $\sum_j N_{j,k} \equiv 1$. For a definition of B-splines and further properties we refer to the monograph [**33**]. There is also a well known recursion formula, which can also be considered as definition of $(N_{j,k})$ and we will recall it here. Let (t_j) be the increasing sequence of boundary points of atoms in $\mathcal{F}_n$, where each point occurs once with the exception of the points 0 and 1 which each occurs k times. Then, we have

$$N_{j,k}(x) = \frac{x - t_j}{t_{j+k-1} - t_j} N_{j,k-1}(x) + \frac{t_{j+k} - x}{t_{j+k} - t_{j+1}} N_{j+1,k-1}(x), \qquad j = 2-k, \ldots, m, \tag{5}$$

with the starting functions $N_{j,1} = \mathbb{1}_{[t_j, t_{j+1})}$ for $j = 1, \ldots, m$. We observe that the basis $(N_{j,k})$ is localized, but, for $k \geq 2$, there is a certain overlap among neighboring functions.

Associated to the B-spline basis $(N_{j,k})$ (we assume the parameter k to be fixed in the sequel and we write N_j for $N_{j,k}$), we define a dual basis (N_j^*) satisfying $N_j^* \in S_k(\mathcal{F}_n)$ for each j and

$$\langle N_j^*, N_i \rangle := \int_0^1 N_j^*(x) N_i(x) \, \mathrm{d}x = \delta_{ij},$$

where δ_{ij} denotes the Kronecker symbol which is 1 when $i = j$ and 0 otherwise. Using the B-spline basis and its dual, we express the projection operator $P_n^{(k)}$ as

$$P_n^{(k)} = \sum_j \langle \cdot, N_j \rangle N_j^* = \sum_{i,j} \langle \cdot, N_j \rangle \langle N_i^*, N_j^* \rangle N_i. \tag{6}$$

We are interested in properties of the sequence of operators $(P_n^{(k)})$ that are true for each filtration $(\mathcal{F}_n)$. Most theorems about martingales have this property. As a first example of such properties for martingales (or rather conditional expectations), we look at the following contractive inequality on L^p for $1 \leq p \leq \infty$, which is easy to see by Jensen's inequality:

$$\sup_n \|\mathbb{E}_n f\|_p \leq \|f\|_p, \qquad f \in L^p, 1 \leq p \leq \infty. \tag{7}$$

This inequality is true for *any* filtration $(\mathcal{F}_n)$ and it implies that the sequence of martingale differences $d_n := \mathbb{E}_n f - \mathbb{E}_{n-1} f$ forms a monotone Schauder basis in the L^p-closure of the linear span of (d_n) for $1 \leq p < \infty$ (where a collection of vectors $(x_j)_{j=1}^\infty$ in a Banach space X is called a Schauder basis of X, if each $x \in X$ can be expressed uniquely as a convergent sum $x = \sum_{j=1}^\infty a_j x_j$ for some scalars (a_j) and the word monotone means that the partial sum projections on this basis have norm one).

When carrying over (7) to spline projections $P_n^{(k)}$ instead of conditional expectations $\mathbb{E}_n$, we cannot hope for a contractive inequality on L^p-spaces for $p \neq 2$, since by [**15, 1**], conditional expectations are the only contractions on L^p that preserve constant functions. But it makes sense to ask whether for every non-negative integer k, there exists a constant C_k so that for any filtration $(\mathcal{F}_n)$,

$$\sup_n \|P_n^{(k)} f\|_p \leq C_k \|f\|_p, \qquad f \in L^p, 1 \leq p \leq \infty. \tag{8}$$

This question turned out to be very difficult and was known for a long time as C. de Boor's conjecture [**10**], but it was eventually found to be true in this generality by A. Shadrin [**34**]. There are many earlier results in this direction. Here we only mention results where the filtration is allowed to be arbitrary, but the order k of polynomials is assumed to be fixed. For $k = 2$, it was shown by Z. Ciesielski [**4**] that in the above inequality, $C_2 = 3$ works, and it was shown in [**28, 29**] that this constant is best possible. For $k = 3, 4$, C. de Boor gave upper bounds for C_k in [**9, 11**]. In view of the known exact value of C_k for $k = 1$ and $k = 2$ and by the lower estimate $C_k \geq 2k-1$ for any k given in [**34**], it is conjectured there that for any k, the best constant C_k in (8) should be $2k - 1$. For a survey of earlier results specializing also in the choice of possible filtrations $(\mathcal{F}_n)$, we refer to [**34**, Section 4.1]. We also remark here, that A. Shadrin's proof [**34**] is very long and complicated and M. v. Golitschek [**23**] gave a shorter and simplified proof of Shadrin's theorem. A crucial ingredient in both proofs is the fact that the matrix $(\langle N_i^*, N_j^* \rangle)$ has the 'checkerboard' property, i.e., $(-1)^{i+j}\langle N_i^*, N_j^*\rangle \geq 0$. This, in turn, is a consequence of the total positivity (cf. [**25**]) of its inverse matrix, the B-spline Gram matrix $B = (\langle N_i, N_j \rangle)$, which means, by definition, that any subdeterminant of B is non-negative.

In the analysis of the operator $P_n^{(k)}$ – as can be seen from (6) – growth estimates on the matrix $(a_{ij}) = (\langle N_i^*, N_j^* \rangle)$ are important. In fact, by [**5**], inequality (8) is equivalent to the existence of two uniform constants $0 < q_k < 1$ and $C_k' < \infty$ so that for all filtrations $(\mathcal{F}_n)$, the matrix (a_{ij}) admits the following geometric decay inequality:

$$|a_{ij}| \leq C_k' \frac{q_k^{|i-j|}}{|\operatorname{supp} N_i| + |\operatorname{supp} N_j|}. \tag{9}$$

The proof uses Demko's theorem [**13**] about the geometric decay of inverse band matrices. For this, note that (a_{ij}) is the inverse of the banded matrix $(\langle N_i, N_j \rangle)$. We also remark that if one applies Demko's theorem to the fact that $\|P_n^{(k)} : L^2 \to L^2\| = 1$, we only get the following estimate, which is weaker than (9)

$$|a_{ij}| \leq C_k' \frac{q_k^{|i-j|}}{|\operatorname{supp} N_i|^{1/2} |\operatorname{supp} N_j|^{1/2}}.$$

We can now ask for additional convergence properties of spline sequences similar to martingales and we begin with almost sure convergence. Almost sure convergence is connected to certain weak-type bounds of associated maximal operators. For martingales (X_j), we consider the maximal function

$$X_n^* := \max_{j \le n} |X_j|,$$

which can be estimated by Doob's inequalities:

(1) weak-type inequality: for all $t > 0$,

$$t\mathbb{P}(X_n^* > t) \le \int_{\{X_n^* > t\}} X_n \,\mathrm{d}\mathbb{P} \le \|X_n\|_1,$$

(2) norm inequality: for all $1 < p < \infty$,

$$\|X_n^*\|_p \le \frac{p}{p-1} \|X_n\|_p.$$

Using Banach's principle (see for instance [**17**, p. 1]), it can be shown that such inequalities for maximal operators imply that for $f \in L^1$,

$$\mathbb{E}_n f \to f \qquad \text{almost surely,} \tag{10}$$

where we recall that we assumed that $\mathcal{F}$ is generated by $\cup \mathcal{F}_n$. We want to extend this version of the martingale convergence theorem to splines as well and the problem is to show that for $f \in L^1$,

$$P_n^{(k)} f \to f \qquad \text{almost surely.}$$

In the special case of piecewise linear splines ($k = 2$), Z. Ciesielski and A. Kamont [**6**] proved that this is true for any interval filtration $(\mathcal{F}_n)$ satisfying (4). The idea for extending this result to arbitrary order k is a pointwise estimate of $P_n^{(k)} f$ by the Hardy-Littlewood maximal function $\mathcal{M} f$ defined by

$$\mathcal{M} f(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \,\mathrm{d}y,$$

where the supremum is taken over all intervals I containing the point x. We know from Real Analysis that this operator satisfies similar inequalities than X_n^*, i.e., we have

$$t\mathbb{P}(\mathcal{M} f > t) \le C\|f\|_1 \qquad \text{and} \qquad \|\mathcal{M} f\|_p \le C \|f\|_p \quad \text{for } 1 < p \le \infty.$$

In order to carry over such estimates to the spline maximal operator $\max_{j \le n} |P_j^{(k)} f(x)|$, we would like to derive a pointwise estimate of the form $|P_n^{(k)} f(x)| \le C_k \mathcal{M} f(x)$, where C_k is a constant that depends on the spline order k but not on the underlying filtration $(\mathcal{F}_n)$. The following short calculation shows that in order to derive such a pointwise inequality, it is sufficient to have a certain refinement of (9). In this calculation, we denote by $\operatorname{conv}(A)$ the smallest convex set that contains A and we begin by inserting formula (6) for $P_n^{(k)}$:

$$\begin{aligned}
|P_n^{(k)} f(x)| &= \Big| \sum_{i,j} \langle f, N_j \rangle a_{ij} N_i(x) \Big| \le \sum_j \sum_{i : x \in \operatorname{supp} N_i} |a_{ij}| \cdot |\langle f, N_j \rangle| \\
&\le \sum_j \sum_{i : x \in \operatorname{supp} N_i} |a_{ij}| \cdot \int_{\operatorname{supp} N_j} |f(y)| \,\mathrm{d}y \\
&\le \sum_j \sum_{i : x \in \operatorname{supp} N_i} |a_{ij}| \cdot \int_{\operatorname{conv}(\operatorname{supp} N_j \cup \operatorname{supp} N_i)} |f(y)| \,\mathrm{d}y \\
&\le \mathcal{M} f(x) \cdot \sum_j \sum_{i : x \in \operatorname{supp} N_i} |a_{ij}| \cdot |\operatorname{conv}(\operatorname{supp} N_i \cup \operatorname{supp} N_j)|.
\end{aligned}$$

Therefore, using the fact that each point x is contained in $\operatorname{supp} N_i$ for exactly k successive indices i, the following estimate on a_{ij}, improving (9), would be sufficient to deduce the desired pointwise inequality for $P_n^{(k)} f$:

$$|a_{ij}| \leq C_k' \frac{q_k^{|i-j|}}{|\operatorname{conv}(\operatorname{supp} N_i \cup \operatorname{supp} N_j)|}, \tag{11}$$

where, as before, $0 < q_k < 1$ and C_k' are constants that depend only on the underlying spline order k. The proof of this inequality is content of Chapter 2. This leads to the proof of $P_n^{(k)} f \to f$ almost surely for $f \in L^1$ and any spline order k, independently of the filtration $(\mathcal{F}_n)$ and this proof is also contained in Chapter 2.

Next, we ask the question about unconditional convergence of $P_n^{(k)} f$ for $f \in L^p$ in L^p-spaces for $1 < p < \infty$. The situation for martingales is the following: Martingales converge unconditionally in L^p, $1 < p < \infty$, i.e., the result of the sum $\sum_j d_j$ of martingale differences does not depend on the order of summation. This fact can be expressed by Burkholder's inequality

$$\Big\| \sum_{j=1}^n \varepsilon_j d_j \Big\|_p \simeq \Big\| \sum_{j=1}^n d_j \Big\|_p, \tag{12}$$

where (ε_j) is an arbitrary sequence of values in $\{+1, -1\}$. By Khintchine's inequality, we can phrase this differently using the square function $S = \big(\sum_j |d_j|^2\big)^{1/2}$ by the inequality

$$\|S\|_p \simeq \Big\| \sum_j d_j \Big\|_p, \tag{13}$$

where the implied constants in both (12) and (13) depend only on p. We want to extend unconditional convergence in L^p to spline differences $(d_j^{(k)})$. Again, we only mention previous results that are true for any filtration $(\mathcal{F}_n)$. In a series of papers by G.G. Gevorkyan, A. Kamont and A. A. Sahakian [**18, 22, 19**], the restriction to special interval filtrations $(\mathcal{F}_n)$ was removed step-by-step to show that piecewise linear spline differences $d_j^{(2)}$ converge unconditionally in L^p independently of the filtration $(\mathcal{F}_n)$. In Chapter 3, we combine methods used in [**19**] with new pointwise and norm estimates for spline differences $d_j^{(k)}$ that are a consequence of inequality (11), to show that for any spline order k, spline differences $d_j^{(k)}$ converge unconditionally in L^p independently of the filtration $(\mathcal{F}_n)$.

The space $L^1[0,1]$ does not have any unconditional Schauder basis, but we can substitute the space L^1 by H^1, the atomic Hardy space. This space is defined as the subspace of functions $f \in L^1$ having the form

$$f = \sum_{n=1}^\infty c_n a_n, \tag{14}$$

where (c_n) is a real sequence satisfying $\sum_{n=1}^\infty |c_n| < \infty$ and a_n are so called atoms, which are basic building block functions satisfying either $a_n \equiv 1$ or there exists an interval $\Gamma_n \subset [0,1]$ with

$$\operatorname{supp} a_n \subset \Gamma_n, \qquad \|a_n\|_\infty \leq |\Gamma_n|^{-1}, \qquad \int_0^1 a_n(x)\,\mathrm{d}x = 0.$$

We equip the space H^1 with the norm

$$\|f\|_{H^1} = \inf \sum_{n=1}^\infty |c_n|,$$

where the infimum is taken over all representations of f of the form (14). For more information on atomic Hardy spaces and in particular their connection to classical Hardy spaces, we refer to [**8**].

Historically, the classical Franklin system, which are spline differences of order $k = 2$ with respect to the dyadic filtration $(\mathcal{G}_n)$ defined above, was the second explicit unconditional basis in H^1 (after L. Carleson's construction [3] of a smooth version of the Haar system). This is a result due to P. Wojtaszczyk [35]. Again, we are interested in how this result generalizes if we consider different filtrations. For this problem it is not true that basis and unconditional basis property extends to any filtration, but we can give a necessary and sufficient condition on $(\mathcal{F}_n)$ for either property. In fact, in the case $k = 2$, this was settled by G. Gevorkyan and A. Kamont in [20] by giving a simple geometric criterion on the interval filtration $(\mathcal{F}_n)$ for basis and unconditional basis property of $(d_n^{(2)})$ in H^1, which we will describe now.

Let $\mathcal{G}$ be a σ-algebra in $[0,1]$ that is generated by a partition $(A_i)_{i=1}^m$ of $[0,1]$ into a sequence of intervals with $\sup A_i = \inf A_{i+1}$ for $i = 1, \ldots, m-1$. Additionally, set $A_i = \emptyset$ for $i \notin \{1, \ldots, m\}$. Then, let (for $\ell \geq 1$)

$$\delta_i^{(\ell)} := \Big| \bigcup_{j=i}^{i+\ell-1} A_j \Big|, \qquad i = 2-\ell, \ldots, m$$

be the length of the union of ℓ neighbouring atoms of $\mathcal{G}$. Finally, define

$$r_\ell(\mathcal{G}) := \max_{i=2-\ell,\ldots,m-1} \max\Big(\frac{\delta_i^{(\ell)}}{\delta_{i+1}^{(\ell)}}, \frac{\delta_{i+1}^{(\ell)}}{\delta_i^{(\ell)}} \Big) \tag{15}$$

as the maximal ratio of those neighbouring lengths. Note that by a simple calculation, $r_{\ell+1}(\mathcal{G}) \leq r_\ell(\mathcal{G}) + 1$.

Combining the results from [20] ($k = 2$) and [21] (general k), we state that for any positive integer k and any interval filtration $(\mathcal{F}_n)$, we have the following equivalence: there exists a constant C so that

$$\sup_n \|P_n^{(k)} f\|_{H^1} \leq C\|f\|_{H^1}, \qquad f \in H^1,$$

if and only if $\sup_n r_k(\mathcal{F}_n) < \infty$.

Moreover, concerning unconditional convergence of spline projections, combining the results from [20] ($k = 2$) and Chapter 4 (general k): Let $k \geq 2$ be an integer and $(\mathcal{F}_n)$ an interval filtration. Then, spline differences converge unconditionally in H^1 if and only if $\sup_n r_{k-1}(\mathcal{F}_n) < \infty$.

Now, we switch our viewpoint slightly and instead of considering

$$S_k(\mathcal{F}_n) = \{f \in C^{k-2}[0,1] : f \text{ is a polynomial of order } k \text{ on each atom of } \mathcal{F}_n\},$$

we now consider its periodic version

$$\widetilde{S}_k(\mathcal{F}_n) = \{f \in C^{k-2}(\mathbb{T}) : f \text{ is a polynomial of order } k \text{ on each atom of } \mathcal{F}_n\},$$

where $\mathbb{T}$ denotes the unit circle. Measure theoretically, there is no distinction between the unit interval $[0,1]$ and the unit circle $\mathbb{T}$, i.e. if we are considering martingales, there is no difference between $[0,1]$ and $\mathbb{T}$. The situation is different if we consider orthogonal projections $\widetilde{P}_n^{(k)}$ onto $\widetilde{S}_k(\mathcal{F}_n)$ because of additional smoothness conditions in the space $\widetilde{S}_k(\mathcal{F}_n)$. Similarly to the interval case, we can define a periodic B-spline basis $(\widetilde{N}_j)$ of $\widetilde{S}_k(\mathcal{F}_n)$ so that it has the same basic properties that the interval B-spline basis, i.e., for each j, $\widetilde{N}_j$ is a non-negative function, has local support and the collection $(\widetilde{N}_j)$ forms a partition of unity.

If we consider a fixed interval representation of $\mathbb{T}$, in contrast to the interval case, the periodic B-spline Gram matrix $(\langle \widetilde{N}_i, \widetilde{N}_j \rangle)$ is not totally positive anymore. This difference is already present when considering piecewise linear splines: it is shown in [7] that for $k = 2$ and the dyadic filtration $(\mathcal{G}_n)$ on the unit interval $[0,1]$, the so called Lebesgue constant of the class of operators $(P_n^{(2)})$, which—by definition—is given by

$$\sup_n \|P_n^{(2)} : L^\infty \to L^\infty\|,$$

has the exact value $2 + (2-\sqrt{3})^2 \approx 2.0718$, whereas if we consider the same filtration on the unit circle $\mathbb{T}$, we get the different answer [30]

$$\sup_n \|\widetilde{P}_n^{(2)} : L^\infty \to L^\infty\| = 2 + \frac{33 - 18\sqrt{3}}{13} \approx 2.1402.$$

In Chapter 5, we show that also for the periodic projection operators $(\widetilde{P}_n^{(k)})$ and arbitrary filtrations $(\mathcal{F}_n)$ as in (4), we have, for $f \in L^1$,

$$\widetilde{P}_n^{(k)} f \to f \qquad \text{almost surely.}$$

We also give a new proof of the fact that the operators $(\widetilde{P}_n^{(k)})$, as operators acting on L^p, $1 \leq p \leq \infty$, are uniformly bounded, which proceeds by relating periodic spline spaces to spline spaces on the interval in a delicate way. It should be noted that by looking at [4], the proof of this fact for $k = 2$ can be done exactly in the same manner than in the non-periodic setting. The already existing proof of this fact for general spline orders k, unfortunately, is unpublished and can be deduced by generalizing A. Shadrin's proof [34] for the interval $[0, 1]$ first to spline projections on the real line (which is done in [12]) and then by viewing periodic functions as defined on the whole real line.

In Chapter 6, we also extend the result contained in Chapter 3 about the unconditionality of spline differences $d_n^{(k)}$ in L^p-spaces in the reflexive range $1 < p < \infty$ to periodic splines, i.e., we show that the periodic spline differences $\widetilde{d}_n^{(k)} = \widetilde{P}_n^{(k)} f - \widetilde{P}_{n-1}^{(k)} f$ also converge unconditionally in this range of L^p-spaces for all filtrations $(\mathcal{F}_n)$. This also extends the earlier piecewise linear periodic result in [26]. One main difficulty to overcome in the course of this proof was that despite the fact that an estimate of the type (9) also holds in the periodic setting, estimate (11) does not extend to the periodic setting.

In Chapter 7, we show that the result for almost sure convergence for L^1-functions f on the unit interval $[0, 1]$

$$P_n^{(k)} f \to f, \qquad \text{almost surely,}$$

extends to tensor product spline projections on $[0, 1]^d$

$$P_{n_1}^{(k_1)} \otimes \cdots \otimes P_{n_d}^{(k_d)} f \to f, \tag{16}$$

provided that f is contained in the Orlicz space $L(\log L)^{d-1}$, i.e. $\int |f|(\log^+ |f|)^{d-1} < \infty$ where $\log^+ x = \max(0, \log x)$. On the one hand, this result is in the spirit of the theorem by Jessen, Marcinkiewicz, Zygmund [24] that shows for $f \in L(\log L)^{d-1}$, almost every point in $[0, 1]^d$ is a strong Lebesgue point of f. We recall that a point $x \in [0, 1]^d$ is called a strong Lebesgue point of the function f, if

$$\frac{1}{|Q_m|} \int_{Q_m} |f(s) - f(x)| \, \mathrm{d}s \to 0,$$

where (Q_m) is a sequence of rectangles parallel to the coordinate axes containing the point x with $\operatorname{diam} Q_m \to 0$. On the other hand, we can compare it to the martingale result by Cairoli [2], who showed that multiparameter conditional expectations of multivariate functions $f \in L(\log L)^{d-1}$ converge almost surely. We also show in Chapter 7 that the space $L(\log L)^{d-1}$ in the assertion of almost sure convergence is somehow best possible, i.e., we show that for any larger Orlicz class Λ of functions than $L(\log L)^{d-1}$, there exists a function $f \in \Lambda$ so that $P_{n_1}^{(k_1)} \otimes \cdots \otimes P_{n_d}^{(k_d)} f$ does not converge almost surely, cf. [32] for the case $k_1 = \cdots = k_d = 1$.

We now go back to (10) and consider again almost sure convergence, but in an extended setting. We begin by noting that for martingales, actually a more general result than (10) is true; in fact, any martingale (X_n) with $\sup \|X_n\|_1 < \infty$ converges almost surely, even without specifying the limit function in advance. One way of identifying the limit of (X_n) proceeds by using a compactness argument in the space of Radon measures followed by an application of the Radon-Nikodým theorem on the absolutely continuous part of the limiting measure, whose density function is the desired a.s. limit of (X_n). We recall that the Radon-Nikodým theorem states that if for two σ-finite measures μ, ν so that ν is absolutely continuous with respect to μ (i.e. $\mu(A) = 0 \implies \nu(A) = 0$), there exists a μ-integrable function f so that for any measurable set A, we have

$$\nu(A) = \int_A f \, \mathrm{d}\mu.$$

When considering martingales, we can also consider vector-valued martingales, where here, vector-valued means Banach-space-valued. For measures with values in Banach spaces X, the Radon-Nikodým theorem is not true anymore in general, but if it is true, X is said to have the Radon-Nikodým property. Examples of Banach spaces with the Radon-Nikodým property include all reflexive Banach spaces and all separable dual spaces. This property of a Banach space X in fact is enough so that any martingale bounded in the Bochner-Lebesgue space L^1_X converges almost surely. Martingale convergence even is a characterization of the Radon-Nikodym property, i.e., the following statements about a Banach space X are equivalent:

(1) X has the Radon-Nikodým property,
(2) every X-valued martingale bounded in L^1_X converges almost surely,

For those results and more about vector measures and vector-valued martingales, see [**14, 31**].

In Chapter 8 and 9, we generalize this characterization theorem to spline projections. In order to state this result, we define that a sequence of functions $(f_n)_{n\geq 0}$ in L^1_X is an (X-valued) *k-martingale spline sequence adapted to* $(\mathcal{F}_n)$ if

$$P_n^{(k)} f_{n+1} = f_n, \qquad n \geq 0.$$

This definition resembles the definition of a martingale with the conditional expectation operator replaced by a spline projection operator.

Then, the spline version of the above result reads as follows: for any positive integer k, the following statements about a Banach space X are equivalent:

(1) X has the Radon-Nikodým property,
(2) every X-valued k-martingale spline sequence bounded in L^1_X converges almost surely,

In Chapter 8, we show the implication (1) $\implies$ (2) and characterize the a.e. limit of every L^1_X-bounded martingale spline sequence intrinsically. In Chapter 9, we show the implication (2) $\implies$ (1) by constructing a non-convergent martingale spline sequence in every Banach space X that does *not* have the Radon-Nikodým property.

In Chapter 10, we extend D. Lépingle's $L^1(\ell^2)$ inequality [**27**]

$$\Big\| \Big(\sum_n \mathbb{E}[f_n | \mathcal{F}_{n-1}]^2 \Big)^{1/2} \Big\|_1 \leq 2 \cdot \Big\| \Big(\sum_n f_n^2 \Big)^{1/2} \Big\|_1, \qquad f_n \in \mathcal{F}_n, \tag{17}$$

to the case where we substitute the conditional expectation operators with orthogonal projection operators $P_n^{(k)}$ onto spline spaces and where we can allow that f_n is contained in a suitable spline space $S_k(\mathcal{F}_n)$. This is done provided the filtration $(\mathcal{F}_n)$ satisfies the regularity condition $\sup_n r_k(\mathcal{F}_n) < \infty$. Recall that the number $r_k(\mathcal{F}_n)$ was defined in (15) as the maximal length ratio of neighbouring B-spline supports in $\mathcal{F}_n$. Using similar techniques, we also obtain a spline version of C. Fefferman's H^1-BMO duality [**16**] under this assumption.

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CHAPTER 2

On almost everywhere convergence of orthogonal spline projections with arbitrary knots

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On almost everywhere convergence of orthogonal spline projections with arbitrary knots

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Abstract

The main result of this paper is a proof that, for any $f \in L_1[a,b]$, a sequence of its orthogonal projections $(P_{\Delta_n}(f))$ onto splines of order k with arbitrary knots Δ_n converges almost everywhere provided that the mesh diameter $|\Delta_n|$ tends to zero, namely

$$f \in L_1[a,b] \Rightarrow P_{\Delta_n}(f,x) \to f(x) \quad \text{a.e. } (|\Delta_n| \to 0).$$

This extends the earlier result that, for $f \in L_p$, we have convergence $P_{\Delta_n}(f) \to f$ in the L_p-norm for $1 \le p \le \infty$, where we interpret L_∞ as the space of continuous functions.

Keywords: Splines; Inverse Gram matrix; Almost everywhere convergence

1. Introduction

Let an interval $[a,b]$ and $k \in \mathbb{N}$ be fixed. For a knot-sequence $\Delta_n = (t_i)_{i=1}^{n+k}$ such that

$$\begin{aligned} &t_i \le t_{i+1}, \qquad t_i < t_{i+k}, \\ &t_1 = \cdots = t_k = a, \qquad b = t_{n+1} = \cdots = t_{n+k}, \end{aligned}$$

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let $(N_i)_{i=1}^n$ be the sequence of L_∞-normalized B-splines of order k on Δ_n forming a partition of unity, with the properties

$$\operatorname{supp} N_i = [t_i, t_{i+k}], \quad N_i \geq 0, \quad \sum_i N_i \equiv 1.$$

For each Δ_n, we define then the space $\mathcal{S}_k(\Delta_n)$ of splines of order k with knots Δ_n as the linear span of (N_i), namely

$$s \in \mathcal{S}_k(\Delta_n) \Leftrightarrow s = \sum_{i=1}^n c_i N_i, \quad c_i \in \mathbb{R},$$

so that $\mathcal{S}_k(\Delta_n)$ is the space of piecewise polynomial functions of degree $\leq k-1$, with $k-1-m_i$ continuous derivatives at t_i, where m_i is multiplicity of t_i. Throughout the paper, we use the following notations:

$$\begin{aligned} I_i &:= [t_i, t_{i+1}], \qquad & h_i &:= |I_i| := t_{i+1} - t_i, \\ E_i &:= [t_i, t_{i+k}], \qquad & \kappa_i &:= |E_i| := t_{i+k} - t_i, \end{aligned}$$

where E_i is the support of the B-spline N_i. With $\operatorname{conv}(A, B)$ standing for the convex hull of two sets A and B, we also set

$$\begin{aligned} I_{ij} &:= \operatorname{conv}(I_i, I_j) = [t_{\min(i,j)}, t_{\max(i,j)+1}], \\ E_{ij} &:= \operatorname{conv}(E_i, E_j) = [t_{\min(i,j)}, t_{\max(i,j)+k}]. \end{aligned}$$

Finally, $|\Delta_n| := \max_i |I_i|$ is the mesh diameter of Δ_n.

Now, let P_{Δ_n} be the orthoprojector onto $\mathcal{S}_k(\Delta_n)$ with respect to the ordinary inner product $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$, i.e.,

$$\langle P_{\Delta_n} f, s\rangle = \langle f, s\rangle, \quad \forall s \in \mathcal{S}_k(\Delta_n),$$

which is well-defined for $f \in L_1[a, b]$.

Some time ago, one of us proved [12] de Boor's conjecture that the max-norm of P_{Δ_n} is bounded independently of the knot-sequence, i.e.,

$$\sup_{\Delta_n} \|P_{\Delta_n}\|_\infty < c_k. \tag{1.1}$$

This readily implies convergence of orthogonal spline projections in the L_p-norm,

$$f \in L_p[a, b] \Rightarrow P_{\Delta_n}(f) \overset{L_p}{\to} f, \quad 1 \leq p \leq \infty, \tag{1.2}$$

where we interpret L_∞ as C, the space of continuous functions. In this paper, we prove that the max-norm boundedness of P_{Δ_n} implies also almost everywhere (a.e.) convergence of orthogonal projections $(P_{\Delta_n}(f))$ with arbitrary knots Δ_n provided that the mesh diameter $|\Delta_n|$ tends to zero.

The main outcome of this article is the following statement.

Theorem 1.1. *For any $k \in \mathbb{N}$ and any sequence of partitions (Δ_n) such that $|\Delta_n| \to 0$, we have*

$$f \in L_1[a, b] \Rightarrow P_{\Delta_n}(f, x) \to f(x) \quad \text{a.e.} \tag{1.3}$$

The proof is based on the standard approach of verifying two conditions that imply a.e. convergence for $f \in L_1$:

(1) there is a dense subset $\mathcal{F}$ of L_1 such that $P_{\Delta_n}(f, x) \to f(x)$ a.e. for $f \in \mathcal{F}$;
(2) the maximal operator $P^*(f, x) := \sup_n |P_{\Delta_n}(f, x)|$ is of the weak (1, 1)-type,

$$m\{x \in [a, b] : P^*(f, x) > t\} < \frac{c_k}{t} \|f\|_1, \tag{1.4}$$

with mA being the Lebesgue measure of A. The first condition is easy: by (1.2), a.e. convergence (in fact, uniform convergence) takes place for continuous functions,

$$f \in C[a, b] \Rightarrow P_{\Delta_n}(f, x) \to f(x) \quad \text{uniformly in } x. \tag{1.5}$$

For the non-trivial part (1.4), we prove a stronger inequality of independent interest, namely that

$$|P_{\Delta_n}(f, x)| \le c_k M(f, x), \tag{1.6}$$

where $M(f, x)$ is the Hardy–Littlewood maximal function. It satisfies a weak (1, 1)-type inequality, hence (1.4) holds too.

The main technical tool which leads to (1.6) is a new estimate for the elements $\{a_{ij}\}$ of the inverse of the Gram matrix of the B-spline functions, which reads as follows.

Theorem 1.2. *For any Δ_n, let $\{a_{ij}\}_{i,j=1}^n$ be the inverse of the B-spline Gram matrix $\{\langle N_i, N_j\rangle\}$. Then,*

$$|a_{ij}| \le K\gamma^{|i-j|} h_{ij}^{-1}, \tag{1.7}$$

where

$$h_{ij} := \max\{h_s : I_s \subset E_{ij}\},$$

and $K > 0$ and $\gamma \in (0, 1)$ are constants that depend only on k, but not on Δ_n.

A pass from (1.7) to (1.6) proceeds as follows. Let K_{Δ_n} be the Dirichlet kernel of the operator P_{Δ_n}, defined by the relation

$$P_{\Delta_n}(f, x) = \int_a^b K_{\Delta_n}(x, y) f(y)\, dy, \quad \forall f \in L_1[a, b].$$

Then, (1.7) implies the inequality

$$|K_{\Delta_n}(x, y)| \le C\,\theta^{|i-j|} |I_{ij}|^{-1}, \quad x \in I_i,\ y \in I_j, \tag{1.8}$$

where $C > 0$ and $\theta \in (0, 1)$. Now, (1.6) is immediately obtained from (1.8).

With a bit more sophisticated arguments, though still standard ones, estimate (1.8) on K_{Δ_n} allows us also to prove convergence of $P_{\Delta_n} f$ at Lebesgue points of f. The latter forms a set of full measure, so we derive this refinement of Theorem 1.1 as a byproduct.

Estimate (1.7) is also useful in other applications, for instance in [10] it is applied to obtain unconditionality of orthonormal spline bases with arbitrary knot-sequences in L_p-spaces for $1 < p < \infty$.

We note that, previously, a.e. convergence of spline orthoprojections was studied by Ciesielski [3] who established (1.3) for dyadic partitions with any $k \in \mathbb{N}$, and by Ciesielski–Kamont [5] who proved this result for any Δ_n with $k = 2$, i.e., for linear splines. Both papers used (1.7) as an intermediate step, however our proof of (1.7) for all k with arbitrary knots Δ_n is based on quite different arguments. The main difference is that the proof of (1.7) for linear splines in [5] does not rely on the mesh-independent bound (1.1) for $\|P_{\Delta_n}\|_\infty$, and can be used to get such a bound for linear splines, whereas our proof depends on (1.1) in an essential manner.

The paper is organized as follows. In Section 2, we show how Theorem 1.2 leads to (1.8) and the latter to (1.6). We complete then the proof of a.e. convergence of $(P_{\Delta_n}(f))$ using the scheme indicated above. In Section 3, as a byproduct, we show that $(P_{\Delta_n}(f))$ converges at Lebesgue points, thus characterizing the convergence set in a sense. Theorem 1.2 is proved then in Section 4 based on Lemma 4.1, which lists several specific properties of the inverse $\{a_{ij}\}$ of the B-spline Gram matrix $G_0 := \{\langle N_i, N_j\rangle\}$. Those properties are proved in Section 5, and they are based mostly on Demko's theorem on the inverses of band matrices, which we apply to the rescaled Gram matrix $G := (\langle M_i, N_j\rangle)$, where $M_i := \frac{k}{\kappa_i} N_i$. The uniform bound $\|G^{-1}\|_\infty < c_k$, being equivalent to (1.1), plays a crucial role here.

2. Proof of Theorem 1.1

Here, we prove the weak-type inequality (1.4), then recall a simple proof of (1.5), and as a result deduce the a.e. convergence for all $f \in L_1$.

We begin with an estimate for the Dirichlet kernel K_{Δ_n}.

Lemma 2.1. *For any Δ_n, the Dirichlet kernel K_{Δ_n} satisfies the inequality*

$$|K_{\Delta_n}(x, y)| \le C\,\theta^{|i-j|}|I_{ij}|^{-1}, \quad x \in I_i,\ y \in I_j, \tag{2.1}$$

where $C > 0$ and $\theta \in (0, 1)$ are constants that depend only on k.

Proof. First note that, with the inverse $\{a_{\ell m}\}$ of the B-spline Gram matrix $\{\langle N_\ell, N_m\rangle\}$, the Dirichlet kernel K_{Δ_n} can be written in the form

$$K_{\Delta_n}(x, y) = \sum_{\ell,m=1}^{n} a_{\ell m} N_\ell(x) N_m(y).$$

For $x \in I_i$ and $y \in I_j$, since $\operatorname{supp} N_\ell = [t_\ell, t_{\ell+k}]$ and $\sum N_\ell(x)N_m(y) \equiv 1$, we obtain

$$|K_{\Delta_n}(x, y)| \le \max_{\substack{i-k+1\le \ell\le i\\ j-k+1\le m\le j}} |a_{\ell m}|.$$

Next, we rewrite inequality (1.7) for $a_{\ell m}$ in terms of $E_{\ell m} = [t_{\min(\ell,m)}, t_{\max(\ell,m)+k}]$: as $h_{\ell m}$ is the largest knot-interval in $E_{\ell m}$, we have $h_{\ell m}^{-1} \le (|\ell - m| + k)|E_{\ell m}|^{-1}$, hence for any real number $\theta \in (\gamma, 1)$,

$$|a_{\ell m}| \le K\gamma^{|\ell-m|}(|\ell - m| + k)|E_{\ell m}|^{-1} \le C_1\theta^{|\ell-m|}|E_{\ell m}|^{-1},$$

where C_1 depends on k and θ. Therefore,

$$|K_{\Delta_n}(x, y)| \le C_1 \max_{\substack{i-k+1\le \ell\le i\\ j-k+1\le m\le j}} \theta^{|\ell-m|}|E_{\ell m}|^{-1}.$$

For indices ℓ and m in the above maximum, we have $I_{ij} \subset E_{\ell m}$, hence $|E_{\ell m}|^{-1} \le |I_{ij}|^{-1}$, and also $|\ell - m| > |i - j| - k$, hence $\theta^{|\ell-m|} \le \theta^{-k}\theta^{|i-j|}$, and inequality (2.1) follows. $\square$

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Definition 2.2. For an integrable f, the Hardy–Littlewood maximal function is defined as

$$M(f,x) := \sup_{I \ni x} |I|^{-1} \int_I |f(t)|\,dt, \tag{2.2}$$

with the supremum taken over all intervals I containing x. As is known [13, p. 5], it satisfies the following weak-type inequality

$$m\{x \in [a,b] : M(f,x) > t\} \le \frac{5}{t}\,\|f\|_1. \tag{2.3}$$

Proposition 2.3. *For any Δ_n, we have*

$$|P_{\Delta_n}(f,x)| \le c_k M(f,x), \qquad x \in [a,b]. \tag{2.4}$$

Proof. Let $x \in [a,b]$, and let the index i be such that $x \in I_i$ and $|I_i| \neq 0$. By definition of the Dirichlet kernel K_{Δ_n},

$$P_{\Delta_n}(f,x) = \int_a^b K_{\Delta_n}(x,y) f(y)\,dy,$$

so using inequality (2.1) from the previous lemma, we obtain

$$|P_{\Delta_n}(f,x)| \le \sum_{j=1}^n \int_{I_j} |K_{\Delta_n}(x,y)|\,|f(y)|\,dy \le C \sum_{j=1}^n \frac{\theta^{|i-j|}}{|I_{ij}|} \int_{I_j} |f(y)|\,dy.$$

Since $I_j \subset I_{ij}$ and $x \in I_i \subset I_{ij}$, the definition (2.2) of the maximal function implies $\int_{I_j} |f(y)|\,dy \le \int_{I_{ij}} |f(y)|\,dy \le |I_{ij}| M(f,x)$. Hence,

$$|P_{\Delta_n}(f,x)| \le C \sum_{j=1}^n \theta^{|i-j|} M(f,x),$$

and (2.4) is proved. □

On combining (2.4) and (2.3), we obtain a weak-type inequality for P^*.

Corollary 2.4. *For the maximal operator $P^*(f,x) := \sup_n |P_{\Delta_n}(f,x)|$, we have*

$$m\{x \in [a,b] : P^*(f,x) > t\} \le \frac{c_k}{t}\,\|f\|_1. \tag{2.5}$$

The next statement is a straightforward corollary of (1.1); we give its proof for completeness.

Proposition 2.5. *We have*

$$f \in C[a,b] \Rightarrow P_{\Delta_n}(f,x) \to f(x) \quad \textit{uniformly}. \tag{2.6}$$

Proof. Since P_{Δ_n} is a linear projector and $\|P_{\Delta_n}\|_\infty \le c_k$ by (1.1), the Lebesgue inequality gives us

$$\|f - P_{\Delta_n} f\|_\infty \le (c_k + 1)\, E_{\Delta_n}(f),$$

where $E_{\Delta_n}(f)$ is the error of the best approximation of f by splines from $\mathcal{S}_k(\Delta_n)$ in the uniform norm. It is known that

$$E_{\Delta_n}(f) \le c_k \omega_k(f, |\Delta_n|),$$

where $\omega_k(f,\delta)$ is the kth modulus of smoothness of f. Since $\omega_k(f,\delta) \to 0$ as $\delta \to 0$, we have the uniform convergence

$$\|f - P_{\Delta_n} f\|_\infty \to 0 \quad (|\Delta_n| \to 0),$$

and that proves (2.6). □

Proof of Theorem 1.1. The derivation of the almost everywhere convergence of $P_{\Delta_n} f$ for $f \in L_1$ from the weak-type inequality (2.5) and convergence on the dense subset (2.6) follows a standard scheme which can be found in [8, pp. 3–4]. We present this argument for completeness.

Let $v \in L^1[a,b]$. We define

$$R(v,x) := \limsup_{n\to\infty} P_{\Delta_n} v(x) - \liminf_{n\to\infty} P_{\Delta_n} v(x)$$

and note that $R(v,x) \le 2P^*(v,x)$, therefore, by (2.5),

$$m\{x \in [a,b] : R(v,x) > \delta\} \le \frac{2c_k}{\delta}\,\|v\|_1. \tag{2.7}$$

Also, for any continuous function g we have $R(g,x) \equiv 0$ by (2.6), and since P_{Δ_n} is linear,

$$R(f,x) \le R(f-g,x) + R(g,x) = R(f-g,x).$$

This implies, for a given $f \in L_1$ and any $g \in C$,

$$m\{x \in [a,b] : R(f,x) > \delta\} \le m\{x \in [a,b] : R(f-g,x) > \delta\} \overset{(2.7)}{\le} \frac{2c}{\delta}\,\|f-g\|_1.$$

Letting $\|f-g\|_1 \to 0$, we obtain, for every $\delta > 0$,

$$m\{x \in [a,b] : R(f,x) > \delta\} = 0,$$

so $R(f,x) = 0$ for almost all $x \in [a,b]$. This means that $P_{\Delta_n} f$ converges almost everywhere. It remains to show that this limit equals f a.e., but this is obtained by replacing $R(f,x)$ by $|\lim_{n\to\infty} P_n f(x) - f(x)|$ in the above argument. □

3. Convergence of $P_{\Delta_n}(f)$ at the Lebesgue points

Here, we show that the estimate (2.1) for the Dirichlet kernel implies convergence of $P_{\Delta_n}(f,x)$ at the Lebesgue points of f. Since by the classical Lebesgue differentiation theorem the set of all Lebesgue points has the full measure, this gives a more precise version of Theorem 1.1.

We use standard arguments similar to those used in [7, Chapter 1, Theorem 2.4] for integral operators, or in [9, Chapter 5.4] for wavelet expansions.

Recall that a point x is said to be a Lebesgue point of f if

$$\lim_{I\ni x,\,|I|\to 0} |I|^{-1} \int_I |f(x) - f(y)|\,dy = 0,$$

where the limit is taken over all intervals I containing the point x, as the diameter of I tends to zero.

Theorem 3.1. *Let x be a Lebesgue point of the integrable function f, and let (Δ_n) be a sequence of partitions of $[a,b]$ with $|\Delta_n| \to 0$. Then,*

$$\lim_{n\to\infty} P_{\Delta_n}(f,x) = f(x).$$

Proof. Let x be a Lebesgue point of f. Since the spline space $\mathcal{S}_k(\Delta_n)$ contains constant functions, we have $\int_a^b K_{\Delta_n}(x,y)\,dy = 1$ for any $x \in [a,b]$, so we need to prove that

$$\int_a^b K_{\Delta_n}(x,y)[f(x)-f(y)]\,dy \to 0 \quad (n \to \infty). \tag{3.1}$$

For $r > 0$, set $B_r(x) := [x-r, x+r] \cap [a,b]$. Now, given $\varepsilon > 0$, let δ be such that

$$|I|^{-1} \int_I |f(x)-f(y)|\,dy < \varepsilon \tag{3.2}$$

for all intervals I with $I \subset B_{2\delta}(x)$ and $I \ni x$. Further, with $\theta \in (0,1)$ from inequality (2.1), take m and $N = N(m)$ such that

$$\theta^m < \varepsilon\delta, \qquad (m+2)|\Delta_n| < \delta \quad \forall n \geq N,$$

and consider any such Δ_n.

(1) Let $|x-y| > \delta$, and let $x \in I_i$ and $y \in I_j$. Then $|i-j| > m$ and $|I_{ij}| > \delta$, hence, by inequality (2.1) for the Dirichlet kernel K_{Δ_n},

$$|K_{\Delta_n}(x,y)| \leq C\theta^m \delta^{-1} \leq C\varepsilon.$$

As a consequence,

$$\int_{|x-y|>\delta} |K_{\Delta_n}(x,y)|\,|f(x)-f(y)|\,dy \leq C\varepsilon \int_a^b |f(x)-f(y)|\,dy \leq 2C\varepsilon\|f\|_1. \tag{3.3}$$

(2) Let $|x-y| \leq \delta$, i.e., $y \in B_\delta(x)$, and let $x \in I_i$. Note that if $I_j \cap B_\delta(x) \neq \emptyset$, then $I_j \subset B_{2\delta}(x)$, hence $I_{ij} \subset B_{2\delta}(x)$ as well, and again, by inequality (2.1),

$$\begin{aligned}\int_{B_\delta(x)} |K_{\Delta_n}(x,y)|\,|f(x)-f(y)|\,dy &\leq \sum_{j:I_j \cap B_\delta(x) \neq \emptyset} \int_{I_j} |K_{\Delta_n}(x,y)|\,|f(x)-f(y)|\,dy \\ &\leq C \sum_{j:I_{ij} \subset B_{2\delta}(x)} \theta^{|i-j|}\Big(|I_{ij}|^{-1}\int_{I_{ij}} |f(x)-f(y)|\,dy\Big).\end{aligned}$$

By (3.2), since $x \in I_{ij} \subset B_{2\delta}(x)$, the terms in the parentheses are all bounded by ε, therefore

$$\int_{|x-y|<\delta} |K_{\Delta_n}(x,y)|\,|f(x)-f(y)|\,dy \leq C\varepsilon \sum_j \theta^{|i-j|} \leq C_1\varepsilon. \tag{3.4}$$

Combining estimates (3.3) and (3.4) for the integration over $|x-y| > \delta$ and $|x-y| < \delta$, respectively, we obtain (3.1), i.e. convergence of $P_{\Delta_n}(f,x)$ to $f(x)$ at Lebesgue points of f, provided $|\Delta_n| \to 0$. $\square$

4. Proof of Theorem 1.2

We will prove (1.7) for $i \leq j$. This proves also the case $i \geq j$, since $h_{ij} = h_{ji}$ and $a_{ij} = a_{ji}$. So, for the entries $\{a_{ij}\}$ of the inverse of the matrix $\{\langle N_i, N_j\rangle\}$, we want to show that

$$|a_{ij}| \leq K\gamma^{|i-j|} h_{ij}^{-1}, \tag{4.1}$$

where h_{ij} is the length of a largest subinterval of $[t_i, t_{j+k}]$. The proof is based on the following lemma.

Lemma 4.1. *For any Δ_n, let $\{a_{ij}\}$ be the inverse of the B-spline Gram matrix $\{\langle N_i, N_j\rangle\}$. Then*

$$|a_{is}| \le K_1 \gamma^{|i-s|}(\max\{\kappa_i, \kappa_s\})^{-1}, \tag{4.2}$$

$$|a_{ij}| \le K_2 \gamma^{|\ell-j|} \sum_{\mu=\ell-(k-1)}^{\ell+k-2} |a_{i\mu}|, \quad i+k \le \ell < j, \tag{4.3}$$

$$|a_{i\mu}| \le K_3 \max_{\mu-(k-1)\le s\le \mu-1} |a_{is}|, \quad i < \mu, \tag{4.4}$$

where $K_i > 0$ and $\gamma \in (0,1)$ are some constants that depend only on k.

Remark 4.2. All three estimates are known in a sense. Inequalities (4.2) and (4.3) follow from Demko's theorem [6] on inverses of band matrices and the fact [12] that the inverse of the Gram matrix $G = \{(M_i, N_j)\}_{i,j=1}^n$ satisfies $\|G^{-1}\|_\infty < c_k$. Actually, (4.2) was explicitly given by Ciesielski [4], while (4.3) is a part of Demko's proof. Inequality (4.4) appeared in Shadrin's manuscript [11], and it does not use the uniform boundedness of $\|G^{-1}\|_\infty$. As those estimates are scattered in the aforementioned papers, we extract the relevant parts from them and present the proofs of (4.2)–(4.4) in Section 5.

Proof of Theorem 1.2. Let I_ℓ be a largest subinterval of $[t_i, t_{j+k}]$, i.e.,

$$h_{ij} = \max\{h_s\}_{s=i}^{j+k-1} = h_\ell.$$

(1) If I_ℓ belongs to the support of N_i or that of N_j, then

$$\max(\kappa_i, \kappa_j) \ge h_\ell = h_{ij},$$

and, by (4.2),

$$|a_{ij}| \le K_1 \gamma^{|i-j|}(\max\{\kappa_i, \kappa_j\})^{-1} \le K_1 \gamma^{|i-j|} h_{ij}^{-1},$$

so (4.1) is true.

(2) Now, assume that I_ℓ does not belong to the supports of either N_i or N_j, i.e.,

$$i + k \le \ell < j.$$

Consider the B-splines $(N_s)_{s=\ell+1-k}^{\ell}$ whose support $[t_s, t_{s+k}]$ contains $I_\ell = [t_\ell, t_{\ell+1}]$. Then

$$\kappa_s \ge h_\ell = h_{ij}, \quad \ell-(k-1) \le s \le \ell.$$

Using estimate (4.2), we obtain for such s

$$|a_{is}| \le K_1 \gamma^{|i-s|} \kappa_s^{-1} \le K_1 \gamma^{|i-s|} h_{ij}^{-1} \le K_1 \gamma^{-k} \gamma^{|i-\ell|} h_{ij}^{-1},$$

i.e.,

$$\max_{\ell-(k-1)\le s\le \ell} |a_{is}| \le C_1 \gamma^{|i-\ell|} h_{ij}^{-1}. \tag{4.5}$$

(3) From (4.3), we have

$$|a_{ij}| \le 2(k-1) K_2 \gamma^{|\ell-j|} \max_{\ell-(k-1)\le \mu \le \ell+k-2} |a_{i\mu}|. \tag{4.6}$$

Note that (4.4) bounds $|a_{i\mu}|$ in terms of the absolute values of the $k-1$ coefficients that precede it, hence by induction and with the understanding that $K_3 > 1$,

$$|a_{i,\ell+r}| \le K_3^r \max_{\ell-(k-1)\le s\le \ell} |a_{is}|, \quad r = 1, 2, \ldots,$$

therefore

$$\max_{\ell-(k-1)\le\mu\le\ell+k-2} |a_{i\mu}| \le K_3^{k-2} \max_{\ell-(k-1)\le s\le\ell} |a_{is}|. \tag{4.7}$$

Combining (4.6), (4.7) and (4.5), gives

$$|a_{ij}| \le 2(k-1)K_2\gamma^{|\ell-j|}K_3^{k-2}C_1\gamma^{|i-\ell|}h_{ij}^{-1} = K\,\gamma^{|i-j|}h_{ij}^{-1},$$

and that proves (4.1), hence (1.7). □

5. Proof of Lemma 4.1

Here, we prove the three parts of Lemma 4.1 as Lemmas 5.5, 5.6 and 5.7, respectively. The proof is based on certain properties of the Gram matrix $G := \{\langle M_i, N_j\rangle\}_{i,j=1}^n$ and its inverse $G^{-1} =: \{b_{ij}\}_{i,j=1}^n$. Here, (M_i) is the sequence of L_1-normalized B-splines on Δ_n,

$$M_i := \frac{k}{\kappa_i} N_i, \qquad \int_{t_i}^{t_{i+k}} M_i(t)\,dt = 1.$$

First, we note that G is a banded matrix with max-norm one, i.e.,

$$\langle M_i, N_j\rangle = 0 \quad \text{for } |i-j| > k-1, \qquad \|G\|_\infty = 1, \tag{5.1}$$

where the latter equality holds due to the fact that $\sum_j |\langle M_i, N_j\rangle| = \langle M_i, \sum_j N_j\rangle = \langle M_i, 1\rangle = 1$. A less obvious property is the boundedness of $\|G^{-1}\|_\infty$.

Theorem 5.1 (*Shadrin [12]*). *For any Δ_n, with $G := \{\langle M_i, N_j\rangle\}_{i,j=1}^n$, we have*

$$\|G^{-1}\|_\infty \le c_k, \tag{5.2}$$

where c_k is a constant that depends only on k.

We recall that (5.2) is equivalent to (1.1), i.e., the ℓ_∞-norm boundedness of the inverse G^{-1} of the Gramian is equivalent to the L_∞-norm boundedness of the orthogonal spline projector P_{Δ_n}, namely, with some constant d_k (e.g., the same as in (5.15)), we have

$$\frac{1}{d_k^2}\,\|G^{-1}\|_\infty \le \|P_{\Delta_n}\|_\infty \le \|G^{-1}\|_\infty.$$

Next, we apply the following theorem to G.

Theorem 5.2 (*Demko [6]*). *Let $A = (\alpha_{ij})$ be an r-banded matrix, i.e., $\alpha_{ij} = 0$ for $|i-j| > r$, and let $\|A\|_p \le c'$ and $\|A^{-1}\|_p \le c''$ for some $p \in [1,\infty]$. Then the elements of the inverse $A^{-1} =: (\alpha_{ij}^{(-1)})$ decay exponentially away from the diagonal, precisely*

$$|\alpha_{ij}^{(-1)}| \le K\gamma^{|i-j|},$$

where $K > 0$ and $\gamma \in (0,1)$ are constants that depend only on c', c'' and r.

We will need two corollaries of this result.

Corollary 5.3. *For any Δ_n, with $G = \{\langle M_i, N_j\rangle\}_{i,j=1}^n$, and $G^{-1} =: \{b_{ij}\}_{i,j=1}^n$, we have*

$$|b_{ij}| \le K_0\gamma^{|i-j|}, \tag{5.3}$$

where $K_0 > 0$ and $\gamma \in (0,1)$ are constants that depend only on k.

Proof. Indeed, by (5.1)–(5.2), we may apply Demko's theorem to the Gram matrix G, with $c' = 1$, $c'' = c_k$, $r = k - 1$, and $p = \infty$, and that gives the statement. □

Corollary 5.4. *For any Δ_n, with $G = \{\langle M_i, N_j\rangle\}_{i,j=1}^n$,*

$$\|G\|_1 < c_1, \qquad \|G^{-1}\|_1 < c_2, \tag{5.4}$$

where c_1, c_2 depend only on k.

Proof. It follows from (5.3) that $\|G^{-1}\|_1 = \max_j \sum_i |b_{ij}|$ is bounded, whereas $\|G\|_1$ is bounded since G is a $(k-1)$-banded matrix with nonnegative entries $\langle M_i, N_j\rangle \le 1$. □

Now we turn to the proof of Lemma 4.1 starting with inequality (4.2).

Lemma 5.5 (*Property* (4.2)). *Let $\{a_{ij}\}$ be the inverse of the B-spline Gram matrix $\{\langle N_i, N_j\rangle\}$. Then*

$$|a_{is}| \le K_1\, \gamma^{|i-s|} (\max\{\kappa_i, \kappa_s\})^{-1}. \tag{5.5}$$

Proof. As we mentioned earlier, this estimate was proved by Ciesielski [4, Property 6]. Here are the arguments. The elements of the two inverses $\{a_{ij}\} = G_0^{-1}$ and $\{b_{ij}\} = G^{-1}$ are connected by the formula

$$a_{ij} = b_{ij}(k/\kappa_j) = b_{ji}(k/\kappa_i). \tag{5.6}$$

Indeed, the identity $N_i = \kappa_i M_i/k$ implies that the matrix $G_0 := \{\langle N_i, N_j\rangle\}$ is related to $G = \{\langle M_i, N_j\rangle\}$ in the form

$$G_0 = DG, \quad \text{where } D = \operatorname{diag}[\kappa_1/k, \dots, \kappa_n/k].$$

Hence, $G_0^{-1} = G^{-1}D^{-1}$, and the first equality in (5.6) follows. The second equality is a consequence of the symmetry of G_0, as then G_0^{-1} is symmetric too, i.e., $a_{ij} = a_{ji}$. Then, in (5.6) we may use the estimate $|b_{ij}| \le K_0\gamma^{|i-j|}$ from (5.3), and (5.5) follows. □

Lemma 5.6 (*Property* (4.3)). *Let $\{a_{ij}\}$ be the inverse of the B-spline Gram matrix $\{\langle N_i, N_j\rangle\}$. Then*

$$|a_{ij}| \le K_2\, \gamma^{|\ell-j|} \sum_{\mu=\ell-(k-1)}^{\ell+k-2} |a_{i\mu}|, \quad i + k \le \ell < j. \tag{5.7}$$

Proof. (1) Since $a_{ij} = b_{ji}(k/\kappa_i)$ by (5.6), it is sufficient to establish the same inequality for the elements b_{ji} of the matrix $G^{-1} = (b_{ij})$:

$$|b_{ji}| \le K_2\, \gamma^{|\ell-j|} \sum_{\mu=\ell-(k-1)}^{\ell+k-2} |b_{\mu i}|. \tag{5.8}$$

We fix i with $1 \le i \le n$, and to simplify notations we write $b_j := b_{ji}$, omitting i in the subscripts. So, the vector $b = (b_1, \dots, b_n)^T$ is the ith column of G^{-1}, hence

$$Gb = e_i. \tag{5.9}$$

(2) The following arguments just repeat those in the proof of Theorem 5.2 used by Demko [6] and extended by de Boor [2].

For $m > i$, set

$$b^{(m)} = (0, 0, \ldots, 0, b_m, b_{m+1}, \ldots, b_n)^T.$$

With $r := k-1$, the Gram matrix $G = \{(M_i, N_j)\}_{i,j=1}^n$ is r-banded, and that together with (5.9) implies

$$\operatorname{supp} G\, b^{(m)} \subset [m-r, m+(r-1)].$$

It follows that $G\, b^{(m)}$ and $G\, b^{(m+2r)}$ have disjoint support, therefore

$$\|G\, b^{(m)}\|_1 + \|G\, b^{(m+2r)}\|_1 = \|G\, b^{(m)} - G\, b^{(m+2r)}\|_1.$$

This yields

$$\begin{aligned} \|G^{-1}\|_1^{-1} (\|b^{(m)}\|_1 + \|b^{(m+2r)}\|_1) &\le \|G\, b^{(m)}\|_1 + \|G\, b^{(m+2r)}\|_1 \\ &= \|G\, b^{(m)} - G\, b^{(m+2r)}\|_1 \\ &\le \|G\|_1 \|b^{(m)} - b^{(m+2r)}\|_1 \\ &= \|G\|_1 (\|b^{(m)}\|_1 - \|b^{(m+2r)}\|_1), \end{aligned}$$

i.e.,

$$\|b^{(m)}\|_1 + \|b^{(m+2r)}\|_1 \le c_3 (\|b^{(m)}\|_1 - \|b^{(m+2r)}\|_1), \tag{5.10}$$

where $c_3 = \|G\|_1 \|G^{-1}\|_1 \ge 1$. This gives

$$\|b^{(m+2r)}\|_1 \le \gamma_0 \|b^{(m)}\|_1, \quad \gamma_0 = \frac{c_3 - 1}{c_3 + 1} < 1,$$

where γ_0 depends only on k since so does $c_3 = c_1 c_2$ by (5.4).

It follows that, for any j, m such that $i < m \le j \le n$, we have

$$\begin{aligned} |b_j| &\le \|b^{(j)}\|_1 \le \gamma_0^{\left\lfloor \frac{j-m}{2r} \right\rfloor} \|b^{(m)}\|_1 \le \gamma_0^{-1} \gamma_0^{|j-m|/2r} \|b^{(m)}\|_1 \\ &=: c_4 \gamma^{|j-m|} \|b^{(m)}\|_1. \end{aligned}$$

Applying (5.10) to the last line, we obtain

$$|b_j| \le c_4 \gamma^{|j-m|} c_3 (\|b^{(m)}\|_1 - \|b^{(m+2r)}\|_1) = c_5 \gamma^{|j-m|} \sum_{\mu=m}^{m+2r-1} |b_\mu|.$$

Taking $m = \ell - r = \ell - (k-1)$, we bring this inequality to the form (5.8) needed:

$$|b_j| = c_5 \gamma^{k-1} \gamma^{|j-\ell|} \sum_{\mu=\ell-(k-1)}^{\ell+k-2} |b_\mu|.$$

Lemma 5.7 (*Property* (4.4)). *Let $\{a_{ij}\}$ be the inverse of the B-spline Gram matrix $\{\langle N_i, N_j\rangle\}$. Then*

$$|a_{im}| \le K_3 \max_{m-(k-1) \le s \le m-1} |a_{is}|, \quad m > i, \tag{5.11}$$

i.e., the absolute value of a coefficient following a_{ii} can be bounded in terms of the absolute values of the $k-1$ coefficients directly preceding that coefficient.

Proof. This estimate appeared in [11, proof of Lemma 7.1]. To adjust that proof to our notations, we note that the basis $\{N_i^*\}$ dual to the B-spline basis $\{N_i\}$ is given by the formula

$$N_i^* = \sum_{j=1}^{n} a_{ij} N_j.$$

Indeed, from the definition of a_{ij}, we have $\langle N_i^*, N_m\rangle = \sum_{j=1}^{n} a_{ij}\langle N_j, N_m\rangle = \delta_{im}$.

(1) We fix i, write $a_j := a_{ij}$ omitting the index i, and for $m > i$, set

$$\psi_{m-(k-1)} := \sum_{j=m-(k-1)}^{n} a_j N_j, \qquad \psi_m := \sum_{j=m}^{n} a_j N_j. \tag{5.12}$$

Then, since $\operatorname{supp} N_j = [t_j, t_{j+k}]$, it follows that

$$\psi_{m-(k-1)}(x) = N_i^*(x), \quad x \in [t_m, b].$$

Therefore, $\psi_{m-(k-1)}$ is orthogonal to $\operatorname{span}\{N_j\}_{j=m}^{n}$, in particular to ψ_m. This gives

$$\|\psi_{m-(k-1)}\|^2_{L_2[t_m,b]} + \|\psi_m\|^2_{L_2[t_m,b]} = \|\psi_{m-(k-1)} - \psi_m\|^2_{L_2[t_m,b]}. \tag{5.13}$$

(2) Further, we have

$$E_m = [t_m, t_{m+k}] \subset [t_m, b],$$

whereas the equality $\psi_{m-(k-1)} - \psi_m = \sum_{j=m-(k-1)}^{m-1} a_j N_j$ implies

$$\operatorname{supp}(\psi_{m-(k-1)} - \psi_m) \cap [t_m, b] = [t_m, t_{m+k-1}] \subset E_m.$$

Therefore, from (5.13), we conclude

$$\|\psi_{m-(k-1)}\|^2_{L_2(E_m)} + \|\psi_m\|^2_{L_2(E_m)} \le \|\psi_{m-(k-1)} - \psi_m\|^2_{L_2(E_m)}. \tag{5.14}$$

(3) Now recall that, by a theorem of de Boor (see [1] or [7, Chapter 5, Lemma 4.1]), there is a constant d_k that depends only on k such that

$$d_k^{-2}|c_m|^2 \le |E_m|^{-1} \left\| \sum_{j=1}^{n} c_j N_j \right\|^2_{L_2(E_m)} \qquad \forall c_j \in \mathbb{R}. \tag{5.15}$$

(This gives the upper bound d_k for the B-spline basis condition number.) So, applying this estimate to the left-hand side of (5.14), where we use (5.12), we derive

$$\begin{aligned}
2\,d_k^{-2}|a_m|^2 &\le |E_m|^{-1}\left(\|\psi_{m-(k-1)}\|^2_{L_2(E_m)} + \|\psi_m\|^2_{L_2(E_m)}\right)\\
&\overset{(5.14)}{\le} |E_m|^{-1}\,\|\psi_{m-(k-1)} - \psi_m\|^2_{L_2(E_m)}\\
&\le \|\psi_{m-(k-1)} - \psi_m\|^2_{L_\infty(E_m)}\\
&\overset{(5.12)}{=} \left\| \sum_{j=m-(k-1)}^{m-1} a_j N_j \right\|^2_{L_\infty(E_m)}\\
&\le \max_{m-(k-1)\le s\le m-1} |a_s|^2,
\end{aligned}$$

i.e.,

$$|a_m| \leq K_3 \max_{m-(k-1)\leq s\leq m-1} |a_s|^2,$$

and that proves (5.11). □

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CHAPTER 3

Unconditionality of orthogonal spline systems in L^p

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Unconditionality of orthogonal spline systems in L^p

by

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Abstract. We prove that given any natural number k and any dense point sequence (t_n), the corresponding orthonormal spline system is an unconditional basis in reflexive L^p.

1. Introduction. In this work, we are concerned with orthonormal spline systems of arbitrary order k with arbitrary partitions. We let $(t_n)_{n=2}^{\infty}$ be a dense sequence of points in the open unit interval $(0,1)$ such that each point occurs at most k times. Moreover, define $t_0 := 0$ and $t_1 := 1$. Such point sequences are called *admissible*.

For $n \geq 2$, we define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with grid points $(t_j)_{j=0}^{n}$, where the points 0 and 1 both have multiplicity k. For each $n \geq 2$, the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$, and therefore there exists $f_n^{(k)} \in \mathcal{S}_n^{(k)}$ that is orthonormal to $\mathcal{S}_{n-1}^{(k)}$. Observe that $f_n^{(k)}$ is unique up to sign. In addition, let $(f_n^{(k)})_{n=-k+2}^{1}$ be the collection of orthonormal polynomials in $L^2[0,1]$ such that the degree of $f_n^{(k)}$ is $k+n-2$. The system of functions $(f_n^{(k)})_{n=-k+2}^{\infty}$ is called the *orthonormal spline system of order k corresponding to* $(t_n)_{n=0}^{\infty}$. We will frequently omit the parameter k and write f_n instead of $f_n^{(k)}$.

The purpose of this article is to prove the following

THEOREM 1.1. *Let $k \in \mathbb{N}$ and $(t_n)_{n\geq 0}$ be an admissible sequence of knots in $[0,1]$. Then the corresponding general orthonormal spline system of order k is an unconditional basis in $L^p[0,1]$ for every $1 < p < \infty$.*

A celebrated result of A. Shadrin [12] states that the orthogonal projection operator onto $\mathcal{S}_n^{(k)}$ is bounded on $L^\infty[0,1]$ by a constant that depends only on k. As a consequence, $(f_n)_{n\geq -k+2}$ is a basis in $L^p[0,1]$, $1 \leq p < \infty$. There are various results on the unconditionality of spline systems restrict-

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ing either the spline order k or the partition $(t_n)_{n\geq 0}$. The first result in this direction, in [1], states that the classical Franklin system—the orthonormal spline system of order 2 corresponding to dyadic knots—is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$. This was extended in [3] to unconditionality of orthonormal spline systems of arbitrary order, but still with dyadic knots. Considerable effort has been made to weaken the restriction to dyadic knot sequences. In the series of papers [7, 9, 8] this restriction was removed step-by-step for general Franklin systems, with the final result that for each admissible point sequence $(t_n)_{n\geq 0}$ with parameter $k = 2$, the associated general Franklin system forms an unconditional basis in $L^p[0,1]$, $1 < p < \infty$. We combine the methods used in [9, 8] with some new inequalities from [11] to prove that orthonormal spline systems are unconditional in $L^p[0,1]$, $1 < p < \infty$, for any spline order k and any admissible point sequence $(t_n)_{n\geq 0}$.

The organization of the present article is as follows. In Section 2, we give some preliminary results concerning polynomials and splines. Section 3 develops some estimates for the orthonormal spline functions f_n using the crucial notion of associating to each function f_n a characteristic interval J_n in a delicate way. Section 4 treats a central combinatorial result concerning the number of indices n such that a given grid interval J can be a characteristic interval of f_n. In Section 5 we prove a few technical lemmata used in the proof of Theorem 1.1, and Section 6 finally proves Theorem 1.1. We remark that the results and proofs in Sections 5 and 6 closely follow [8].

2. Preliminaries. Let k be a positive integer. The parameter k will always be used for the order of the underlying polynomials or splines. We use the notation $A(t) \sim B(t)$ to indicate the existence of two constants $c_1, c_2 > 0$ that depend only on k, such that $c_1 B(t) \leq A(t) \leq c_2 B(t)$ for all t, where t denotes all implicit and explicit dependences that the expressions A and B might have. If the constants c_1, c_2 depend on an additional parameter p, we write $A(t) \sim_p B(t)$. Correspondingly, we use the symbols $\lesssim$, $\gtrsim$, $\lesssim_p$, $\gtrsim_p$. For a subset E of the real line, we denote by $|E|$ its Lebesgue measure and by $\mathbb{1}_E$ its characteristic function.

First, we recall a few elementary properties of polynomials.

Proposition 2.1. *Let $0 < \rho < 1$. Let I be an interval and A be a subset of I with $|A| \geq \rho |I|$. Then, for every polynomial Q of order k on I,*

$$\max_{t\in I} |Q(t)| \lesssim_\rho \sup_{t\in A} |Q(t)| \quad \textit{and} \quad \int_I |Q(t)|\,dt \lesssim_\rho \int_A |Q(t)|\,dt.$$

Lemma 2.2. *Let V be an open interval and f be a function satisfying $\int_V |f(t)|\,dt \leq \lambda |V|$ for some $\lambda > 0$. Then, denoting by $T_V f$ the orthogonal* *projection of $f \cdot \mathbb{1}_V$ onto the space of polynomials of order k on V,*

$$\|T_V f\|_{L^2(V)}^2 \lesssim \lambda^2 |V|. \tag{2.1}$$

Moreover,

$$\|T_V f\|_{L^p(V)} \lesssim \|f\|_{L^p(V)}, \qquad 1 \le p \le \infty. \tag{2.2}$$

Proof. Let l_j, $0 \le j \le k-1$, be the jth Legendre polynomial on $[-1,1]$ with the normalization $l_j(1) = 1$. In view of the integral identity

$$l_j(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2-1}\cos\varphi)^j \, d\varphi, \quad x \in \mathbb{C} \setminus \{-1, 1\},$$

l_j is uniformly bounded by 1 on $[-1,1]$. We have the orthogonality relation

$$\int_{-1}^{1} l_i(x) l_j(x) \, dx = \frac{2}{2j+1} \delta(i,j), \qquad 0 \le i, j \le k-1, \tag{2.3}$$

where $\delta(\cdot,\cdot)$ denotes the Kronecker delta. Now let $\alpha := \inf V$ and $\beta := \sup V$. For

$$l_j^V(x) := 2^{1/2} |V|^{-1/2} l_j\Big(\frac{2x - \alpha - \beta}{\beta - \alpha}\Big), \quad x \in [\alpha, \beta],$$

relation (2.3) still holds for the sequence $(l_j^V)_{j=0}^{k-1}$, that is,

$$\int_\alpha^\beta l_i^V(x) l_j^V(x) \, dx = \frac{2}{2j+1} \delta(i,j), \qquad 0 \le i, j \le k-1.$$

So, $T_V f$ can be represented in the form

$$T_V f = \sum_{j=0}^{k-1} \frac{2j+1}{2} \langle f, l_j^V \rangle l_j^V.$$

Thus we obtain

$$\begin{aligned} \|T_V f\|_{L^2(V)} &\le \sum_{j=0}^{k-1} \frac{2j+1}{2} |\langle f, l_j^V \rangle| \, \|l_j^V\|_{L^2(V)} = \sum_{j=0}^{k-1} \sqrt{\frac{2j+1}{2}} \, |\langle f, l_j^V \rangle| \\ &\le \|f\|_{L^1(V)} \sum_{j=0}^{k-1} \sqrt{\frac{2j+1}{2}} \, \|l_j^V\|_{L^\infty(V)} \lesssim \|f\|_{L^1(V)} |V|^{-1/2}. \end{aligned}$$

Now, (2.1) is a consequence of the assumption $\int_V |f(t)| \, dt \le \lambda |V|$. If we set $p' = p/(p-1)$, the second inequality (2.2) follows from

$$\|T_V f\|_{L^p(V)} \le \sum_{j=0}^{k-1} \frac{2j+1}{2} \|f\|_{L^p(V)} \|l_j^V\|_{L^{p'}(V)} \|l_j^V\|_{L^p(V)} \lesssim \|f\|_{L^p(V)},$$

since $\|l_j^V\|_{L^p(V)} \lesssim |V|^{1/p - 1/2}$ for $0 \le j \le k-1$ and $1 \le p \le \infty$. ∎

We now let

$$\mathcal{T}=(0=\tau_1=\cdots=\tau_k<\tau_{k+1}\leq\cdots\leq\tau_M<\tau_{M+1}=\cdots=\tau_{M+k}=1) \tag{2.4}$$

be a partition of $[0,1]$ consisting of knots of multiplicity at most k, that is, $\tau_i<\tau_{i+k}$ for all $1\leq i\leq M$. Let $\mathcal{S}_{\mathcal{T}}^{(k)}$ be the space of polynomial splines of order k with knots $\mathcal{T}$. The basis of L^∞-normalized B-spline functions in $\mathcal{S}_{\mathcal{T}}^{(k)}$ is denoted by $(N_{i,k})_{i=1}^M$ or for short $(N_i)_{i=1}^M$. Corresponding to this basis, there exists a biorthogonal basis of $\mathcal{S}_{\mathcal{T}}^{(k)}$, denoted by $(N_{i,k}^*)_{i=1}^M$ or $(N_i^*)_{i=1}^M$. Moreover, we write $\nu_i=\tau_{i+k}-\tau_i$.

We now recall a few important results on the B-splines N_i and their dual functions N_i^*.

PROPOSITION 2.3. *Let $1\leq p\leq\infty$ and $g=\sum_{j=1}^M a_jN_j$. Then*

$$|a_j|\lesssim|J_j|^{-1/p}\|g\|_{L^p(J_j)},\qquad 1\leq j\leq M, \tag{2.5}$$

where J_j is the subinterval $[\tau_i,\tau_{i+1}]$ of $[\tau_j,\tau_{j+k}]$ of maximal length. Additionally,

$$\|g\|_p\sim\Big(\sum_{j=1}^M|a_j|^p\nu_j\Big)^{1/p}=\|(a_j\nu_j^{1/p})_{j=1}^M\|_{\ell^p}. \tag{2.6}$$

Moreover, if $h=\sum_{j=1}^M b_jN_j^*$, then

$$\|h\|_p\lesssim\Big(\sum_{j=1}^M|a_j|^p\nu_j^{1-p}\Big)^{1/p}=\|(a_j\nu_j^{1/p-1})_{j=1}^M\|_{\ell^p}. \tag{2.7}$$

The two inequalites (2.5) and (2.6) are Lemmata 4.1 and 4.2 in [6, Chapter 5], respectively. Inequality (2.7) is a consequence of the celebrated result of Shadrin [12] that the orthogonal projection operator onto $\mathcal{S}_{\mathcal{T}}^{(k)}$ is bounded on L^∞ independently of $\mathcal{T}$. For a deduction of (2.7) from this result, see [4, Property P.7].

The next task is to estimate the inverse of the Gram matrix $(\langle N_{i,k},N_{j,k}\rangle)_{i,j=1}^M$. Before we do that, we recall the concept of totally positive matrices: Let $Q_{m,n}$ be the set of strictly increasing sequences of m integers from the set $\{1,\ldots,n\}$, and A be an $n\times n$-matrix. For $\alpha,\beta\in Q_{m,n}$, we denote by $A[\alpha;\beta]$ the submatrix of A consisting of the rows indexed by α and the columns indexed by β. Furthermore, we let α' (the complement of α) be the uniquely determined element of $Q_{n-m,n}$ that consists of all integers in $\{1,\ldots,n\}$ not occurring in α. In addition, we use the notation $A(\alpha;\beta):=A[\alpha';\beta']$.

DEFINITION 2.4. Let A be an $n\times n$-matrix. Then A is called *totally positive* if

$$\det A[\alpha;\beta]\geq 0\quad\text{ for }\alpha,\beta\in Q_{m,n},\ 1\leq m\leq n.$$

The cofactor formula $b_{ij} = (-1)^{i+j} \det A(j;i)/\det A$ for the inverse $B = (b_{ij})_{i,j=1}^M$ of the matrix A leads to

PROPOSITION 2.5. *The inverse $B = (b_{ij})$ of a totally positive matrix $A = (a_{ij})$ has the checkerboard property:*

$$(-1)^{i+j} b_{ij} \geq 0 \quad \textit{for all } i, j.$$

THEOREM 2.6 ([5]). *Let $k \in \mathbb{N}$ and $\mathcal{T}$ be an arbitrary partition of $[0,1]$ as in (2.4). Then the Gram matrix $A = (\langle N_{i,k}, N_{j,k}\rangle)_{i,j=1}^M$ of the B-spline functions is totally positive.*

This theorem is a consequence of the so called basic composition formula [10, Chapter 1, equation (2.5)] and the fact that the kernel $N_{i,k}(x)$, depending on the variables i and x, is totally positive [10, Chapter 10, Theorem 4.1]. As a consequence, the inverse $B = (b_{ij})_{i,j=1}^M$ of A has the checkerboard property by Proposition 2.5.

THEOREM 2.7 ([11]). *Let $k \in \mathbb{N}$, let $\mathcal{T}$ be the partition defined as in (2.4) and $(b_{ij})_{i,j=1}^M$ be the inverse of the Gram matrix $(\langle N_{i,k}, N_{j,k}\rangle)_{i,j=1}^M$ of the B-spline functions $N_{i,k}$ of order k corresponding to $\mathcal{T}$. Then*

$$|b_{ij}| \leq C \frac{\gamma^{|i-j|}}{\tau_{\max(i,j)+k} - \tau_{\min(i,j)}}, \qquad 1 \leq i, j \leq M,$$

where the constants $C > 0$ and $0 < \gamma < 1$ depend only on k.

Let $f \in L^p[0,1]$ for some $1 \leq p < \infty$. Since the orthonormal spline system $(f_n)_{n \geq -k+2}$ is a basis in $L^p[0,1]$, we can write $f = \sum_{n=-k+2}^\infty a_n f_n$. Based on this expansion, we define the *square function* $Sf := \big(\sum_{n=-k+2}^\infty |a_n f_n|^2\big)^{1/2}$ and the *maximal function* $Mf := \sup_m \big|\sum_{n \leq m} a_n f_n\big|$. Moreover, given a measurable function g, we denote by $\mathcal{M}g$ the *Hardy–Littlewood maximal function* of g, defined as

$$\mathcal{M}g(x) := \sup_{I \ni x} |I|^{-1} \int_I |g(t)|\, dt,$$

where the supremum is taken over all intervals I containing x.

A corollary of Theorem 2.7 is the following relation between M and $\mathcal{M}$:

THEOREM 2.8 ([11]). *If $f \in L^1[0,1]$, we have*

$$Mf(t) \lesssim \mathcal{M}f(t), \qquad t \in [0,1].$$

3. Properties of orthogonal spline functions. This section deals with the calculation and estimation of one explicit orthonormal spline function $f_n^{(k)}$ for fixed $k \in \mathbb{N}$ and $n \geq 2$ induced by the admissible sequence $(t_n)_{n=0}^\infty$. Let i_0 be an index with $k+1 \leq i_0 \leq M$. The partition $\mathcal{T}$ is defined as follows:

$$\mathcal{T} = (0 = \tau_1 = \cdots = \tau_k < \tau_{k+1} \le \cdots \le \tau_{i_0} \\ \le \cdots \le \tau_M < \tau_{M+1} = \cdots = \tau_{M+k} = 1),$$

and $\widetilde{\mathcal{T}}$ is defined to be $\mathcal{T}$ with τ_{i_0} removed. In the same way we denote by $(N_i : 1 \le i \le M)$ the B-spline functions corresponding to $\mathcal{T}$, and by $(\widetilde{N}_i : 1 \le i \le M-1)$ those corresponding to $\widetilde{\mathcal{T}}$. Böhm's formula [2] gives the following relationship between N_i and $\widetilde{N}_i$:

$$\begin{cases} \widetilde{N}_i(t) = N_i(t) & \text{if } 1 \le i \le i_0-k-1, \\ \widetilde{N}_i(t) = \dfrac{\tau_{i_0}-\tau_i}{\tau_{i+k}-\tau_i} N_i(t) + \dfrac{\tau_{i+k+1}-\tau_{i_0}}{\tau_{i+k+1}-\tau_{i+1}} N_{i+1}(t) & \text{if } i_0-k \le i \le i_0-1, \\ \widetilde{N}_i(t) = N_{i+1}(t) & \text{if } i_0 \le i \le M-1. \end{cases} \tag{3.1}$$

To calculate the orthonormal spline functions corresponding to $\widetilde{\mathcal{T}}$ and $\mathcal{T}$, we first determine a function $g \in \operatorname{span}\{N_i : 1 \le i \le M\}$ such that $g \perp \widetilde{N}_j$ for all $1 \le j \le M-1$. That is, we assume that g is of the form

$$g = \sum_{j=1}^{M} \alpha_j N_j^*,$$

where $(N_j^* : 1 \le j \le M)$ is the system biorthogonal to $(N_i : 1 \le i \le M)$. In order for g to be orthogonal to $\widetilde{N}_j$, $1 \le j \le M-1$, it has to satisfy the identities

$$0 = \langle g, \widetilde{N}_i \rangle = \sum_{j=1}^{M} \alpha_j \langle N_j^*, \widetilde{N}_i \rangle, \qquad 1 \le i \le M-1.$$

Using (3.1), this implies $\alpha_j = 0$ if $1 \le i \le i_0 - k - 1$ or $i_0 + 1 \le i \le M$. For $i_0 - k \le i \le i_0 - 1$, we have the recursion formula

$$\alpha_{i+1} \frac{\tau_{i+k+1} - \tau_{i_0}}{\tau_{i+k+1} - \tau_{i+1}} + \alpha_i \frac{\tau_{i_0} - \tau_i}{\tau_{i+k} - \tau_i} = 0, \tag{3.2}$$

which determines the sequence (α_j) up to a multiplicative constant. We choose

$$\alpha_{i_0-k} = \prod_{\ell=i_0-k+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell}$$

for symmetry reasons. This starting value and the recursion (3.2) yield the explicit formula

$$\alpha_j = (-1)^{j-i_0+k} \Bigg(\prod_{\ell=i_0-k+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_\ell} \Bigg) \Bigg(\prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell} \Bigg), \quad i_0-k \le j \le i_0. \tag{3.3}$$

So,

$$g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^* = \sum_{j=i_0-k}^{i_0} \sum_{\ell=1}^{M} \alpha_j b_{j\ell} N_\ell,$$

where $(b_{j\ell})_{j,\ell=1}^M$ is the inverse of the Gram matrix $(\langle N_j, N_\ell\rangle)_{j,\ell=1}^M$. We remark that the sequence (α_j) alternates in sign and since the matrix $(b_{j\ell})_{j,\ell=1}^M$ is checkerboard, we see that the B-spline coefficients of g, namely

$$w_\ell := \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell}, \qquad 1 \le \ell \le M, \tag{3.4}$$

satisfy

$$\Big|\sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell}\Big| = \sum_{j=i_0-k}^{i_0} |\alpha_j b_{j\ell}|, \qquad 1 \le j \le M. \tag{3.5}$$

In Definition 3.1 below, we assign to each orthonormal spline function a characteristic interval that is a grid point interval $[\tau_i, \tau_{i+1}]$ and lies close to the newly inserted point τ_{i_0}. We will see later that the choice of this interval is crucial proving important properties that are needed to show that the system $(f_n^{(k)})_{n=-k+2}^\infty$ is an unconditional basis in L^p, $1 < p < \infty$, for all admissible knot sequences $(t_n)_{n\ge 0}$. This approach was already used by G. G. Gevorkyan and A. Kamont [8] in the proof that general Franklin systems are unconditional in L^p, $1 < p < \infty$, where the characteristic intervals were called J-intervals. Since we give a slightly different construction here, we name them characteristic intervals.

Definition 3.1. Let $\mathcal{T}, \widetilde{\mathcal{T}}$ be as above and τ_{i_0} the new point in $\mathcal{T}$ that is not present in $\widetilde{\mathcal{T}}$. We define the *characteristic interval J corresponding to* τ_{i_0} as follows.

(1) Let

$$\Lambda^{(0)} := \Big\{ i_0 - k \le j \le i_0 : |[\tau_j, \tau_{j+k}]| \le 2 \min_{i_0-k\le \ell \le i_0} |[\tau_\ell, \tau_{\ell+k}]| \Big\}$$

be the set of all indices j for which the support of the B-spline function N_j is approximately minimal. Observe that $\Lambda^{(0)}$ is nonempty.

(2) Define

$$\Lambda^{(1)} := \Big\{ j \in \Lambda^{(0)} : |\alpha_j| = \max_{\ell\in\Lambda^{(0)}} |\alpha_\ell| \Big\}.$$

For an arbitrary, but fixed index $j^{(0)} \in \Lambda^{(1)}$, set $J^{(0)} := [\tau_{j^{(0)}}, \tau_{j^{(0)}+k}]$.

(3) The interval $J^{(0)}$ can now be written as the union of k grid intervals

$$J^{(0)} = \bigcup_{\ell=0}^{k-1} [\tau_{j^{(0)}+\ell}, \tau_{j^{(0)}+\ell+1}] \qquad \text{with } j^{(0)} \text{ as above.}$$

We define the *characteristic interval* $J = J(\tau_{i_0})$ to be one of the above k intervals that has maximal length.

We remark that in the definition of $\Lambda^{(0)}$, we may replace the factor 2 by any other constant $C > 1$. It is essential, though, that $C > 1$ in order to obtain the following theorem which is crucial for further investigations.

THEOREM 3.2. *With the above definition* (3.4) *of* w_ℓ *for* $1 \le \ell \le M$ *and the index* $j^{(0)}$ *given in Definition* 3.1,

$$|w_{j^{(0)}}| \gtrsim b_{j^{(0)},j^{(0)}}. \tag{3.6}$$

Before we start the proof of this theorem, we state a few remarks and lemmata. For the choice of $j^{(0)}$ in Definition 3.1, we have, by construction, the following inequalities: for all $i_0 - k \le \ell \le i_0$ with $\ell \ne j^{(0)}$,

$$|\alpha_\ell| \le |\alpha_{j^{(0)}}| \quad \text{or} \quad |[\tau_\ell, \tau_{\ell+k}]| > 2 \min_{i_0-k\le s\le i_0} |[\tau_s, \tau_{s+k}]|. \tag{3.7}$$

We recall the identity

$$|\alpha_j| = \Big(\prod_{\ell=i_0-k+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_\ell}\Big)\Big(\prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell}\Big), \qquad i_0 - k \le j \le i_0. \tag{3.8}$$

Since by (3.5),

$$|w_{j^{(0)}}| = \sum_{j=i_0-k}^{i_0} |\alpha_j b_{j,j^{(0)}}| \ge |\alpha_{j^{(0)}}|\, |b_{j^{(0)},j^{(0)}}|,$$

in order to show (3.6), we prove the inequality

$$|\alpha_{j^{(0)}}| \ge D_k > 0$$

with a constant D_k only depending on k. By (3.8), this inequality follows from the more elementary inequalities

$$\begin{aligned} \tau_{i_0} - \tau_\ell &\gtrsim \tau_{\ell+k} - \tau_{i_0}, & i_0 - k + 1 \le \ell \le j^{(0)} - 1,\\ \tau_{\ell+k} - \tau_{i_0} &\gtrsim \tau_{i_0} - \tau_\ell, & j^{(0)} + 1 \le \ell \le i_0 - 1. \end{aligned} \tag{3.9}$$

We will only prove the second line of (3.9) for all choices of $j^{(0)}$. The first line is proved by a similar argument. We observe that if $j^{(0)} \ge i_0 - 1$, then there is nothing to prove, so we assume

$$j^{(0)} \le i_0 - 2. \tag{3.10}$$

Moreover, we need only show the single inequality

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \gtrsim \tau_{i_0} - \tau_{j^{(0)}+1}, \tag{3.11}$$

since if we assume (3.11), then for any $j^{(0)} + 1 \le \ell \le i_0 - 1$,

$$\tau_{\ell+k} - \tau_{i_0} \ge \tau_{j^{(0)}+k+1} - \tau_{i_0} \gtrsim \tau_{i_0} - \tau_{j^{(0)}+1} \ge \tau_{i_0} - \tau_\ell.$$

We now choose j to be the minimal index in the range $i_0 \geq j > j^{(0)}$ such that

$$|\alpha_j| \leq |\alpha_{j^{(0)}}|. \tag{3.12}$$

If there is no such j, we set $j = i_0 + 1$.

If $j \leq i_0$, we employ (3.8) to deduce that (3.12) is equivalent to

$$\begin{aligned}(\tau_{j+k} - \tau_j)^{1-\delta(j,i_0)} \prod_{\ell=j^{(0)}\vee(i_0-k+1)}^{j-1} (\tau_{i_0} - \tau_\ell) \\ \leq (\tau_{j^{(0)}+k} - \tau_{j^{(0)}})^{1-\delta(j^{(0)},i_0-k)} \prod_{\ell=j^{(0)}+1}^{j\wedge(i_0-1)} (\tau_{\ell+k} - \tau_{i_0}),\end{aligned} \tag{3.13}$$

where $\delta(\cdot,\cdot)$ is the Kronecker delta. Furthermore, let m in the range $i_0 - k \leq m \leq i_0$ be such that $\tau_{m+k} - \tau_m = \min_{i_0-k\leq s\leq i_0}(\tau_{s+k} - \tau_s)$.

Now, from the minimality of j and (3.7), we obtain

$$\tau_{\ell+k} - \tau_\ell > 2(\tau_{m+k} - \tau_m), \qquad j^{(0)} + 1 \leq \ell \leq j - 1. \tag{3.14}$$

Thus, by definition,

$$m \leq j^{(0)} \quad \text{or} \quad m \geq j. \tag{3.15}$$

Lemma 3.3. *In the above notation, if $m \leq j^{(0)}$ and $j - j^{(0)} \geq 2$, then we have* (3.11), *or more precisely,*

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \geq \tau_{i_0} - \tau_{j^{(0)}+1}. \tag{3.16}$$

Proof. We expand the left hand side of (3.16) as

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} = \tau_{j^{(0)}+k+1} - \tau_{j^{(0)}+1} - (\tau_{i_0} - \tau_{j^{(0)}+1}).$$

By (3.14) (observe that $j - j^{(0)} \geq 2$), we conclude that

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \geq 2(\tau_{m+k} - \tau_m) - (\tau_{i_0} - \tau_{j^{(0)}+1}).$$

Since $m + k \geq i_0$ and $m \leq j^{(0)}$, we finally obtain

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \geq \tau_{i_0} - \tau_{j^{(0)}+1}. \blacksquare$$

Lemma 3.4. *Let $j^{(0)}$, j and m be as above. If $j^{(0)} + 1 \leq \ell \leq j - 1$ and $m \geq j$, we have*

$$\tau_{i_0} - \tau_\ell \geq \tau_{\ell+1+k} - \tau_{i_0}.$$

Proof. Let $j^{(0)} + 1 \leq \ell \leq j - 1$. Then from (3.14) we obtain

$$\tau_{i_0} - \tau_\ell = \tau_{\ell+1+k} - \tau_\ell - (\tau_{\ell+1+k} - \tau_{i_0}) \geq 2(\tau_{m+k} - \tau_m) - (\tau_{\ell+1+k} - \tau_{i_0}). \tag{3.17}$$

Since we have assumed $m \geq j \geq \ell + 1$, we get $m + k \geq \ell + 1 + k$, and additionally we have $m \leq i_0$ by definition of m. Thus (3.17) yields

$$\tau_{i_0} - \tau_\ell \geq \tau_{\ell+1+k} - \tau_{i_0}.$$

Since the index ℓ was arbitrary in the range $j^{(0)} + 1 \le \ell \le j - 1$, the proof of the lemma is complete. ∎

Proof of Theorem 3.2. We employ the above definition of $j^{(0)}$, j, and m and split our analysis into a few cases, distinguishing various possibilities for $j^{(0)}$ and j. In each case we will show (3.11).

Case 1: *There is no $j > j^{(0)}$ such that $|\alpha_j| \le |\alpha_{j^{(0)}}|$.* In this case, (3.15) implies $m \le j^{(0)}$. Since $j^{(0)} \le i_0 - 2$ by (3.10), we apply Lemma 3.3 to deduce (3.11).

Case 2: $i_0 - k + 1 \le j^{(0)} < j \le i_0 - 1$. Using the restrictions on $j^{(0)}$ and j, we see that (3.13) becomes

$$(\tau_{j^{(0)}+k} - \tau_{j^{(0)}}) \prod_{\ell=j^{(0)}+1}^{j} (\tau_{\ell+k} - \tau_{i_0}) \ge (\tau_{j+k} - \tau_j) \prod_{\ell=j^{(0)}}^{j-1} (\tau_{i_0} - \tau_\ell).$$

This implies

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \ge \frac{(\tau_{j+k} - \tau_j)(\tau_{i_0} - \tau_{j^{(0)}})}{\tau_{j^{(0)}+k} - \tau_{j^{(0)}}} \prod_{\ell=j^{(0)}+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+1+k} - \tau_{i_0}}.$$

Since by definition of $j^{(0)}$, we have in particular $\tau_{j^{(0)}+k} - \tau_{j^{(0)}} \le 2(\tau_{j+k} - \tau_j)$, we conclude further that

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \ge \frac{\tau_{i_0} - \tau_{j^{(0)}+1}}{2} \prod_{\ell=j^{(0)}+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+1+k} - \tau_{i_0}}. \tag{3.18}$$

If $j = j^{(0)} + 1$, the assertion (3.11) follows from (3.18), since the product is then empty.

If $j \ge j^{(0)} + 2$ and $m \le j^{(0)}$, we use Lemma 3.3 to obtain (3.11).

If $j \ge j^{(0)} + 2$ and $m \ge j$, we apply Lemma 3.4 to the terms in the product appearing in (3.18) to deduce (3.11).

This finishes the proof of Case 2.

Case 3: $i_0 - k + 1 \le j^{(0)} < j = i_0$. Recall that $j^{(0)} \le i_0 - 2 = j - 2$ by (3.10). If $m \le j^{(0)}$, Lemma 3.3 gives (3.11). So we assume $m \ge j$. Since $i_0 = j$ and $m \le i_0$, we have $m = j$. The restrictions on $j^{(0)}$, j imply that condition (3.13) is nothing else than

$$(\tau_{j^{(0)}+k} - \tau_{j^{(0)}}) \prod_{\ell=j^{(0)}+1}^{i_0-1} (\tau_{\ell+k} - \tau_{i_0}) \ge \prod_{\ell=j^{(0)}}^{i_0-1} (\tau_{i_0} - \tau_\ell).$$

Thus, in order to show (3.11), it is enough to prove that there exists a constant $D_k > 0$ only depending on k such that

$$\frac{\tau_{i_0} - \tau_{j^{(0)}}}{\tau_{j^{(0)}+k} - \tau_{j^{(0)}}} \prod_{\ell=j^{(0)}+2}^{i_0-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_{i_0}} \geq D_k. \tag{3.19}$$

First observe that by Lemma 3.4,

$$\tau_{i_0} - \tau_{j^{(0)}} \geq \tau_{j^{(0)}+k+2} - \tau_{i_0} \geq \tau_{j^{(0)}+k} - \tau_{i_0}.$$

Inserting this inequality in the left hand side of (3.19) and applying Lemma 3.4 directly to the terms in the product, we obtain (3.19).

Case 4: $i_0 - k = j^{(0)} < j = i_0$. We have $j^{(0)} \leq i_0 - 2$ by (3.10). If $m \leq j^{(0)}$, just apply Lemma 3.3 to obtain (3.11). Thus we assume $m \geq j$. Since $i_0 = j$ and $m \leq i_0$, we have $m = j$. The restrictions on $j^{(0)}$, j imply that (3.13) takes the form

$$\prod_{\ell=i_0-k+1}^{i_0-1} (\tau_{\ell+k} - \tau_{i_0}) \geq \prod_{\ell=i_0-k+1}^{i_0-1} (\tau_{i_0} - \tau_\ell).$$

Thus, to show (3.11), it is enough to prove that there exists a constant $D_k > 0$ only depending on k such that

$$\prod_{\ell=i_0-k+2}^{i_0-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_{i_0}} \geq D_k.$$

But this is a consequence of Lemma 3.4, finishing the proof of Case 4.

Case 5: $i_0 - k = j^{(0)} < j \leq i_0 - 1$. In this case, (3.11) becomes

$$\tau_{i_0+1} - \tau_{i_0} \gtrsim \tau_{i_0} - \tau_{i_0-k+1}, \tag{3.20}$$

and (3.13) is nothing else than

$$\prod_{\ell=i_0-k+1}^{j} (\tau_{\ell+k} - \tau_{i_0}) \geq (\tau_{j+k} - \tau_j) \prod_{\ell=i_0-k+1}^{j-1} (\tau_{i_0} - \tau_\ell). \tag{3.21}$$

For $j = i_0 - k + 1$, (3.20) follows easily from (3.21). If we assume $j - j^{(0)} \geq 2$ and $m \leq j^{(0)}$, we just apply Lemma 3.3 to obtain (3.11). If $j - j^{(0)} \geq 2$ and $m \geq j$, then (3.20) is equivalent to the existence of a constant $D_k > 0$ only depending on k such that

$$\frac{(\tau_{j+k} - \tau_j) \prod_{\ell=i_0-k+2}^{j-1} (\tau_{i_0} - \tau_\ell)}{\prod_{\ell=i_0-k+2}^{j} (\tau_{\ell+k} - \tau_{i_0})} \geq D_k.$$

This follows from the obvious inequality $\tau_{j+k} - \tau_j \geq \tau_{j+k} - \tau_{i_0}$ and from Lemma 3.4. Thus, the proof of Case 5 is complete, thereby concluding the proof of Theorem 3.2. ∎

We will use this result to prove lemmata connecting the L^p norm of the function g and the corresponding characteristic interval J. Before we start, we need another simple

LEMMA 3.5. *Let $C = (c_{ij})_{i,j=1}^n$ be a symmetric positive definite matrix. Then for $(d_{ij})_{i,j=1}^n = C^{-1}$ we have*

$$c_{ii}^{-1} \leq d_{ii}, \qquad 1 \leq i \leq n.$$

Proof. Since C is symmetric, it is diagonalizable:

$$C = S\Lambda S^T,$$

for some orthogonal matrix $S = (s_{ij})_{i,j=1}^n$ and for the diagonal matrix Λ consisting of the eigenvalues $\lambda_1, \dots, \lambda_n$ of C. These eigenvalues are positive, since C is positive definite. Clearly,

$$C^{-1} = S\Lambda^{-1}S^T.$$

Let i be an arbitrary integer in the range $1 \leq i \leq n$. Then

$$c_{ii} = \sum_{\ell=1}^n s_{i\ell}^2 \lambda_\ell \quad \text{and} \quad d_{ii} = \sum_{\ell=1}^n s_{i\ell}^2 \lambda_\ell^{-1}.$$

Since $\sum_{\ell=1}^n s_{i\ell}^2 = 1$ and the function $x \mapsto x^{-1}$ is convex on $(0, \infty)$, we conclude by Jensen's inequality that

$$c_{ii}^{-1} = \Big(\sum_{\ell=1}^n s_{i\ell}^2 \lambda_\ell\Big)^{-1} \leq \sum_{\ell=1}^n s_{i\ell}^2 \lambda_\ell^{-1} = d_{ii}. \ \blacksquare$$

LEMMA 3.6. *Let $\mathcal{T}$, $\widetilde{\mathcal{T}}$ be as above and $g = \sum_{j=1}^M w_j N_j$ be the function in* span$\{N_i : 1 \leq i \leq M\}$ *that is orthogonal to every $\widetilde{N}_i$, $1 \leq i \leq M-1$, with $(w_j)_{j=1}^M$ given in* (3.4)*. Moreover, let $\varphi = g/\|g\|_2$ be the L^2-normalized orthogonal spline function corresponding to the mesh point τ_{i_0}. Then*

$$\|\varphi\|_{L^p(J)} \sim \|\varphi\|_p \sim |J|^{1/p-1/2}, \qquad 1 \leq p \leq \infty,$$

where J is the characteristic interval associated to the point τ_{i_0}, given in Definition 3.1.

Proof. As a consequence of (2.5), we get

$$\|g\|_{L^p(J)} \gtrsim |J|^{1/p} |w_{j^{(0)}}|. \tag{3.22}$$

By Theorem 3.2, $|w_{j^{(0)}}| \gtrsim b_{j^{(0)},j^{(0)}}$, where we recall that $(b_{ij})_{i,j=1}^M$ is the inverse of the Gram matrix $(a_{ij})_{i,j=1}^M = (\langle N_i, N_j \rangle)_{i,j=1}^M$. Now we invoke Lemma 3.5 and (2.6) to infer from (3.22) that

$$\begin{aligned}\|g\|_{L^p(J)} &\gtrsim |J|^{1/p} b_{j^{(0)},j^{(0)}} \geq |J|^{1/p} a_{j^{(0)},j^{(0)}}^{-1} \\ &= |J|^{1/p} \|N_{j^{(0)}}\|_2^{-2} \gtrsim |J|^{1/p} \nu_{j^{(0)}}^{-1}.\end{aligned}$$

Since, by construction, J is the maximal subinterval of $J^{(0)}$ and there are exactly k subintervals of $J^{(0)}$, we finally get

$$\|g\|_{L^p(J)} \gtrsim |J|^{1/p-1}. \tag{3.23}$$

On the other hand, $g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^*$, so we use (2.7) to obtain

$$\|g\|_p \lesssim \Big(\sum_{j=i_0-k}^{i_0} |\alpha_j|^p \nu_j^{1-p}\Big)^{1/p}.$$

Since $|\alpha_j| \leq 1$ for all j and $\nu_{j^{(0)}}$ is minimal (up to the factor 2) among the values $\nu_j,\ i_0 - k \leq j \leq i_0$, we can estimate this further by

$$\|g\|_p \lesssim \nu_{j^{(0)}}^{1/p-1}.$$

We now use the inequality $|J| \leq \nu_{j^{(0)}} = |J^{(0)}|$ from the construction of J to get

$$\|g\|_p \lesssim |J|^{1/p-1}. \tag{3.24}$$

The assertion of the lemma now follows from (3.23) and (3.24) after renormalization. ∎

We denote by $d_{\mathcal{T}}(x)$ the number of points in $\mathcal{T}$ between x and J counting the endpoints of J. Correspondingly, for an interval $V \subset [0,1]$, we denote by $d_{\mathcal{T}}(V)$ the number of points in $\mathcal{T}$ between V and J counting the endpoints of both J and V.

Lemma 3.7. *Let $\mathcal{T}, \widetilde{\mathcal{T}}$ be as above and $g = \sum_{j=1}^{M} w_j N_j$ be orthogonal to every $\widetilde{N}_i$, $1 \leq i \leq M-1$, with $(w_j)_{j=1}^M$ as in (3.4). Moreover, let $\varphi = g/\|g\|_2$ be the normalized orthogonal spline function corresponding to τ_{i_0}, and $\gamma < 1$ the constant from Theorem 2.7 depending only on the spline order k. Then*

$$|w_j| \lesssim \frac{\gamma^{d_{\mathcal{T}}(\tau_j)}}{|J| + \operatorname{dist}(\operatorname{supp} N_j, J) + \nu_j} \qquad \text{for all } 1 \leq j \leq M. \tag{3.25}$$

Moreover, if $x < \inf J$, then

$$\|\varphi\|_{L^p(0,x)} \lesssim \frac{\gamma^{d_{\mathcal{T}}(x)} |J|^{1/2}}{(|J| + \operatorname{dist}(x, J))^{1-1/p}}, \qquad 1 \leq p \leq \infty. \tag{3.26}$$

Similarly, for $x > \sup J$,

$$\|\varphi\|_{L^p(x,1)} \lesssim \frac{\gamma^{d_{\mathcal{T}}(x)} |J|^{1/2}}{(|J| + \operatorname{dist}(x, J))^{1-1/p}}, \qquad 1 \leq p \leq \infty. \tag{3.27}$$

Proof. We begin by showing (3.25). By definition of w_j and α_ℓ (see (3.4) and (3.3)), we have

$$|w_j| \lesssim \max_{i_0-k \leq \ell \leq i_0} |b_{j\ell}|.$$

Now we invoke Theorem 2.7 to deduce

$$|w_j| \lesssim \frac{\max_{i_0-k\le\ell\le i_0}\gamma^{|\ell-j|}}{\min_{i_0-k\le\ell\le i_0}(\tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)})} \lesssim \frac{\gamma^{d_{\mathcal{T}}(\tau_j)}}{\min_{i_0-k\le\ell\le i_0}(\tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)})}, \tag{3.28}$$

where the second inequality follows from the location of J in the interval $[\tau_{i_0-k},\tau_{i_0+k}]$. It remains to estimate the minimum in the denominator. Fix ℓ with $i_0-k\le\ell\le i_0$. First we observe that

$$\tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)} \ge \tau_{j+k}-\tau_j = |\operatorname{supp} N_j| = \nu_j. \tag{3.29}$$

Moreover, by definition of J,

$$\tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)} \ge \min_{i_0-k\le r\le i_0}(\tau_{r+k}-\tau_r) \ge |J^{(0)}|/2 \ge |J|/2. \tag{3.30}$$

If now $j\ge\ell$, then

$$\begin{aligned}\tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)} &= \tau_{j+k}-\tau_\ell \ge \tau_{j+k}-\tau_{i_0}\\ &\ge \max(\tau_j-\sup J^{(0)},0),\end{aligned} \tag{3.31}$$

since $\tau_{i_0}\le\sup J^{(0)}$. But $\max(\tau_j-\sup J^{(0)},0)=\operatorname{dist}([\tau_j,\tau_{j+k}],J^{(0)})$ due to the fact that $\inf J^{(0)}\le\tau_{i_0}\le\tau_{\ell+k}\le\tau_{j+k}$ for the current choice of j. Additionally, $\operatorname{dist}([\tau_j,\tau_{j+k}],J)\le|J^{(0)}|+d([\tau_j,\tau_{j+k}],J^{(0)})$. So, as a consequence of (3.31),

$$\tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)} \ge \operatorname{dist}([\tau_j,\tau_{j+k}],J)-|J^{(0)}|. \tag{3.32}$$

An analogous calculation proves (3.32) also in the case $j\le\ell$. We now combine (3.28) with (3.29), (3.30) and (3.32) to obtain (3.25).

Next we consider the integral $\big(\int_0^x |g(t)|^p\,dt\big)^{1/p}$ for $x<\inf J$. The analogous estimate (3.27) follows from a similar argument. Let τ_s be the first grid point in $\mathcal{T}$ to the right of x and observe that $\operatorname{supp} N_r\cap[0,\tau_s)=\emptyset$ for $r\ge s$. Then

$$\|g\|_{L^p(0,x)} \le \|g\|_{L^p(0,\tau_s)} \le \Big\|\sum_{i=1}^{s-1} w_iN_i\Big\|_p.$$

By (2.6),

$$\|g\|_{L^p(0,x)} \le \big\|\big(w_i\nu_i^{1/p}\big)_{i=1}^{s-1}\big\|_{\ell^p}.$$

We now use (3.25) for w_i to get

$$\|g\|_{L^p(0,x)} \lesssim \Big\|\Big(\frac{\gamma^{d_{\mathcal{T}}(\tau_i)}\nu_i^{1/p}}{|J|+\operatorname{dist}(\operatorname{supp} N_i,J)+\nu_i}\Big)_{i=1}^{s-1}\Big\|_{\ell^p}.$$

Since $\nu_i \le |J| + \operatorname{dist}(\operatorname{supp} N_i, J) + \nu_i$ for all $1 \le i \le M$ and $\operatorname{dist}(\operatorname{supp} N_i, J) + \nu_i \ge \operatorname{dist}(x, J)$ for all $1 \le i \le s-1$, the last display yields

$$\|g\|_{L^p(0,x)} \lesssim (|J| + \operatorname{dist}(x,J))^{-1+1/p} \|(\gamma^{d_{\mathcal{T}}(\tau_i)})_{i=1}^{s-1}\|_{\ell^p}.$$

The last ℓ^p-norm is a geometric sum with largest term $\gamma^{d_{\mathcal{T}}(x)}$, so

$$\|g\|_{L^p(0,x)} \lesssim \frac{\gamma^{d_{\mathcal{T}}(x)}}{(|J| + \operatorname{dist}(x,J))^{1-1/p}}.$$

This concludes the proof, since we have seen in the proof of Lemma 3.6 that $\|g\|_2 \sim |J|^{-1/2}$. ∎

Remark 3.8. Analogously we obtain

$$\begin{aligned}\sup_{\tau_{j-1} \le t \le \tau_j} |\varphi(t)| &\lesssim \max_{j-k \le i \le j-1} \frac{\gamma^{d_{\mathcal{T}}(\tau_i)} |J|^{1/2}}{|J| + \operatorname{dist}(\operatorname{supp} N_i, J) + \nu_i} \\ &\lesssim \frac{\gamma^{d_{\mathcal{T}}(\tau_j)} |J|^{1/2}}{|J| + \operatorname{dist}(J, [\tau_{j-1}, \tau_j]) + |[\tau_{j-1}, \tau_j]|},\end{aligned}$$

since $[\tau_{j-1}, \tau_j] \subset \operatorname{supp} N_i$ whenever $j-k \le i \le j-1$.

4. Combinatorics of characteristic intervals. Let $(t_n)_{n=0}^\infty$ be an admissible sequence of points and $(f_n)_{n=-k+2}^\infty$ the corresponding orthonormal spline functions of order k. For $n \ge 2$, the associated partitions $\mathcal{T}_n$ to f_n are defined to consist of the grid points $(t_j)_{j=0}^n$, the knots $t_0 = 0$ and $t_1 = 1$ having both multiplicity k in $\mathcal{T}_n$. If $n \ge 2$, we denote by $J_n^{(0)}$ and J_n the characteristic intervals $J^{(0)}$ and J from Definition 3.1 associated to the new grid point t_n. If $-k+2 \le n \le 1$, we additionally set $J_n := [0,1]$. For any $x \in [0,1]$, we define $d_n(x)$ to be the number of grid points in $\mathcal{T}_n$ between x and J_n counting the endpoints of J_n. Moreover, for a subinterval V of $[0,1]$, we denote by $d_n(V)$ the number of knots in $\mathcal{T}_n$ between V and J_n counting the endpoints of both V and J_n. Finally, if

$$\begin{aligned}\mathcal{T}_n = (0 = \tau_{n,1} = \cdots = \tau_{n,k} < \tau_{n,k+1} \\ \le \cdots \le \tau_{n,n+k-1} < \tau_{n,n+k} = \cdots = \tau_{n,n+2k-1} = 1),\end{aligned}$$

and if $t_n = \tau_{n,i_0}$, then we denote by $t_n^{+\ell}$ the point $\tau_{n,i_0+\ell}$.

For the proof of the central Lemma 4.2 of this section, we need a combinatorial lemma of Erdős and Szekeres:

Lemma 4.1 (Erdős–Szekeres). *Let n be an integer. Every sequence $(x_1, \ldots, x_{(n-1)^2+1})$ of real numbers of length $(n-1)^2+1$ contains a monotone sequence of length n.*

We now use this result to prove a lemma about the combinatorics of the characteristic intervals J_n:

Lemma 4.2. *Let $x, y \in (t_n)_{n=0}^\infty$ be such that $x < y$ and $0 \le \beta \le 1/2$. Then there exists a constant F_k only depending on k such that*

$$N_0 := \operatorname{card}\{n : J_n \subseteq [x,y],\ |J_n| \ge (1-\beta)|[x,y]|\} \le F_k,$$

where card E *denotes the cardinality of the set E.*

Proof. If n is such that $J_n \subseteq [x,y]$ and $|J_n| \ge (1-\beta)|[x,y]|$, then, by definition of J_n, we have $t_n \in [0,(1-\beta)x+\beta y] \cup [\beta x + (1-\beta)y, 1]$. Thus, by the pigeon-hole principle, in one of the two sets $[0,(1-\beta)x+\beta y]$ and $[\beta x+(1-\beta)y,1]$, there are at least

$$N_1 := \left\lfloor \frac{N_0 - 1}{2} \right\rfloor + 1$$

indices n with $J_n \subset [x,y]$ and $|J_n| \ge (1-\beta)|[x,y]|$. Assume without loss of generality that this set is $[\beta x+(1-\beta)y,1]$. Now, let $(n_i)_{i=1}^{N_1}$ be an increasing sequence of indices such that $t_{n_i} \in [\beta x + (1-\beta)y, 1]$ and $J_{n_i} \subset [x,y]$, $|J_{n_i}| \ge (1-\beta)|[x,y]|$ for every $1 \le i \le N_1$. Observe that for such i, J_{n_i} is to the left of t_{n_i}. By the Erdős–Szekeres Lemma 4.1, the sequence $(t_{n_i})_{i=1}^{N_1}$ contains a monotone subsequence $(t_{m_i})_{i=1}^{N_2}$ of length

$$N_2 := \lfloor \sqrt{N_1 - 1} \rfloor + 1.$$

If $(t_{m_i})_{i=1}^{N_2}$ is increasing, then $N_2 \le k$. Indeed, if $N_2 \ge k+1$, there are at least k points (namely $t_{m_1}, \dots, t_{m_k}$) in the sequence $\mathcal{T}_{m_{k+1}}$ between $\inf J_{m_{k+1}}$ and $t_{m_{k+1}}$. This is in conflict with the location of $J_{m_{k+1}}$.

If $(t_{m_i})_{i=1}^{N_2}$ is decreasing, we let

$$s_1 \le \cdots \le s_L$$

be an enumeration of the elements in $\mathcal{T}_{m_1}$ such that $\inf J_{m_1} \le s \le t_{m_1}$. By definition of J_{m_1}, we obtain $L \le k+1$. Thus, there are at most k intervals $[s_\ell, s_{\ell+1}]$, $1 \le \ell \le L-1$, contained in $[\inf J_{m_1}, t_{m_1}]$. Again, by the pigeon-hole principle, there exists one index $1 \le \ell \le L-1$ such that the interval $[s_\ell, s_{\ell+1}]$ contains (at least)

$$N_3 := \left\lfloor \frac{N_2 - 1}{k} \right\rfloor + 1$$

points of the sequence $(t_{m_i})_{i=1}^{N_2}$. Let $(t_{r_i})_{i=1}^{N_3}$ be a subsequence of length N_3 of such points. Furthermore, define

$$N_4 := \lfloor N_3/k \rfloor.$$

Since $(t_{r_i})_{i=1}^{N_3}$ is decreasing, we have a collection of N_4 disjoint intervals

$$I_\mu := (t_{r_{\mu\cdot k}}, t_{r_{\mu\cdot k}}^{+k}) \subseteq [s_\ell, s_{\ell+1}], \qquad 1 \le \mu \le N_4.$$

Consequently, there exists (at least) one index μ such that

$$|I_\mu| \le |[s_\ell, s_{\ell+1}]|/N_4.$$

We next observe that the definition of J_{m_1} yields

$$|J_{m_1}| \geq |[s_\ell, s_{\ell+1}]|.$$

We thus get

$$\begin{aligned} |J^{(0)}_{r_{\mu\cdot k}}| \geq |J_{r_{\mu\cdot k}}| &\geq (1-\beta)|[x,y]| \geq (1-\beta)|J_{m_1}| \\ &\geq (1-\beta)|[s_\ell, s_{\ell+1}]| \geq (1-\beta)N_4|I_\mu|. \end{aligned} \tag{4.1}$$

On the other hand, the construction of $J^{(0)}_{r_{\mu\cdot k}}$ implies in particular

$$|J^{(0)}_{r_{\mu\cdot k}}| \leq 2(t^{+k}_{r_{\mu\cdot k}} - t_{r_{\mu\cdot k}}) = 2|I_\mu|. \tag{4.2}$$

The inequalities (4.1) and (4.2) imply $N_4 \leq 2/(1-\beta) \leq 4$. Since N_4 only depends on k, this proves the assertion of the lemma. ∎

5. Technical estimates

LEMMA 5.1. *Let* $f = \sum_{n=-k+2}^{\infty} a_n f_n$ *and* V *be an open subinterval of* $[0,1]$. *Then*

$$\int_{V^c} \sum_{j\in\Gamma} |a_j f_j(t)|\,dt \lesssim \int_V \Big(\sum_{j\in\Gamma} |a_j f_j(t)|^2\Big)^{1/2} dt, \tag{5.1}$$

where $\Gamma := \{j : J_j \subset V \text{ and } -k+2 \leq j < \infty\}$.

Proof. If $|V| = 1$, then (5.1) holds trivially, so we assume that $|V| < 1$. We define $x := \inf V$, $y := \sup V$ and fix $n \in \Gamma$. The definition of Γ implies $n \geq 2$, since $J_j = [0,1]$ for $-k+2 \leq j \leq 1$. We only estimate the integral in (5.1) over $[y,1]$; the integral over $[0,x]$ is estimated similarly. Lemma 3.7 implies

$$\int_y^1 |f_n(t)|\,dt \lesssim \gamma^{d_n(y)}|J_n|^{1/2}.$$

Applying Lemma 3.6 yields

$$\int_y^1 |f_n(t)|\,dt \lesssim \gamma^{d_n(y)} \int_{J_n} |f_n(t)|\,dt. \tag{5.2}$$

Now choose $\beta = 1/4$ and let J_n^β be the unique closed interval that satisfies

$$|J_n^\beta| = \beta|J_n| \quad \text{and} \quad \inf J_n^\beta = \inf J_n.$$

Since f_n is a polynomial of order k on J_n, we apply Proposition 2.1 to (5.2) and estimate further

$$\int_y^1 |a_n f_n(t)|\,dt \lesssim \gamma^{d_n(y)} \int_{J_n^\beta} |a_n f_n(t)|\,dt \leq \gamma^{d_n(y)} \int_{J_n^\beta} \Big(\sum_{j\in\Gamma} |a_j f_j(t)|^2\Big)^{1/2} dt. \tag{5.3}$$

Define $\Gamma_s := \{j \in \Gamma : d_j(y) = s\}$ for $s \geq 0$. For fixed $s \geq 0$ and $j_1, j_2 \in \Gamma_s$, we have either

$$J_{j_1} \cap J_{j_2} = \emptyset \quad \text{or} \quad \sup J_{j_1} = \sup J_{j_2}.$$

So, Lemma 4.2 implies that there exists a constant F_k, only depending on k, such that each $t \in V$ belongs to at most F_k intervals J_j^β, $j \in \Gamma_s$. Thus, summing over $j \in \Gamma_s$, from (5.3) we get

$$\begin{aligned}\sum_{j\in\Gamma_s} \int_y^1 |a_j f_j(t)|\,dt &\lesssim \sum_{j\in\Gamma_s} \gamma^s \int_{J_j^\beta} \Big(\sum_{\ell\in\Gamma} |a_\ell f_\ell(t)|^2\Big)^{1/2} dt \\ &\lesssim \gamma^s \int_V \Big(\sum_{\ell\in\Gamma} |a_\ell f_\ell(t)|^2\Big)^{1/2} dt.\end{aligned}$$

Finally, we sum over $s \geq 0$ to obtain (5.1). ■

Let g be a real-valued function defined on $[0,1]$. We denote by $[g > \lambda]$ the set $\{x \in [0,1] : g(x) > \lambda\}$ for any $\lambda > 0$.

Lemma 5.2. *Let $f = \sum_{n=-k+2}^\infty a_n f_n$ with only finitely many nonzero coefficients a_n, $\lambda > 0$, $r < 1$ and*

$$E_\lambda = [Sf > \lambda], \quad B_{\lambda,r} = [\mathcal{M}\mathbb{1}_{E_\lambda} > r].$$

Then

$$E_\lambda \subset B_{\lambda,r}.$$

Proof. Fix $t \in E_\lambda$. Since $Sf = \big(\sum_{n=-k+2}^\infty |a_n f_n|^2\big)^{1/2}$ is continuous except possibly at finitely many grid points, where it is continuous from the right, there exists an interval $I \subset E_\lambda$ such that $t \in I$. This implies

$$\begin{aligned}(\mathcal{M}\mathbb{1}_{E_\lambda})(t) &= \sup_{t\ni U} |U|^{-1} \int_U \mathbb{1}_{E_\lambda}(x)\,dx \\ &= \sup_{t\ni U} \frac{|E_\lambda \cap U|}{|U|} \geq \frac{|E_\lambda \cap I|}{|I|} = \frac{|I|}{|I|} = 1 > r,\end{aligned}$$

so $t \in B_{\lambda,r}$, proving the lemma. ■

Lemma 5.3. *Under the assumptions of Lemma 5.2, define*

$$\Lambda = \{n : J_n \not\subset B_{\lambda,r} \text{ and } -k+2 \leq n < \infty\} \quad \text{and} \quad g = \sum_{n\in\Lambda} a_n f_n.$$

Then

$$\int_{E_\lambda} Sg(t)^2\,dt \lesssim_r \int_{E_\lambda^c} Sg(t)^2\,dt. \tag{5.4}$$

Proof. If $B_{\lambda,r} = [0,1]$, the index set Λ is empty, and thus (5.4) holds trivially; so assume $B_{\lambda,r} \neq [0,1]$. Then we apply Lemma 3.6 (for $n \geq 2$) and the fact that $J_n = [0,1]$ for $n \le 1$ to obtain

$$\int_{E_\lambda} Sg(t)^2\,dt = \sum_{n\in\Lambda} \int_{E_\lambda} |a_n f_n(t)|^2\,dt \lesssim \sum_{n\in\Lambda} \int_{J_n} |a_n f_n(t)|^2\,dt.$$

We split the last expression into

$$I_1 := \sum_{n\in\Lambda} \int_{J_n\cap E_\lambda^c} |a_n f_n(t)|^2\,dt, \qquad I_2 := \sum_{n\in\Lambda} \int_{J_n\cap E_\lambda} |a_n f_n(t)|^2\,dt.$$

For I_1, we clearly have

$$I_1 \le \sum_{n\in\Lambda} \int_{E_\lambda^c} |a_n f_n(t)|^2\,dt = \int_{E_\lambda^c} Sg(t)^2\,dt. \tag{5.5}$$

It remains to estimate I_2. First we observe that by Lemma 5.2, $E_\lambda \subset B_{\lambda,r}$. Since the set $B_{\lambda,r} = [\mathcal{M}\mathbb{1}_{E_\lambda} > r]$ is open in $[0,1]$, we decompose it into a countable collection $(V_j)_{j=1}^\infty$ of disjoint open subintervals of $[0,1]$. Utilizing this decomposition, we estimate

$$I_2 \le \sum_{n\in\Lambda} \sum_{j:\,|J_n\cap V_j|>0} \int_{J_n\cap V_j} |a_n f_n(t)|^2\,dt. \tag{5.6}$$

If $n \in \Lambda$ and $|J_n \cap V_j| > 0$, then, by definition of Λ, J_n is an interval containing at least one endpoint $x \in \{\inf V_j, \sup V_j\}$ of V_j for which

$$\mathcal{M}\mathbb{1}_{E_\lambda}(x) \le r.$$

This implies

$$|E_\lambda \cap J_n \cap V_j| \le r|J_n \cap V_j| \quad \text{or equivalently} \quad |E_\lambda^c \cap J_n \cap V_j| \ge (1-r)|J_n\cap V_j|.$$

This inequality and the fact that $|f_n|^2$ is a polynomial of order $2k-1$ on J_n allow us to use Proposition 2.1 to deduce from (5.6) that

$$\begin{aligned} I_2 &\lesssim_r \sum_{n\in\Lambda} \sum_{j:\,|J_n\cap V_j|>0} \int_{E_\lambda^c\cap J_n\cap V_j} |a_n f_n(t)|^2\,dt \\ &\le \sum_{n\in\Lambda} \int_{E_\lambda^c\cap J_n\cap B_{\lambda,r}} |a_n f_n(t)|^2\,dt \\ &\le \sum_{n\in\Lambda} \int_{E_\lambda^c} |a_n f_n(t)|^2\,dt = \int_{E_\lambda^c} Sg(t)^2\,dt. \end{aligned}$$

Combined with (5.5), this completes the proof. ∎

Lemma 5.4. *Let V be an open subinterval of $[0,1]$, $x := \inf V$, $y := \sup V$ and $f = \sum_{n=-k+2}^\infty a_n f_n \in L^p[0,1]$ for $1 < p < 2$ with $\operatorname{supp} f \subset V$. Let $R > 1$ satisfy $R\gamma < 1$ for the constant γ from Theorem 2.7. Then*

$$\sum_{n=\mathrm{n}(V)}^\infty R^{pd_n(V)} |a_n|^p \|f_n\|^p_{L^p(\widetilde{V}^c)} \lesssim_{p,R} \|f\|_p^p, \tag{5.7}$$

where $\mathrm{n}(V) = \min\{n : \mathcal{T}_n \cap V \neq \emptyset\}$ *and* $\widetilde{V} = (\widetilde{x}, \widetilde{y})$ *with* $\widetilde{x} = x - 2|V|$ *and* $\widetilde{y} = y + 2|V|$.

Proof. First observe that $\widetilde{V}^c = [0, \widetilde{x}] \cup [\widetilde{y}, 1]$. We estimate only the part corresponding to $[0, \widetilde{x}]$ and assume that $\widetilde{x} > 0$. The other part is treated analogously.

Let $m \geq 0$ and define

$$T_m := \{n \in \mathbb{N} : n \geq \mathrm{n}(V),\ \mathrm{card}\{i \leq n : \widetilde{x} \leq t_i \leq x\} = m\}. \tag{5.8}$$

We remark that T_m is finite, since the sequence $(t_n)_{n=0}^{\infty}$ is dense in $[0, 1]$.

We now split T_m into the following six subsets:

$$\begin{aligned}
T_m^{(1)} &= \{n \in T_m : J_n \subset [\widetilde{x}, x]\}, \\
T_m^{(2)} &= \{n \in T_m : \widetilde{x} \in J_n,\ |J_n \cap [\widetilde{x}, x]| \geq |V|,\ J_n \not\subset [\widetilde{x}, x]\}, \\
T_m^{(3)} &= \{n \in T_m : J_n \subset [0, \widetilde{x}] \text{ or} \\
&\qquad\qquad (\widetilde{x} \in J_n \text{ with } |J_n \cap [\widetilde{x}, x]| \leq |V| \text{ and } J_n \not\subset [\widetilde{x}, x])\}, \\
T_m^{(4)} &= \{n \in T_m : x \in J_n,\ |J_n \cap [\widetilde{x}, x]| \geq |V|,\ J_n \not\subset [\widetilde{x}, x]\}, \\
T_m^{(5)} &= \{n \in T_m : J_n \subset [x, \widetilde{y}] \text{ or} \\
&\qquad\qquad (x \in J_n \text{ with } |J_n \cap [\widetilde{x}, x]| \leq |V| \text{ and } J_n \not\subset [\widetilde{x}, x])\}, \\
T_m^{(6)} &= \{n \in T_m : J_n \subset [\widetilde{y}, 1] \text{ or } (\widetilde{y} \in J_n \text{ with } J_n \not\subset [x, \widetilde{y}])\}.
\end{aligned}$$

We treat each of these separately. Before examining sums like the one in (5.7) with n restricted to one of the above sets, we note that for all n we have, by definition of $a_n = \langle f, f_n \rangle$ and the support assumption on f,

$$|a_n|^p \leq \int_V |f(t)|^p\,dt \cdot \Big(\int_V |f_n(t)|^{p'}\,dt\Big)^{p-1}, \tag{5.9}$$

where $p' = p/(p-1)$ denotes the conjugate Hölder exponent to p.

Case 1: $n \in T_m^{(1)} = \{n \in T_m : J_n \subset [\widetilde{x}, x]\}$. Let $\widetilde{T}_m^{(1)} := T_m^{(1)} \setminus \{\min T_m^{(1)}\}$. By definition, the interval J_n is at most $k-1$ grid points in $\mathcal{T}_n$ away from t_n. Since the number m of grid points between $\widetilde{x}$ and x is constant for all $n \in T_m$, there are only $2(k-1)$ possibilities for J_n with $n \in \widetilde{T}_m^{(1)}$. By Lemma 4.2 applied with $\beta = 0$, every J_n is a characteristic interval of at most F_k points t_m, and thus

$$\mathrm{card}\, T_m^{(1)} \leq 2(k-1)F_k + 1. \tag{5.10}$$

By Lemmata 3.7 and 3.6 respectively,

$$\int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{p d_n(\widetilde{x})} \|f_n\|_p^p \quad \text{and} \quad \int_V |f_n(t)|^{p'}\,dt \lesssim \gamma^{p' d_n(V)} \|f_n\|_{p'}^{p'} \tag{5.11}$$

for $n \in T_m^{(1)}$. Furthermore, $d_n(\widetilde{x}) + d_n(V) = m$ by definition of d_n, the location of J_n and the fact that $n \in T_m^{(1)}$. So, using (5.9), (5.11) and Lemma 3.6, we get

$$\begin{aligned}
&\sum_{n\in T_m^{(1)}} R^{pd_n(V)}|a_n|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\
&\qquad \le \sum_{n\in T_m^{(1)}} R^{pd_n(V)} \int_V |f(t)|^p\,dt \cdot \Big(\int_V |f_n(t)|^{p'}\,dt\Big)^{p-1} \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\
&\qquad \lesssim \sum_{n\in T_m^{(1)}} R^{pd_n(V)}\gamma^{p(d_n(\widetilde{x})+d_n(V))}\|f_n\|_p^p\|f_n\|_{p'}^p \int_V |f(t)|^p\,dt \\
&\qquad \lesssim \sum_{n\in T_m^{(1)}} (R\gamma)^{pm} \int_V |f(t)|^p\,dt.
\end{aligned}$$

Finally, we employ (5.10) to obtain

$$\sum_{n\in T_m^{(1)}} R^{pd_n(V)}|a_n|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim (R\gamma)^{pm} \int_V |f(t)|^p\,dt, \tag{5.12}$$

which concludes the proof of Case 1.

Case 2: $n \in T_m^{(2)} = \{n \in T_m : \widetilde{x} \in J_n,\ |J_n \cap [\widetilde{x}, x]| \ge |V|,\ J_n \not\subset [\widetilde{x}, x]\}$. In this case we have $d_n(V) = m$, and thus Lemma 3.7 implies

$$\int_V |f_n(t)|^{p'}\,dt \le \|f_n\|_{L^\infty(V)}^{p'}|V| \lesssim \gamma^{p'm}|J_n|^{-p'/2}|V|.$$

We use (5.9) and this estimate to obtain

$$\begin{aligned}
|a_n|^p\|f_n\|_p^p &\le \int_V |f(t)|^p\,dt \cdot \Big(\int_V |f_n(t)|^{p'}\,dt\Big)^{p-1}\|f_n\|_p^p \\
&\lesssim \int_V |f(t)|^p\,dt \cdot \gamma^{pm}|J_n|^{-p/2}|V|^{p-1}\|f_n\|_p^p.
\end{aligned}$$

Lemma 3.6 further yields

$$\begin{aligned}
|a_n|^p\|f_n\|_p^p &\lesssim \gamma^{pm}|J_n|^{-p/2+1-p/2}|V|^{p-1}\int_V |f(t)|^p\,dt \\
&\le \gamma^{pm}|J_n|^{1-p}|V|^{p-1}\|f\|_p^p.
\end{aligned} \tag{5.13}$$

If $n_0 < n_1 < \cdots < n_s$ is an enumeration of all elements in $T_m^{(2)}$, by definition of $T_m^{(2)}$ we have

$$J_{n_0} \supset J_{n_1} \supset \cdots \supset J_{n_s} \quad \text{and} \quad |J_{n_s}| \ge |V|.$$

Thus, Lemma 4.2 and the fact that $1 < p < 2$ imply

$$\sum_{n\in T_m^{(2)}} |J_n|^{1-p} \sim_p |J_{n_s}|^{1-p} \leq |V|^{1-p}. \tag{5.14}$$

We finally use (5.13) and (5.14) to conclude that

$$\begin{aligned}\sum_{n\in T_m^{(2)}} R^{pd_n(V)}|a_n|^p\|f_n\|_p^p &\lesssim (R\gamma)^{pm}|V|^{p-1}\|f\|_p^p \sum_{n\in T_m^{(2)}} |J_n|^{1-p} \\ &\lesssim_p (R\gamma)^{pm}\|f\|_p^p.\end{aligned} \tag{5.15}$$

Case 3: $n \in T_m^{(3)} = \{n \in T_m : J_n \subset [0,\widetilde{x}] \text{ or } (\widetilde{x} \in J_n \text{ with } |J_n \cap [\widetilde{x}, x]| \leq |V| \text{ and } J_n \not\subset [\widetilde{x}, x])\}$. For $n \in T_m^{(3)}$, we denote by $(x_i)_{i=1}^m$ the finite sequence of points in $\mathcal{T}_n \cap [\widetilde{x}, x]$ in increasing order and counting multiplicities. If there exists $n \in T_m^{(3)}$ such that x_1 is the right endpoint of J_n and $\widetilde{x} \in J_n$, we define $x^* := x_1$. If not, we set $x^* := \widetilde{x}$. By definition of $T_m^{(3)}$ and x^*, we have

$$|V| \leq |[x^*, x]| \leq 2|V|. \tag{5.16}$$

Furthermore, for all $n \in T_m^{(3)}$,

$$J_n \subset [0, x^*] \quad \text{and} \quad |[x^*, x] \cap \mathcal{T}_n| = m.$$

Moreover,

$$m + d_n(x^*) - k \leq d_n(V) \leq m + d_n(x^*), \tag{5.17}$$

where the exact value of $d_n(V)$ depends on the multiplicity of x^* in $\mathcal{T}_n$ (which cannot exceed k). By Lemma 3.7 and (5.17) we have

$$\sup_{t\in V} |f_n(t)| \lesssim \gamma^{m+d_n(x^*)} \frac{|J_n|^{1/2}}{|J_n| + \operatorname{dist}(x, J_n)}.$$

Hence

$$\int_V |f_n(t)|^{p'}\, dt \lesssim |V| \cdot \gamma^{p'(m+d_n(x^*))} \frac{|J_n|^{p'/2}}{(|J_n| + \operatorname{dist}(x, J_n))^{p'}}. \tag{5.18}$$

Employing (5.9), (5.18) and Lemma 3.6 gives

$$\begin{aligned} R^{pd_n(V)}|a_n|^p\|f_n\|_p^p &\leq R^{pd_n(V)} \int_V |f(t)|^p\, dt \cdot \Big(\int_V |f_n(t)|^{p'}\, dt\Big)^{p-1} \|f_n\|_p^p \\ &\lesssim R^{pd_n(V)}\|f\|_p^p|V|^{p-1}\gamma^{p(m+d_n(x^*))} \frac{|J_n|^{p/2}}{(|J_n| + \operatorname{dist}(x, J_n))^p}\|f_n\|_p^p \\ &\lesssim R^{pd_n(V)}\|f\|_p^p|V|^{p-1}\gamma^{p(m+d_n(x^*))} \frac{|J_n|}{(|J_n| + \operatorname{dist}(x, J_n))^p}. \end{aligned}$$

Inequality (5.17) then yields

$$R^{pd_n(V)}|a_n|^p\|f_n\|_p^p \le (R\gamma)^{p(m+d_n(x^*))}\|f\|_p^p|V|^{p-1}\frac{|J_n|}{(|J_n|+\operatorname{dist}(x,J_n))^p}. \tag{5.19}$$

We now have to sum this inequality. In order to do this we split our analysis depending on the value of $d_n(x^*)$. For fixed $j \in \mathbb{N}_0$ we consider $n \in T_m^{(3)}$ with $d_n(x^*) = j$. Let $\beta = 1/4$. Then, by Lemma 4.2, each point t (which is not a grid point) belongs to at most F_k intervals J_n^β with $n \in T_m^{(3)}$ and $d_n(x^*) = j$. Here J_n^β is the unique closed interval with

$$|J_n^\beta| = \beta|J_n| \quad \text{and} \quad \inf J_n^\beta = \inf J_n.$$

Furthermore, for $t \in J_n$, we have

$$|J_n| + \operatorname{dist}(x, J_n) \ge x - t.$$

Hence

$$\begin{aligned}\sum_{\substack{n\in T_m^{(3)}\\ d_n(x^*)=j}} \frac{|J_n|\,|V|^{p-1}}{(|J_n| + \operatorname{dist}(x,J_n))^p} &\le \beta^{-1} \sum_{\substack{n\in T_m^{(3)}\\ d_n(x^*)=j}} \int_{J_n^\beta} \frac{|V|^{p-1}}{(x-t)^p}\,dt\\ &\le \frac{F_k}{\beta}|V|^{p-1}\int_{-\infty}^{x^*}(x-t)^{-p}\,dt \lesssim_p \frac{|V|^{p-1}}{(x-x^*)^{p-1}} \le 1,\end{aligned}$$

where in the last step we used (5.16). Combining (5.19) and the last inequality and summing over j (here we use the fact that $R\gamma < 1$), we arrive at

$$\sum_{n\in T_m^{(3)}} R^{pd_n(V)}|a_n|^p\|f_n\|_p^p \lesssim_{p,R} (R\gamma)^{pm}\|f\|_p^p. \tag{5.20}$$

Case 4: $n \in T_m^{(4)} = \{n \in T_m : x \in J_n, |J_n \cap [\widetilde{x}, x]| \ge |V|, J_n \not\subset [\widetilde{x}, x]\}$. We can ignore the cases $m = 0$ and ($m = 1$ and $[\widetilde{x}, x] \cap \mathcal{T}_n = \{x\}$) since these are settled in Case 2. We define $\widetilde{T}_m^{(4)}$ to be the set of all remaining indices from $T_m^{(4)}$. Let $n \in \widetilde{T}_m^{(4)}$. Then the definition of $T_m^{(4)}$ implies

$$d_n(V) = d_n([x,y]) = 0. \tag{5.21}$$

Moreover, there exists at least one point of $\mathcal{T}_n$ in V (since $n \ge \mathrm{n}(V)$ for $n \in T_m$) and at least one point of $\mathcal{T}_n$ in $[\widetilde{x}, x]$ (since $m \ge 1$). Thus we have

$$|V| \le |J_n| \le 3|V|. \tag{5.22}$$

Since $x \in J_n$ for all $n \in \widetilde{T}_m^{(4)}$, the family $\{J_n : n \in \widetilde{T}_m^{(4)}\}$ is a decreasing collection of sets. Inequality (5.22) and a multiple application of Lemma 4.2 with sufficiently large β gives a constant c_k depending only on k such that

$$\operatorname{card} \widetilde{T}_m^{(4)} \le c_k. \tag{5.23}$$

We employ Lemmata 3.7 and 3.6 to get

$$\int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{pm}|J|^{p/2-p+1} = \gamma^{pm}|J|^{1-p/2} \lesssim \gamma^{pm}\|f_n\|_p^p. \tag{5.24}$$

Hence

$$\begin{aligned}\sum_{n\in\widetilde{T}_m^{(4)}} & R^{pd_n(V)}|a_n|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\ &\lesssim \sum_{n\in\widetilde{T}_m^{(4)}} \int_V |f(t)|^p\,dt \cdot \Big(\int_V |f_n(t)|^{p'}\,dt\Big)^{p-1} \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\ &\lesssim \sum_{n\in\widetilde{T}_m^{(4)}} \int_V |f(t)|^p\,dt \cdot \|f_n\|_{p'}^p \gamma^{pm}\|f_n\|_p^p \le \sum_{n\in\widetilde{T}_m^{(4)}} \gamma^{pm}\|f\|_p^p,\end{aligned}$$

where we used (5.21) and (5.9) in the first inequality, (5.24) in the second and Lemma 3.6 in the last one. Consequently, considering (5.23), the last display implies

$$\sum_{n\in\widetilde{T}_m^{(4)}} R^{pd_n(V)}|a_n|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{pm}\|f\|_p^p. \tag{5.25}$$

Case 5: $n \in T_m^{(5)} = \{n \in T_m : J_n \subset [x,\widetilde{y}]$ or $(x \in J_n$ with $|J_n \cap [\widetilde{x},x]| \le |V|$ and $J_n \not\subset [\widetilde{x},x])\}$. If there exists $n \in T_m^{(5)}$ with $x_m = \inf J_n$, then we define $x' = x_m$. If there exists no such index, we set $x' = x$. We now fix $n \in T_m^{(5)}$. By definition of x' and $\widetilde{x}$,

$$m + d_n(x') - k \le d_n(\widetilde{x}) \le m + d_n(x'). \tag{5.26}$$

The exact relation between $d_n(\widetilde{x})$ and $d_n(x')$ depends on the multiplicity of the point x' in the grid $\mathcal{T}_n$. By definition of $T_m^{(5)}$,

$$\operatorname{dist}(\widetilde{x}, J_n) \le 5|V| \quad \text{and} \quad |V| \le \operatorname{dist}(\widetilde{x}, J_n).$$

Moreover,

$$|J_n| \le |[x',\widetilde{y}]| \le 4|V| \quad \text{and} \quad d_n(V) \le d_n(x'). \tag{5.27}$$

The last two displays now imply

$$|J_n| + \operatorname{dist}(\widetilde{x}, J_n) \sim |V|.$$

Lemma 3.7, together with the former observation, yields

$$\int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{pd_n(\widetilde{x})} \frac{|J_n|^{p/2}}{(|J_n| + \operatorname{dist}(\widetilde{x}, J_n))^{p-1}} \lesssim \gamma^{pd_n(\widetilde{x})} \frac{|J_n|^{p/2}}{|V|^{p-1}}.$$

Inserting (5.26) in this inequality, we get

$$\int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{p(d_n(x')+m)}\frac{|J_n|^{p/2}}{|V|^{p-1}}. \tag{5.28}$$

For each $n \in T_m^{(5)}$, we split $[x', \widetilde{y}]$ into three disjoint subintervals I_ℓ, $1 \le \ell \le 3$, defined by

$$I_1 := [x', \inf J_n], \qquad I_2 := J_n, \qquad I_3 := [\sup J_n, \widetilde{y}].$$

Correspondingly, we set

$$a_{n,\ell} := \int_{I_\ell \cap V} f(t) f_n(t)\,dt, \qquad \ell = 1,2,3.$$

We start by analyzing the choice $\ell = 2$ and first observe that by definition of I_2,

$$|a_{n,2}|^p \le \|f_n\|_{p'}^p \int_{J_n} |f(t)|^p\,dt. \tag{5.29}$$

We split the index set $T_m^{(5)}$ further and look at the set of those $n \in T_m^{(5)}$ such that $d_n(x') = j$ for fixed $j \in \mathbb{N}_0$. These indices n may be arranged in packets such that the intervals J_n from one packet have the same left endpoint and the maximal intervals of different packets are disjoint. Observe that the intervals J_n from one packet form a decreasing collection of sets. Let J_{n_0} be the maximal interval of one packet. Define $\mathcal{I}_j := \{n \in T_m^{(5)} : d_n(x') = j$, $J_n \subset J_{n_0}\}$. Then we use (5.27) and (5.29) to estimate

$$\begin{aligned} E_{2,j} &:= \sum_{n\in\mathcal{I}_j} R^{pd_n(V)} |a_{n,2}|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\ &\le \sum_{n\in\mathcal{I}_j} R^{pj} \|f_n\|_{p'}^p \int_{J_n} |f(t)|^p\,dt \cdot \int_0^{\widetilde{x}} |f_n(t)|^p\,dt. \end{aligned}$$

Continuing, we use (5.28) to get

$$E_{2,j} \lesssim R^{pj} \int_{J_{n_0}} |f(t)|^p\,dt \cdot \sum_{n\in\mathcal{I}_j} \|f_n\|_{p'}^p \gamma^{p(d_n(x')+m)} \frac{|J_n|^{p/2}}{|V|^{p-1}}.$$

By Lemma 3.6, $\|f_n\|_{p'} \sim |J|^{1/p'-1/2}$, and thus

$$E_{2,j} \lesssim (R\gamma)^{pj}\gamma^{pm} \int_{J_{n_0}} |f(t)|^p\,dt \cdot \sum_{n\in\mathcal{I}_j} \frac{|J_n|^{p-1}}{|V|^{p-1}}.$$

We apply Lemma 4.2 to the above sum to conclude that

$$E_{2,j} \lesssim_p (R\gamma)^{pj}\gamma^{pm} \int\limits_{J_{n_0}} |f(t)|^p\,dt \cdot \frac{|J_{n_0}|^{p-1}}{|V|^{p-1}} \lesssim (R\gamma)^{pj}\gamma^{pm} \int\limits_{J_{n_0}} |f(t)|^p\,dt,$$

where in the last inequality we used (5.27). Now, summing over all maximal intervals J_{n_0} and over j finally yields (note that $R\gamma < 1$)

$$\sum_{n\in T_m^{(5)}} R^{pd_n(V)}|a_{n,2}|^p \int\limits_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim_{p,R} \gamma^{pm}\|f\|_p^p. \tag{5.30}$$

This completes the proof of the case $\ell = 2$.

Now consider $\ell = 3$. Fix $j \in \mathbb{N}_0$ and let $(n_{j,r})_{r=1}^\infty$ be the subsequence of all $n \in T_m^{(5)}$ with $d_n(x') = j$. For two such indices $n_1 < n_2$ we have either

$$(\inf J_{n_1} = \inf J_{n_2} \text{ and } J_{n_2} \subset J_{n_1}) \quad\text{or}\quad \sup J_{n_2} \le \inf J_{n_1}.$$

Observe that $J_{n_2} = J_{n_1}$ is possible, but by Lemma 4.2 (with $\beta = 0$) only F_k times, with F_k only depending on k. Therefore, with $\beta_{n_{j,r}} := \sup J_{n_{j,r}}$ for $r \ge 1$ and $\beta_{n_{j,0}} := \widetilde{y}$,

$$d_{n_{j,s}}(\beta_{n_{j,r}}) \ge \frac{s-r}{F_k} - 1, \qquad s \ge r \ge 1.$$

Thus for $s \ge r \ge 1$ by Lemmata 3.7 and 3.6 we obtain

$$\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f_{n_{j,s}}(t)|^{p'}\,dt \lesssim \gamma^{p' d_{n_{j,s}}(\beta_{n_{j,r}})}\|f_{n_{j,s}}\|_{p'}^{p'} \lesssim \gamma^{p'\frac{s-r}{F_k}}\|f_{n_{j,s}}\|_{p'}^{p'}, \tag{5.31}$$

and similarly, using also (5.26),

$$\int\limits_0^{\widetilde{x}} |f_{n_{j,s}}|^p\,dt \lesssim \gamma^{pd_{n_{j,s}}(\widetilde{x})}\|f_{n_{j,s}}\|_p^p \lesssim \gamma^{p(m+d_{n_{j,s}}(x'))}\|f_{n_{j,s}}\|_p^p. \tag{5.32}$$

Choosing $\kappa := \gamma^{1/(2F_k)} < 1$, we deduce that

$$\begin{aligned}
|a_{n_{j,s},3}|^p &= \Big| \int\limits_{\beta_{n_{j,s}}}^{\widetilde{y}} f(t)f_{n_{j,s}}(t)\,dt\Big|^p = \Big|\sum_{r=1}^s \kappa^{s-r}\kappa^{r-s}\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} f(t)f_{n_{j,s}}(t)\,dt\Big|^p \\
&\le \Big(\sum_{r=1}^s \kappa^{p'(s-r)}\Big)^{p/p'} \sum_{r=1}^s \kappa^{p(r-s)}\Big|\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} f(t)f_{n_{j,s}}(t)\,dt\Big|^p \\
&\lesssim \sum_{r=1}^s \kappa^{p(r-s)} \int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f(t)|^p\,dt \cdot \Big(\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f_{n_{j,s}}(t)|^{p'}\,dt\Big)^{p/p'}.
\end{aligned}$$

We now use inequality (5.31) to obtain

$$|a_{n_{j,s},3}|^p \lesssim \sum_{r=1}^{s} \gamma^{p\frac{s-r}{2F_k}} \int_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f(t)|^p\,dt \cdot \|f_{n_{j,s}}\|_{p'}^p. \tag{5.33}$$

Combining (5.33) and (5.32) yields

$$E_{3,j} := \sum_{\substack{n\in T_m^{(5)}\\ d_n(x')=j}} R^{pd_n(V)}|a_{n,3}|^p\|f\|^p_{L^p(0,\widetilde{x})} = \sum_{s\geq 1} R^{pj}|a_{n_{j,s},3}|^p\|f_{n_{j,s}}\|^p_{L^p(0,\widetilde{x})}$$

$$\lesssim \sum_{s\geq 1} R^{pj}\sum_{r=1}^{s}\gamma^{p\frac{s-r}{2F_k}}\|f_{n_{j,s}}\|^p_{p'}\int_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}}|f(t)|^p\,dt\cdot\gamma^{p(m+j)}\|f_{n_{j,s}}\|_p^p.$$

Using again Lemma 3.6 gives

$$E_{3,j} \lesssim \gamma^{pm}(R\gamma)^{pj}\sum_{r\geq 1}\int_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}}|f(t)|^p\,dt\cdot\sum_{s\geq r}\gamma^{p\frac{s-r}{2F_k}} \lesssim \gamma^{pm}(R\gamma)^{pj}\|f\|_p^p.$$

Summing over j finally yields

$$\sum_{n\in T_m^{(5)}} R^{pd_n(V)}|a_{n,3}|^p\|f\|^p_{L^p(0,\widetilde{x})} \lesssim_{p,R} \gamma^{pm}\|f\|_p^p, \tag{5.34}$$

since $R\gamma < 1$. This finishes the proof of the case $\ell = 3$.

We now come to the final part, $\ell = 1$. Fix j and n such that $d_n(x') = j$ and let $L_{1,n}, \ldots, L_{j,n}$ be the grid intervals in the grid $\mathcal{T}_n$ between x' and J_n, from left to right. Observe that f_n is a polynomial on each $L_{i,n}$. We define

$$b_{i,n} := \int_{L_{i,n}} f(t)f_n(t)\,dt, \qquad 1\leq i\leq j.$$

For n with $d_n(x') = j$, we clearly have $a_{n,1} = \sum_{i=1}^j b_{i,n}$, and Hölder's inequality implies

$$|b_{i,n}|^p \leq \int_{L_{i,n}}|f(t)|^p\,dt\cdot\Big(\int_{L_{i,n}}|f_n(t)|^{p'}\,dt\Big)^{p/p'}. \tag{5.35}$$

Remark 3.8 yields the bound

$$\sup_{t\in L_{i,n}}|f_n(t)| \lesssim \gamma^{j-i}\frac{|J_n|^{1/2}}{|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|},$$

and inserting this in (5.35) gives

$$|b_{i,n}|^p \leq \int_{L_{i,n}}|f(t)|^p\,dt\cdot\gamma^{p(j-i)}\frac{|J_n|^{p/2}|L_{i,n}|^{p-1}}{(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^p}. \tag{5.36}$$

Observe that we have the elementary inequality

$$\frac{|J_n|^{p/2}|L_{i,n}|^{p-1}}{(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^p}\,\frac{|J_n|^{p/2}}{|V|^{p-1}} \leq \frac{|J_n|}{|V|^{p-1}}(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^{p-2}. \tag{5.37}$$

Combining (5.36), (5.37) and (5.28) allows us to estimate (recall that we have assumed that $d_n(x')=j$)

$$\begin{aligned}
&R^{pd_n(V)}|b_{i,n}|^p\cdot\int_0^{\widetilde{x}}|f_n(t)|^p\,dt \\
&\lesssim R^{pj}\gamma^{p(j-i)}\int_{L_{i,n}}|f(t)|^p\,dt\cdot\frac{|J_n|^{p/2}|L_{i,n}|^{p-1}}{(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^p}\cdot\gamma^{p(j+m)}\frac{|J_n|^{p/2}}{|V|^{p-1}} \\
&\lesssim R^{pj}\gamma^{p(2j+m-i)}\frac{|J_n|}{|V|^{p-1}}(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^{p-2}\int_{L_{i,n}}|f(t)|^p\,dt.
\end{aligned} \tag{5.38}$$

For fixed j and i we consider those indices n such that $d_n(x')=j$, and the corresponding intervals $L_{i,n}$. These intervals can be collected in packets such that all intervals from one packet have the same left endpoint and the maximal intervals of different packets are disjoint. For $\beta=1/4$, we denote by J_n^β the unique interval that has the same right endpoint as J_n and length $\beta|J_n|$. The intervals J_n corresponding to $L_{i,n}$'s from one packet can now be grouped in the same way as the $L_{i,n}$'s, and thus Lemma 4.2 implies the existence of a constant F_k depending only on k such that every point $t\in[0,1]$ belongs to at most F_k intervals J_n^β corresponding to the intervals $L_{i,n}$ from one packet.

We now consider one such packet and denote by u^* the left endpoint of (all) intervals $L_{i,n}$ in the packet. Then for $t\in J_n^\beta$ we have

$$|J_n|+\operatorname{dist}(L_{i,n},J_n)+|L_{i,n}|\geq|t-u^*|. \tag{5.39}$$

If L_i^* is the maximal interval of the packet, (5.38) and (5.39) yield

$$\begin{aligned}
&\sum_{n:\,L_{i,n}\text{ in one packet}} R^{pd_n(V)}|b_{i,n}|^p\|f_n\|^p_{L^p(0,\widetilde{x})} \\
&\quad\lesssim\frac{R^{pj}\gamma^{p(2j+m-i)}}{|V|^{p-1}}\sum_n|J_n|(|J_n|+\operatorname{dist}(L_{i,n},J_n)+|L_{i,n}|)^{p-2}\int_{L_{i,n}}|f(t)|^p\,dt \\
&\quad\lesssim\frac{R^{pj}\gamma^{p(2j+m-i)}}{|V|^{p-1}}\int_{L_i^*}|f(t)|^p\,dt\cdot\sum_n\int_{J_n^\beta}|t-u^*|^{p-2}\,dt.
\end{aligned}$$

Since every point t belongs to at most F_k intervals J_n^β in one packet of $L_{i,n}$'s, by using $J_n \subset [x', \widetilde{y}\,]$ and $p < 2$ we get

$$\begin{aligned}\sum_{n\colon L_{i,n}\text{ in one packet}} & R^{pd_n(V)}|b_{i,n}|^p\|f_n\|^p_{L^p(0,\widetilde{x})} \\ &\lesssim \frac{R^{pj}\gamma^{p(2j+m-i)}}{|V|^{p-1}} \int_{L_i^*} |f(t)|^p\,dt \cdot \int_{u^*}^{\widetilde{y}} |t-u^*|^{p-2}\,dt \\ &\lesssim R^{pj}\gamma^{p(2j+m-i)} \int_{L_i^*} |f(t)|^p\,dt,\end{aligned}$$

where in the last inequality we used (5.27). Since the maximal intervals L_i^* of different packets are disjoint, we can sum over all packets (for fixed j and i) to obtain

$$\sum_{\substack{n\in T_m^{(5)}\\ d_n(x')=j}} R^{pd_n(V)}|b_{i,n}|^p\|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim R^{pj}\gamma^{p(2j+m-i)}\|f\|_p^p. \tag{5.40}$$

Let $\kappa := \gamma^{1/2} < 1$. Then for n such that $d_n(x') = j$ we have

$$|a_{n,1}|^p = \Big|\sum_{i=1}^j b_{i,n}\Big|^p = \Big|\sum_{i=1}^j \kappa^{j-i}\kappa^{i-j}b_{i,n}\Big|^p \lesssim_p \sum_{i=1}^j \kappa^{p(i-j)}|b_{i,n}|^p. \tag{5.41}$$

Combining (5.41) with (5.40) we get

$$\begin{aligned}\sum_{\substack{n\in T_m^{(5)}\\ d_n(x')=j}} & R^{pd_n(V)}|a_{1,n}|^p\|f_n\|^p_{L^p(0,\widetilde{x})} \\ &\lesssim_p \sum_{i=1}^j \kappa^{p(i-j)} \sum_{\substack{n\in T_m^{(5)}\\ d_n(x')=j}} R^{pd_n(V)}|b_{i,n}|^p\|f_n\|^p_{L^p(0,\widetilde{x})} \\ &\lesssim \sum_{i=1}^j \kappa^{p(i-j)}R^{pj}\gamma^{p(2j+m-i)}\|f\|_p^p \lesssim (R\gamma)^{pj}\gamma^{pm}\|f\|_p^p.\end{aligned}$$

Since $R\gamma < 1$, we sum over j to conclude that finally

$$\sum_{n\in T_m^{(5)}} R^{pd_n(V)}|a_{n,1}|^p\|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim_{p,R} \gamma^{pm}\|f\|_p^p. \tag{5.42}$$

This finishes the proof of the case $\ell = 1$.

We can now combine the inequalities for $\ell = 1, 2, 3$, that is, (5.42), (5.30) and (5.34), to complete the analysis of Case 5 with the estimate

$$\sum_{n\in T_m^{(5)}} R^{pd_n(V)}|a_n|^p\|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim_{p,R} \gamma^{pm}\|f\|_p^p. \tag{5.43}$$

Case 6: $n \in T_m^{(6)} = \{n \in T_m : J_n \subset [\widetilde{y}, 1]$ or $(\widetilde{y} \in J_n$ with $J_n \not\subset [x, \widetilde{y}])\}$. Similarly to (5.8), we use the symmetric splitting of the indices n into

$$T_{\mathrm{r},s} := \{n \geq \mathrm{n}(V) : |[y, \widetilde{y}] \cap \mathcal{T}_n| = s\},$$

where r stands for "right". These collections are again split into six subcollections $T_{\mathrm{r},s}^{(i)}$, $1 \leq i \leq 6$, where the two of interest are

$$\begin{aligned} T_{\mathrm{r},s}^{(2)} &= \{n \in T_{r,s} : \widetilde{y} \in J_n, |J_n \cap [y, \widetilde{y}]| \geq |V|, J_n \not\subset [y, \widetilde{y}]\}, \\ T_{\mathrm{r},s}^{(3)} &= \big\{n \in T_{r,s} : J_n \subset [\widetilde{y}, 1] \text{ or} \\ &\qquad\qquad (\widetilde{y} \in J_n \text{ with } |J_n \cap [y, \widetilde{y}]| \leq |V| \text{ and } J_n \not\subset [y, \widetilde{y}])\big\}. \end{aligned}$$

The results (5.15) and (5.20) for $T_m^{(2)}$ and $T_m^{(3)}$ respectively had the form

$$\sum_{n \in T_m^{(2)} \cup T_m^{(3)}} R^{pd_n(V)} |a_n|^p \|f_n\|_p^p \lesssim_{p,R} (R\gamma)^{pm} \|f\|_p^p.$$

Observe that the p-norm of f_n on the left hand side is over the whole interval $[0, 1]$. The same argument as for $T_m^{(2)}$ and $T_m^{(3)}$ yields

$$\sum_{n \in T_{\mathrm{r},s}^{(2)} \cup T_{\mathrm{r},s}^{(3)}} R^{pd_n(V)} |a_n|^p \|f_n\|_p^p \lesssim_{p,R} (R\gamma)^{ps} \|f\|_p^p. \tag{5.44}$$

Now, since

$$\bigcup_{m \geq 0} T_m^{(6)} \subset \bigcup_{s \geq 0} T_{\mathrm{r},s}^{(2)} \cup T_{\mathrm{r},s}^{(3)},$$

inequality (5.44) implies

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{n \in T_m^{(6)}} R^{pd_n(V)} |a_n|^p \|f_n\|_p^p \\ &\qquad \leq \sum_{s=0}^{\infty} \sum_{n \in T_{r,s}^{(2)} \cup T_{r,s}^{(3)}} R^{pd_n(V)} |a_n|^p \|f_n\|_p^p \lesssim_{p,R} \|f\|_p^p. \end{aligned} \tag{5.45}$$

After summing (5.12), (5.15), (5.20), (5.25) and (5.43) over m, we add inequality (5.45) to obtain finally

$$\sum_{n \geq \mathrm{n}(V)} R^{pd_n(V)} |a_n|^p \|f_n\|_{L^p(0, \widetilde{x})}^p \lesssim_{p,R} \|f\|_p^p.$$

The symmetric inequality

$$\sum_{n \geq \mathrm{n}(V)} R^{pd_n(V)} |a_n|^p \|f_n\|_{L^p(\widetilde{y}, 1)}^p \lesssim_{p,R} \|f\|_p^p$$

is proved analogously, and thus the proof of the lemma is complete. ∎

6. Proof of the main theorem. In this section, we prove our main result, Theorem 1.1, that is, unconditionality of orthonormal spline systems corresponding to an arbitrary admissible point sequence $(t_n)_{n\geq 0}$ in reflexive L^p.

Proof of Theorem 1.1. We recall the notation

$$Sf(t) = \Big(\sum_{n=-k+2}^{\infty} |a_n f_n(t)|^2\Big)^{1/2}, \quad Mf(t) = \sup_{m\geq -k+2}\Big|\sum_{n=-k+2}^{m} a_n f_n(t)\Big|$$

when

$$f = \sum_{n=-k+2}^{\infty} a_n f_n.$$

Since $(f_n)_{n=-k+2}^{\infty}$ is a basis in $L^p[0,1]$, $1 \leq p < \infty$, Khintchine's inequality implies that a necessary and sufficient condition for $(f_n)_{n=-k+2}^{\infty}$ to be an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$ is

$$\|Sf\|_p \sim_p \|f\|_p, \quad f \in L^p[0,1]. \tag{6.1}$$

We will prove (6.1) for $1 < p < 2$ since the case $p > 2$ then follows by a duality argument.

We first prove the inequality

$$\|f\|_p \lesssim_p \|Sf\|_p. \tag{6.2}$$

Let $f \in L^p[0,1]$ with $f = \sum_{n=-k+2}^{\infty} a_n f_n$. We may assume that the sequence $(a_n)_{n\geq -k+2}$ has only finitely many nonzero entries. We will prove (6.2) by showing that $\|Mf\|_p \lesssim_p \|Sf\|_p$.

We first observe that

$$\|Mf\|_p^p = p\int_0^{\infty} \lambda^{p-1}\psi(\lambda)\, d\lambda \tag{6.3}$$

with $\psi(\lambda) := [Mf > \lambda]$. We will decompose f into two parts φ_1, φ_2 and estimate the distribution functions $\psi_i(\lambda) := [M\varphi_i > \lambda/2]$, $i \in \{1,2\}$, separately. To define φ_i, for $\lambda > 0$ we set

$$\begin{aligned} E_\lambda &:= [Sf > \lambda], & B_\lambda &:= [\mathcal{M}\mathbb{1}_{E_\lambda} > 1/2],\\ \Gamma &:= \{n : J_n \subset B_\lambda, -k+2 \leq n < \infty\}, & \Lambda &:= \Gamma^c; \end{aligned}$$

recall that J_n is the characteristic interval corresponding to the grid point t_n and the function f_n. Then, let

$$\varphi_1 := \sum_{n\in\Gamma} a_n f_n \quad \text{and} \quad \varphi_2 := \sum_{n\in\Lambda} a_n f_n.$$

Now we estimate $\psi_1 = [M\varphi_1 > \lambda/2]$:

$$\begin{aligned}\psi_1(\lambda) &= |\{t\in B_\lambda : M\varphi_1(t) > \lambda/2\}| + |\{t\notin B_\lambda : M\varphi_1(t) > \lambda/2\}| \\ &\leq |B_\lambda| + \frac{2}{\lambda}\int_{B_\lambda^c} M\varphi_1(t)\,dt \leq |B_\lambda| + \frac{2}{\lambda}\int_{B_\lambda^c}\sum_{n\in\Gamma}|a_n f_n(t)|\,dt.\end{aligned}$$

We decompose B_λ into a disjoint collection of open subintervals of $[0,1]$ and apply Lemma 5.1 to each of those intervals to deduce that

$$\begin{aligned}\psi_1(\lambda) &\lesssim |B_\lambda| + \frac{1}{\lambda}\int_{B_\lambda} Sf(t)\,dt = |B_\lambda| + \frac{1}{\lambda}\int_{B_\lambda\setminus E_\lambda} Sf(t)\,dt + \frac{1}{\lambda}\int_{E_\lambda\cap B_\lambda} Sf(t)\,dt \\ &\leq |B_\lambda| + |B_\lambda\setminus E_\lambda| + \frac{1}{\lambda}\int_{E_\lambda} Sf(t)\,dt,\end{aligned}$$

where in the last inequality we simply used the definition of E_λ. Since the Hardy–Littlewood maximal function operator $\mathcal{M}$ is of weak type $(1,1)$, we have $|B_\lambda| \lesssim |E_\lambda|$, and thus finally

$$\psi_1(\lambda) \lesssim |E_\lambda| + \frac{1}{\lambda}\int_{E_\lambda} Sf(t)\,dt. \tag{6.4}$$

We now estimate $\psi_2(\lambda)$. From Theorem 2.8 and the fact that $\mathcal{M}$ is a bounded operator on $L^2[0,1]$ we obtain

$$\begin{aligned}\psi_2(\lambda) &\lesssim \frac{1}{\lambda^2}\|\mathcal{M}\varphi_2\|_2^2 \lesssim \frac{1}{\lambda^2}\|\varphi_2\|_2^2 = \frac{1}{\lambda^2}\|S\varphi_2\|_2^2 \\ &= \frac{1}{\lambda^2}\Big(\int_{E_\lambda} S\varphi_2(t)^2\,dt + \int_{E_\lambda^c} S\varphi_2(t)^2\,dt\Big).\end{aligned}$$

We apply Lemma 5.3 to the first summand to get

$$\psi_2(\lambda) \lesssim \frac{1}{\lambda^2}\int_{E_\lambda^c} S\varphi_2(t)^2\,dt. \tag{6.5}$$

Thus, combining (6.4) and (6.5) gives

$$\psi(\lambda) \leq \psi_1(\lambda) + \psi_2(\lambda) \lesssim |E_\lambda| + \frac{1}{\lambda}\int_{E_\lambda} Sf(t)\,dt + \frac{1}{\lambda^2}\int_{E_\lambda^c} Sf(t)^2\,dt.$$

Inserting this into (6.3), we obtain

$$\begin{aligned}\|Mf\|_p^p &\lesssim p\int_0^\infty \lambda^{p-1}|E_\lambda|\,d\lambda + p\int_0^\infty \lambda^{p-2}\int_{E_\lambda} Sf(t)\,dt\,d\lambda \\ &\quad + p\int_0^\infty \lambda^{p-3}\int_{E_\lambda^c} Sf(t)^2\,dt\,d\lambda \\ &= \|Sf\|_p^p + p\int_0^1 Sf(t)\int_0^{Sf(t)} \lambda^{p-2}\,d\lambda\,dt + p\int_0^1 Sf(t)^2\int_{Sf(t)}^\infty \lambda^{p-3}\,d\lambda\,dt,\end{aligned}$$

and thus, since $1 < p < 2$,

$$\|Mf\|_p \lesssim_p \|Sf\|_p.$$

So, the inequality $\|f\|_p \lesssim_p \|Sf\|_p$ is proved.

We now turn to the proof of

$$\|Sf\|_p \lesssim_p \|f\|_p, \qquad 1 < p < 2. \tag{6.6}$$

It is enough to show that S is of weak type (p,p) whenever $1 < p < 2$. This is because S is (clearly) also of strong type 2 and we can use the Marcinkiewicz interpolation theorem to obtain (6.6). Thus we have to show

$$|[Sf > \lambda]| \lesssim_p \|f\|_p^p/\lambda^p, \qquad f \in L^p[0,1],\ \lambda > 0. \tag{6.7}$$

We fix f and $\lambda > 0$, define $G_\lambda := [\mathcal{M}f > \lambda]$ and observe that

$$|G_\lambda| \lesssim_p \|f\|_p^p/\lambda^p, \tag{6.8}$$

since $\mathcal{M}$ is of weak type (p,p), and, by the Lebesgue differentiation theorem,

$$|f| \le \lambda \quad \text{a.e. on } G_\lambda^c. \tag{6.9}$$

We decompose the open set $G_\lambda \subset [0,1]$ into a collection $(V_j)_{j=1}^\infty$ of disjoint open subintervals of $[0,1]$ and split f into

$$h := f \cdot \mathbb{1}_{G_\lambda^c} + \sum_{j=1}^\infty T_{V_j} f, \qquad g := f - h,$$

where for fixed index j, $T_{V_j}f$ is the projection of $f \cdot \mathbb{1}_{V_j}$ onto the space of polynomials of order k on the interval V_j.

We treat the functions h, g separately. The definition of h implies

$$\|h\|_2^2 = \int_{G_\lambda^c} |f(t)|^2\,dt + \sum_{j=1}^\infty \int_{V_j} (T_{V_j}f)(t)^2\,dt,$$

since the intervals V_j are disjoint. We apply (6.9) to the first summand and (2.1) to the second to obtain

$$\|h\|_2^2 \lesssim \lambda^{2-p} \int_{G_\lambda^c} |f(t)|^p\,dt + \lambda^2 |G_\lambda|,$$

and thus, in view of (6.8),

$$\|h\|_2^2 \lesssim_p \lambda^{2-p} \|f\|_p^p.$$

Hence

$$|[Sh > \lambda/2]| \le \frac{4}{\lambda^2}\|Sh\|_2^2 = \frac{4}{\lambda^2}\|h\|_2^2 \lesssim_p \frac{\|f\|_p^p}{\lambda^p},$$

which concludes the proof of (6.7) for h.

We turn to the proof of (6.7) for g. Since $p < 2$, we have

$$Sg(t)^p = \Big(\sum_{n=-k+2}^{\infty} |\langle g, f_n\rangle|^2 f_n(t)^2\Big)^{p/2} \leq \sum_{n=-k+2}^{\infty} |\langle g, f_n\rangle|^p |f_n(t)|^p. \tag{6.10}$$

For each j, we define $\widetilde{V}_j$ to be the open interval with the same center as V_j but with 5 times its length. Then set $\widetilde{G}_\lambda := \bigcup_{j=1}^\infty \widetilde{V}_j \cap [0,1]$ and observe that $|\widetilde{G}_\lambda| \leq 5|G_\lambda|$. We get

$$|[Sg > \lambda/2]| \leq |\widetilde{G}_\lambda| + \frac{2^p}{\lambda^p} \int_{\widetilde{G}_\lambda^c} Sg(t)^p \, dt.$$

By (6.8) and (6.10), this becomes

$$|[Sg > \lambda/2]| \lesssim_p \lambda^{-p}\Big(\|f\|_p^p + \sum_{n=-k+2}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, f_n\rangle|^p |f_n(t)|^p \, dt\Big).$$

But by definition of g and (2.2),

$$\|g\|_p^p = \sum_j \int_{V_j} |f(t) - T_{V_j} f(t)|^p \, dt \lesssim_p \sum_j \int_{V_j} |f(t)|^p \, dt \lesssim \|f\|_p^p,$$

so to prove $|[Sg > \lambda/2]| \leq \lambda^{-p}\|f\|_p^p$, it is enough to show that

$$\sum_{n=-k+2}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, f_n\rangle|^p |f_n(t)|^p \, dt \lesssim \|g\|_p^p. \tag{6.11}$$

We let $g_j := g \cdot \mathbb{1}_{V_j}$. The supports of g_j are disjoint and we have $\|g\|_p^p = \sum_{j=1}^\infty \|g_j\|_p^p$. Furthermore $g = \sum_{j=1}^\infty g_j$ with convergence in L^p. Thus for each n,

$$\langle g, f_n\rangle = \sum_{j=1}^\infty \langle g_j, f_n\rangle,$$

and it follows from the definition of g_j that

$$\int_{V_j} g_j(t) p(t) \, dt = 0$$

for each polynomial p on V_j of order k. This implies that $\langle g_j, f_n\rangle = 0$ for $n < \mathrm{n}(V_j)$, where

$$\mathrm{n}(V) := \min\{n : \mathcal{T}_n \cap V \neq \emptyset\}.$$

Thus, for all $R > 1$ and every n,

$$
\begin{aligned}
(6.12) \qquad |\langle g, f_n\rangle|^p &= \Big| \sum_{j:\, n \geq \mathrm{n}(V_j)} \langle g_j, f_n\rangle \Big|^p \\
&\leq \Big(\sum_{j:\, n \geq \mathrm{n}(V_j)} R^{d_n(V_j)} |\langle g_j, f_n\rangle| R^{-d_n(V_j)} \Big)^p \\
&\leq \Big(\sum_{j:\, n \geq \mathrm{n}(V_j)} R^{pd_n(V_j)} |\langle g_j, f_n\rangle|^p \Big) \Big(\sum_{j:\, n \geq \mathrm{n}(V_j)} R^{-p'd_n(V_j)} \Big)^{p/p'},
\end{aligned}
$$

where $p' = p/(p-1)$. If we fix $n \geq \mathrm{n}(V_j)$, there is at least one point of the partition $\mathcal{T}_n$ contained in V_j. This implies that for each fixed $s \geq 0$, there are at most two indices j such that $n \geq \mathrm{n}(V_j)$ and $d_n(V_j) = s$. Therefore,

$$
\Big(\sum_{j:\, n \geq \mathrm{n}(V_j)} R^{-p'd_n(V_j)} \Big)^{p/p'} \lesssim_p 1,
$$

and from (6.12) we obtain

$$
|\langle g, f_n\rangle|^p \lesssim_p \sum_{j:\, n \geq \mathrm{n}(V_j)} R^{pd_n(V_j)} |\langle g_j, f_n\rangle|^p.
$$

Now we insert this inequality in (6.11) to get

$$
\begin{aligned}
&\sum_{n=-k+2}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, f_n\rangle|^p |f_n(t)|^p \, dt \\
&\qquad \lesssim_p \sum_{n=-k+2}^{\infty} \sum_{j:\, n \geq \mathrm{n}(V_j)} R^{pd_n(V_j)} |\langle g_j, f_n\rangle|^p \int_{\widetilde{G}_\lambda^c} |f_n(t)|^p \, dt \\
&\qquad \leq \sum_{n=-k+2}^{\infty} \sum_{j:\, n \geq \mathrm{n}(V_j)} R^{pd_n(V_j)} |\langle g_j, f_n\rangle|^p \int_{\widetilde{V}_j^c} |f_n(t)|^p \, dt \\
&\qquad \leq \sum_{j=1}^{\infty} \sum_{n \geq \mathrm{n}(V_j)} R^{pd_n(V_j)} |\langle g_j, f_n\rangle|^p \int_{\widetilde{V}_j^c} |f_n(t)|^p \, dt.
\end{aligned}
$$

We choose $R > 1$ such that $R\gamma < 1$ for $\gamma < 1$ from Theorem 2.7 and apply Lemma 5.4 to obtain

$$
\sum_{n=-k+2}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, f_n\rangle|^p |f_n(t)|^p \, dt \lesssim_p \sum_{j=1}^{\infty} \|g_j\|_p^p = \|g\|_p^p,
$$

proving (6.11) and hence $\|Sf\|_p^p \lesssim_p \|f\|_p^p$. Thus the proof of Theorem 1.1 is complete. ∎

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CHAPTER 4

Unconditionality of orthogonal spline systems in H^1

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Unconditionality of orthogonal spline systems in H^1

by

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Abstract. We give a simple geometric characterization of knot sequences for which the corresponding orthonormal spline system of arbitrary order k is an unconditional basis in the atomic Hardy space $H^1[0,1]$.

1. Introduction. This paper belongs to a series of papers studying properties of orthonormal spline systems with arbitrary knots. The detailed study of such systems started in the 1960's with Z. Ciesielski's papers [2, 3] on properties of the Franklin system, which is an orthonormal system consisting of continuous piecewise linear functions with dyadic knots. Next, the 1972 results by J. Domsta [11] made it possible to extend the study to orthonormal spline systems of higher order—and higher smoothness—with dyadic knots. These systems occurred to be bases or unconditional bases in several function spaces like $L^p[0,1]$, $1 \le p < \infty$, $C[0,1]$, $H^p[0,1]$, $0 < p \le 1$, Sobolev spaces $W^{p,k}[0,1]$; they also give characterizations of BMO and VMO spaces, and various spaces of smooth functions (Hölder functions, Zygmund class, Besov spaces). One should mention here the work of Z. Ciesielski, J. Domsta, S. V. Bochkarev, P. Wojtaszczyk, S.-Y. A. Chang, P. Sjölin, J.-O. Strömberg (for more detailed references see e.g. [13], [15], [16]). Nowadays, results of this kind are known for wavelets.

The extension of these results to orthonormal spline systems with arbitrary knots began with the case of piecewise linear systems, i.e. general Franklin systems, or orthonormal spline systems of order 2. This was possible due to precise estimates of the inverse of the Gram matrix of piecewise linear B-spline bases with arbitrary knots, as presented in [19]. First results in this direction were obtained in [5] and [13]. We would like to mention here two results by G. G. Gevorkyan and A. Kamont. First, each general Franklin system is an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$ (see [14]).

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Second, there is a simple geometric characterization of knot sequences for which the corresponding general Franklin system is a basis or an unconditional basis in $H^1[0,1]$ (see [15]). For both of these results, an essential tool is the association of a so called characteristic interval to each general Franklin function f_n.

The case of splines of higher order is much more difficult. The existence of a uniform bound for L^∞-norms of orthogonal projections on spline spaces of order k with arbitrary order (i.e. a bound depending on k, but independent of the sequence of knots)—was a long-standing problem known as C. de Boor's conjecture (1973) (cf. [8]). The case of $k = 2$ was settled earlier by Z. Ciesielski [2], the cases $k = 3, 4$ were solved by C. de Boor himself (1968, 1981) in [7, 9], but the positive answer in the general case was given by A. Yu. Shadrin [22] only in 2001. A much simplified and shorter proof was recently obtained by M. v. Golitschek (2014) in [24]. An immediate consequence of A.Yu. Shadrin's result is that if a sequence of knots is dense in $[0,1]$, then the corresponding orthonormal spline system of order k is a basis in $L^p[0,1]$, $1 \le p < \infty$, and in $C[0,1]$. Moreover, Z. Ciesielski [4] obtained several consequences of Shadrin's result, one of them being an estimate for the inverse of the B-spline Gram matrix. Using this estimate, G. G. Gevorkyan and A. Kamont [16] extended a part of their result from [15] to orthonormal spline systems of arbitrary order and obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order k is a basis in $H^1[0,1]$. Further extension required more precise estimates for the inverse of B-spline Gram matrices, of the type known for the piecewise linear case. Such estimates were obtained recently by M. Passenbrunner and A. Yu. Shadrin [21]. Using these estimates, M. Passenbrunner [20] proved that for each sequence of knots, the corresponding orthonormal spline system of order k is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$.

The main result of the present paper is a characterization of those knot sequences for which the corresponding orthonormal spline system of order k is an unconditional basis in $H^1[0,1]$.

The paper is organized as follows. In Section 2 we give the necessary definitions and we formulate the main result of this paper, Theorem 2.4. In Sections 3 and 4 we recall or prove several facts needed to establish our results. In particular, in Section 4 we recall precise pointwise estimates for orthonormal spline systems with arbitrary knots, the associated characteristic intervals and some combinatorial facts for characteristic intervals. Then Section 5 contains some auxiliary results, and the proof of Theorem 2.4 is given in Section 6.

The results contained in this paper were obtained independently by two teams, G. Gevorkyan & K. Keryan and A. Kamont & M. Passenbrunner at the same time, so we have decided to produce a joint paper.

2. Definitions and the main result. Let $k \geq 2$ be an integer. In this work, we are concerned with orthonormal spline systems of order k with arbitrary partitions. We let $\mathcal{T} = (t_n)_{n=2}^{\infty}$ be a dense sequence of points in the open unit interval $(0,1)$ such that each point occurs at most k times. Moreover, define $t_0 := 0$ and $t_1 := 1$. Such point sequences are called *k-admissible*. For $-k+2 \leq n \leq 1$, let $\mathcal{S}_n^{(k)}$ be the space of polynomials of order $n+k-1$ (or degree $n+k-2$) on the interval $[0,1]$ and $(f_n^{(k)})_{n=-k+2}^{1}$ be the collection of orthonormal polynomials in $L^2 \equiv L^2[0,1]$ such that the degree of $f_n^{(k)}$ is $n+k-2$. For $n \geq 2$, let $\mathcal{T}_n$ be the ordered sequence of points consisting of the grid points $(t_j)_{j=0}^{n}$ repeated according to their multiplicities and where the knots 0 and 1 have multiplicity k, i.e.,

$$\begin{aligned}\mathcal{T}_n = (0 = \tau_{n,1} = \cdots = \tau_{n,k} < \tau_{n,k+1} \\ \leq \cdots \leq \tau_{n,n+k-1} < \tau_{n,n+k} = \cdots = \tau_{n,n+2k-1} = 1).\end{aligned}$$

In that case, we also define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with grid points $\mathcal{T}_n$. For each $n \geq 2$, the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$, and therefore there exists $f_n^{(k)} \in \mathcal{S}_n^{(k)}$ orthonormal to $\mathcal{S}_{n-1}^{(k)}$. Observe that $f_n^{(k)}$ is unique up to sign.

Definition 2.1. The system of functions $(f_n^{(k)})_{n=-k+2}^{\infty}$ is called the *orthonormal spline system of order k corresponding to the sequence* $(t_n)_{n=0}^{\infty}$.

We will frequently omit the parameter k and write f_n and $\mathcal{S}_n$ instead of $f_n^{(k)}$ and $\mathcal{S}_n^{(k)}$, respectively.

Note that the case $k = 2$ corresponds to orthonormal systems of piecewise linear functions, i.e. general Franklin systems.

We are interested in characterizing sequences $\mathcal{T}$ of knots such that the system $(f_n^{(k)})_{n=-k+2}^{\infty}$ is an unconditional basis in $H^1 = H^1[0,1]$. By $H^1 = H^1[0,1]$ we mean the atomic Hardy space on $[0,1]$ (see [6]). A function $a : [0,1] \to \mathbb{R}$ is called an *atom* if either $a \equiv 1$ or there exists an interval Γ such that:

(i) $\operatorname{supp} a \subset \Gamma$,
(ii) $\|a\|_\infty \leq |\Gamma|^{-1}$,
(iii) $\int_0^1 a(x)\,dx = \int_\Gamma a(x)\,dx = 0$.

Then, by definition, H^1 consists of all functions f with a representation

$$f = \sum_{n=1}^{\infty} c_n a_n$$

for some atoms $(a_n)_{n=1}^\infty$ and real scalars $(c_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty |c_n| < \infty$. The space H^1 becomes a Banach space under the norm

$$\|f\|_{H^1} := \inf \sum_{n=1}^{\infty} |c_n|,$$

where the inf is taken over all atomic representations $\sum c_n a_n$ of f.

To formulate our result, we need to introduce some regularity conditions for a sequence $\mathcal{T}$.

For $n \geq 2$, $\ell \leq k$ and $k - \ell + 1 \leq i \leq n + k - 1$, we define $D_{n,i}^{(\ell)}$ to be the interval $[\tau_{n,i}, \tau_{n,i+\ell}]$.

Definition 2.2. Let $\ell \leq k$ and $(t_n)_{n=0}^\infty$ be an ℓ-admissible (and therefore k-admissible) point sequence. This sequence is called *ℓ-regular with parameter $\gamma \geq 1$* if

$$\frac{|D_{n,i}^{(\ell)}|}{\gamma} \leq |D_{n,i+1}^{(\ell)}| \leq \gamma |D_{n,i}^{(\ell)}|, \qquad n \geq 2,\ k - \ell + 1 \leq i \leq n + k - 2.$$

In other words, (t_n) is ℓ-regular if there is a uniform finite bound $\gamma \geq 1$ such that for all n, the ratios of the lengths of neighboring supports of B-spline functions (cf. Section 3.2) of order ℓ in the grid $\mathcal{T}_n$ are bounded by γ.

The following characterization for $(f_n^{(k)})$ to be a basis in H^1 is the main result of [16]:

Theorem 2.3 ([16]). *Let $k \geq 1$ and let (t_n) be a k-admissible sequence of knots in $[0, 1]$ with the corresponding orthonormal spline system $(f_n^{(k)})$ of order k. Then $(f_n^{(k)})$ is a basis in H^1 if and only if (t_n) is k-regular with some parameter $\gamma \geq 1$,*

In this paper, we prove a characterization for $(f_n^{(k)})$ to be an unconditional basis in H^1. The main result of our paper is the following:

Theorem 2.4. *Let (t_n) be a k-admissible sequence of points. Then the corresponding orthonormal spline system $(f_n^{(k)})$ is an unconditional basis in H^1 if and only if (t_n) is $(k-1)$-regular with some parameter $\gamma \geq 1$.*

Let us note that in case $k = 2$, i.e. for general Franklin systems, both Theorems 2.3 and 2.4 were obtained by G. G. Gevorkyan and A. Kamont [15]. (In the terminology of the current paper, strong regularity from [15] is 1-regularity, and strong regularity for pairs from [15] is 2-regularity.)

The proof of Theorem 2.4 follows the same general scheme as the proof of Theorem 2.2 in [15]. In Section 5 we introduce four conditions (A)–(D) for series with respect to orthonormal spline systems of order k corresponding to a k-admissible sequence of points. Then we study relations between these conditions under various regularity assumptions on the underlying sequence of points. Finally, we prove Theorem 2.4 in Section 6.

3. Preliminaries. The parameter $k \geq 2$ will always be used for the order of the underlying polynomials or splines. We use the notation $A(t) \sim B(t)$ to indicate the existence of two constants $c_1, c_2 > 0$ such that $c_1 B(t) \leq A(t) \leq c_2 B(t)$ for all t, where t denotes all implicit and explicit dependencies that the expressions A and B might have. If the constants c_1, c_2 depend on an additional parameter p, we write $A(t) \sim_p B(t)$. Correspondingly, we use the symbols $\lesssim, \gtrsim, \lesssim_p, \gtrsim_p$. For a subset E of the real line, we denote by $|E|$ its Lebesgue measure and by $\mathbb{1}_E$ the characteristic function of E. If $f : \Omega \to \mathbb{R}$ is a real valued function and λ is a real parameter, we write $[f > \lambda] := \{\omega \in \Omega : f(\omega) > \lambda\}$.

3.1. Properties of regular sequences of points. The following lemma describes geometric decay of intervals in regular sequences (recall the notation $D_{n,i}^{(\ell)} = [\tau_{n,i}, \tau_{n,i+\ell}]$):

LEMMA 3.1. *Let (t_n) be a k-admissible sequence of points that is ℓ-regular for some $1 \leq \ell \leq k$ with parameter γ and let $D_{n_1,i_1}^{(\ell)} \supset \cdots \supset D_{n_{2\ell},i_{2\ell}}^{(\ell)}$ be a strictly decreasing sequence of sets defined above. Then*

$$|D_{n_{2\ell},i_{2\ell}}^{(\ell)}| \leq \frac{\gamma^\ell}{1+\gamma^\ell} |D_{n_1,i_1}^{(\ell)}|.$$

Proof. We set $V_j := D_{n_j,i_j}^{(\ell)}$ for $1 \leq j \leq 2\ell$. Then, by definition, V_1 contains $\ell + 1$ grid points from $\mathcal{T}_{n_1}$ and at least 3ℓ grid points of $\mathcal{T}_{n_{2\ell}}$. As a consequence, there exists an interval $D_{n_{2\ell},m}^{(\ell)}$ for some m that satisfies

$$\operatorname{int}(D_{n_{2\ell},m}^{(\ell)} \cap V_{2\ell}) = \emptyset, \qquad D_{n_{2\ell},m}^{(\ell)} \subset V_1, \qquad \operatorname{dist}(D_{n_{2\ell},m}^{(\ell)}, V_{2\ell}) = 0.$$

The ℓ-regularity of (t_n) now implies

$$|V_{2\ell}| \leq \gamma^\ell |D_{n_{2\ell},m}^{(\ell)}| \leq \gamma^\ell (|V_1| - |V_{2\ell}|),$$

i.e., $|V_{2\ell}| \leq \frac{\gamma^\ell}{1+\gamma^\ell} |V_1|$, which proves the assertion of the lemma. ■

3.2. Properties of B-spline functions. We define $(N_{n,i}^{(k)})_{i=1}^{n+k-1}$ to be the collection of B-spline functions of order k corresponding to the partition $\mathcal{T}_n$. Those functions are normalized so that they form a partition of unity, i.e., $\sum_{i=1}^{n+k-1} N_{n,i}^{(k)}(x) = 1$ for all $x \in [0, 1]$. Associated to this basis, there exists a biorthogonal basis of $\mathcal{S}_n$, denoted by $(N_{n,i}^{(k)*})_{i=1}^{n+k-1}$. If the parameters k and n are clear from the context, we also denote those functions by $(N_i)_{i=1}^{n+k-1}$ and $(N_i^*)_{i=1}^{n+k-1}$, respectively.

We will need the following well known formula for the derivative of a linear combination of B-spline functions: if $g = \sum_{j=1}^{n+k-1} a_j N_{n,j}^{(k)}$, then

$$g' = (k-1) \sum_{j=2}^{n+k-1} (a_j - a_{j-1}) \frac{N_{n,j}^{(k-1)}}{|D_{n,j}^{(k-1)}|}. \tag{3.1}$$

We now recall an elementary property of polynomials.

PROPOSITION 3.2. *Let $0 < \rho < 1$. Let I be an interval and $A \subset I$ be a subset of I with $|A| \geq \rho |I|$. Then, for every polynomial Q of order k on I,*

$$\max_{t\in I} |Q(t)| \lesssim_{\rho,k} \sup_{t\in A} |Q(t)| \quad \text{and} \quad \int_I |Q(t)|\,dt \lesssim_{\rho,k} \int_A |Q(t)|\,dt.$$

We recall a few important results on B-splines (N_i) and their dual functions (N_i^*).

PROPOSITION 3.3. *Let $1 \leq p \leq \infty$ and $g = \sum_{j=1}^{n+k-1} a_j N_j$, where $(N_i)_{i=1}^{n+k-1}$ are the B-splines of order k corresponding to the partition $\mathcal{T}_n$. Then*

$$|a_j| \lesssim_k |J_j|^{-1/p} \|g\|_{L^p(J_j)}, \qquad 1 \leq j \leq n+k-1, \tag{3.2}$$

where J_j is a subinterval $[\tau_{n,i}, \tau_{n,i+1}]$ of $[\tau_{n,j}, \tau_{n,j+k}]$ of maximal length. Furthermore,

$$\|g\|_p \sim_k \Big(\sum_{j=1}^{n+k-1} |a_j|^p |D_{n,j}^{(k)}| \Big)^{1/p} = \|(a_j |D_{n,j}^{(k)}|^{1/p})_{j=1}^{n+k-1}\|_{\ell^p}. \tag{3.3}$$

Moreover, if $h = \sum_{j=1}^{n+k-1} b_j N_j^*$, then

$$\|h\|_p \lesssim_k \Big(\sum_{j=1}^{n+k-1} |b_j|^p |D_{n,j}^{(k)}|^{1-p} \Big)^{1/p} = \|(b_j |D_{n,j}^{(k)}|^{1/p-1})_{j=1}^{n+k-1}\|_{\ell^p}. \tag{3.4}$$

The inequalites (3.2) and (3.3) are Lemmas 4.1 and 4.2 in [10, Chapter 5], respectively. Inequality (3.4) is a consequence of Shadrin's theorem [22] that the orthogonal projection onto $\mathcal{S}_n^{(k)}$ is bounded on L^∞ independently of n and $\mathcal{T}_n$. For a deduction of (3.4) from this result, see [4, Property P.7].

We next consider estimates for the inverse $(b_{ij})_{i,j=1}^{n+k-1}$ of the Gram matrix $(\langle N_i, N_j \rangle)_{i,j=1}^{n+k-1}$. Later, we will need a special property of this matrix, of being *checkerboard*, i.e.,

$$(-1)^{i+j} b_{ij} \geq 0 \quad \text{for all } i,j. \tag{3.5}$$

This is a simple consequence of the total positivity of the Gram matrix (cf. [7, 18]). Moreover, we need the lower estimate for $b_{i,i}$,

$$|D_{n,i}^{(k)}|^{-1} \lesssim_k b_{i,i}. \tag{3.6}$$

This is a consequence of the total positivity of the B-spline Gram matrix, the L^2-stability of B-splines and the following lemma:

LEMMA 3.4 ([20]). *Let $C = (c_{ij})_{i,j=1}^n$ be a symmetric positive definite matrix. Then for $(d_{ij})_{i,j=1}^n = C^{-1}$ we have*

$$c_{ii}^{-1} \le d_{ii}, \qquad 1 \le i \le n.$$

3.3. Some results for orthonormal spline systems. We now recall two results concerning orthonormal spline series.

THEOREM 3.5 ([21]). *Let $(f_n)_{n=-k+2}^\infty$ be the orthonormal spline system of order k corresponding to an arbitrary k-admissible point sequence $(t_n)_{n=0}^\infty$. Then, for every $f \in L^1 \equiv L^1[0,1]$, the series $\sum_{n=-k+2}^\infty \langle f, f_n\rangle f_n$ converges to f almost everywhere.*

Let $f \in L^p \equiv L^p[0,1]$ for some $1 \le p < \infty$. Since the orthonormal spline system $(f_n)_{n\ge -k+2}$ is a basis in L^p, we can write $f = \sum_{n=-k+2}^\infty a_n f_n$. Based on this expansion, we define the *square function* $Pf := (\sum_{n=-k+2}^\infty |a_n f_n|^2)^{1/2}$ and the *maximal function* $Sf := \sup_m |\sum_{n\le m} a_n f_n|$. Moreover, given a measurable function g, we denote by $\mathcal{M}g$ the *Hardy–Littlewood maximal function* of g defined as

$$\mathcal{M}g(x) := \sup_{I\ni x} |I|^{-1} \int_I |g(t)|\, dt,$$

where the supremum is taken over all intervals I containing x. The connection between the maximal function Sf and the Hardy–Littlewood maximal function is given by the following result:

THEOREM 3.6 ([21]). *If $f \in L^1$, then*

$$Sf(t) \lesssim_k \mathcal{M}f(t), \qquad t \in [0,1].$$

4. Properties of orthogonal spline functions and characteristic intervals

4.1. Estimates for f_n. This section concerns the calculation and estimation of one explicit orthonormal spline function $f_n^{(k)}$ for fixed $k \in \mathbb{N}$ and $n \ge 2$ induced by a k-admissible sequence $(t_n)_{n=0}^\infty$. Most of the results are taken from [20].

Here, we change our notation slightly. We fix n and let i_0 with $k+1 \le i_0 \le n+k-1$ be such that $\mathcal{T}_{n-1}$ equals $\mathcal{T}_n$ with the point τ_{i_0} removed. In the points of the partition $\mathcal{T}_n$, we omit the parameter n, and thus $\mathcal{T}_n$ is given by

$$\mathcal{T}_n = (0 = \tau_1 = \cdots = \tau_k < \tau_{k+1} \le \cdots \le \tau_{i_0} \le \cdots \le \tau_{n+k-1} < \tau_{n+k} = \cdots = \tau_{n+2k-1} = 1).$$

We denote by $(N_i : 1 \le i \le n+k-1)$ the B-spline functions corresponding to $\mathcal{T}_n$.

An (unnormalized) orthogonal spline function $g \in \mathcal{S}_n^{(k)}$ that is orthogonal to $\mathcal{S}_{n-1}^{(k)}$, as calculated in [20], is given by

$$g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^* = \sum_{j=i_0-k}^{i_0} \sum_{\ell=1}^{n+k-1} \alpha_j b_{j\ell} N_\ell, \tag{4.1}$$

where $(b_{j\ell})_{j,\ell=1}^{n+k-1}$ is the inverse of the Gram matrix $(\langle N_j, N_\ell\rangle)_{j,\ell=1}^{n+k-1}$ and

(4.2)
$$\alpha_j = (-1)^{j-i_0+k} \Big(\prod_{\ell=i_0-k+1}^{j-1} \frac{\tau_{i_0}-\tau_\ell}{\tau_{\ell+k}-\tau_\ell}\Big)\Big(\prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k}-\tau_{i_0}}{\tau_{\ell+k}-\tau_\ell}\Big), \quad i_0-k \le j \le i_0.$$

We remark that the sequence (α_j) alternates in sign, and since the matrix $(b_{j\ell})_{j,\ell=1}^{n+k-1}$ is checkerboard, the B-spline coefficients of g, that is,

$$w_\ell := \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell}, \qquad 1 \le \ell \le n+k-1, \tag{4.3}$$

satisfy

$$\Big| \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell} \Big| = \sum_{j=i_0-k}^{i_0} |\alpha_j b_{j\ell}|, \qquad 1 \le j \le n+k-1. \tag{4.4}$$

In the definition below, we assign to each orthonormal spline function a characteristic interval that is a grid point interval $[\tau_i, \tau_{i+1}]$ and lies close to the newly inserted point τ_{i_0}. The choice of this interval is crucial for proving important properties of the system $(f_n^{(k)})_{n=-k+2}^\infty$. This approach has its origins in [14], where it is proved that general Franklin systems are unconditional bases in L^p, $1 < p < \infty$.

Definition 4.1. Let $\mathcal{T}_n, \mathcal{T}_{n-1}$ be as above and τ_{i_0} be the new point in $\mathcal{T}_n$ that is not present in $\mathcal{T}_{n-1}$. We define the *characteristic interval J_n corresponding to the pair $(\mathcal{T}_n, \mathcal{T}_{n-1})$* as follows.

(1) Let
$$\Lambda^{(0)} := \Big\{ i_0 - k \le j \le i_0 : |[\tau_j, \tau_{j+k}]| \le 2 \min_{i_0-k\le \ell \le i_0} |[\tau_\ell, \tau_{\ell+k}]| \Big\}$$
be the set of all j for which the support of the B-spline function N_j is approximately minimal. Observe that $\Lambda^{(0)}$ is nonempty.

(2) Define

$$\Lambda^{(1)} := \Big\{ j \in \Lambda^{(0)} : |\alpha_j| = \max_{\ell \in \Lambda^{(0)}} |\alpha_\ell| \Big\}.$$

For any fixed index $j^{(0)} \in \Lambda^{(1)}$, set $J^{(0)} := [\tau_{j^{(0)}}, \tau_{j^{(0)}+k}]$.

(3) The interval $J^{(0)}$ can now be written as the union of k grid intervals

$$J^{(0)} = \bigcup_{\ell=0}^{k-1} [\tau_{j^{(0)}+\ell}, \tau_{j^{(0)}+\ell+1}] \quad \text{with } j^{(0)} \text{ as above.}$$

We define the *characteristic interval* J_n to be one of the above k intervals that has maximal length.

A few clarifying comments are in order. Roughly speaking, we first take the B-spline support $[\tau_j, \tau_{j+k}]$ including the new point τ_{i_0} with minimal length and then we choose as J_n the largest grid point interval in $[\tau_j, \tau_{j+k}]$. This definition guarantees the concentration of f_n on J_n in terms of the L^p-norm (cf. Lemma 4.3) and the exponential decay of f_n away from J_n (cf. Lemma 4.4), which are crucial for further investigations. An important ingredient in the proof of Lemma 4.3 is Proposition 3.3, which justifies why we choose the largest grid point interval as J_n. Further important properties of the collection (J_n) of characteristic intervals are that they form a nested family of sets and for a subsequence of decreasing characteristic intervals, their lengths decay geometrically (cf. Lemma 4.5).

Next we remark that the constant 2 in step (1) of Definition 4.1 could also be an arbitrary number $C > 1$, but $C = 1$ is not allowed. This is in contrast to the definition of characteristic intervals in [14] for piecewise linear orthogonal functions ($k = 2$), where precisely $C = 1$ is chosen, step (2) is omitted and $j^{(0)}$ is an arbitrary index in $\Lambda^{(0)}$.

At first glance, it might seem natural to carry over the same definition to arbitrary spline orders k, but at a certain point in the proof of Theorem 4.2 below, we estimate $\alpha_{j^{(0)}}$ by the constant $C - 1$ from below, which has to be strictly greater than zero in order to establish (4.5). Since Theorem 4.2 is also used in the proofs of both Lemmas 4.3 and 4.4, this is the reason for a different definition of characteristic intervals, in particular for step (2) of Definition 4.1.

Theorem 4.2 ([20]). *With the above definition* (4.3) *of w_ℓ for $1 \le \ell \le n + k - 1$ and with $j^{(0)}$ given in Definition* 4.1,

$$|w_{j^{(0)}}| \gtrsim_k b_{j^{(0)},j^{(0)}}. \tag{4.5}$$

Lemma 4.3 ([20]). *Let $\mathcal{T}_n$, $\mathcal{T}_{n-1}$ be as above and g be the function given in* (4.1). *Then $f_n = g/\|g\|_2$ is the L^2-normalized orthogonal spline function* *corresponding to* $(\mathcal{T}_n, \mathcal{T}_{n-1})$ *and*

$$\|f_n\|_{L^p(J_n)} \sim_k \|f_n\|_p \sim_k |J_n|^{1/p-1/2} \sim_k |J_n|^{1/2}\|g\|_p, \qquad 1 \le p \le \infty,$$

where J_n *is the characteristic interval associated to* $(\mathcal{T}_n, \mathcal{T}_{n-1})$.

We denote by $d_n(x)$ the number of points in $\mathcal{T}_n$ between x and J_n counting endpoints of J_n. Correspondingly, for an interval $V \subset [0,1]$, we denote by $d_n(V)$ the number of points in $\mathcal{T}_n$ between V and J_n counting endpoints of both J_n and V.

Lemma 4.4 ([20]). *Let* $\mathcal{T}_n, \mathcal{T}_{n-1}$ *be as above,* $g = \sum_{j=1}^{n+k-1} w_j N_j$ *be the function in* (4.1) *with* $(w_j)_{j=1}^{n+k-1}$ *as in* (4.3)*, and* $f_n = g/\|g\|_2$. *Then there exists a constant* $0 < q < 1$ *that depends only on* k *such that*

(4.6)
$$|w_j| \lesssim_k \frac{q^{d_n(\tau_j)}}{|J_n| + \operatorname{dist}(\operatorname{supp} N_j, J_n) + |D^k_{n,j}|} \quad \text{for all } 1 \le j \le n+k-1.$$

Moreover, if $x < \inf J_n$*, we have*

$$\|f_n\|_{L^p(0,x)} \lesssim_k \frac{q^{d_n(x)}|J_n|^{1/2}}{(|J_n| + \operatorname{dist}(x, J_n))^{1-1/p}}, \qquad 1 \le p \le \infty. \tag{4.7}$$

Similarly, for $x > \sup J_n$,

$$\|f_n\|_{L^p(x,1)} \lesssim_k \frac{q^{d_n(x)}|J_n|^{1/2}}{(|J_n| + \operatorname{dist}(x, J_n))^{1-1/p}}, \qquad 1 \le p \le \infty. \tag{4.8}$$

4.2. Combinatorics of characteristic intervals. Next, we recall a combinatorial result about the relative positions of different characteristic intervals:

Lemma 4.5 ([20]). *Let* $x, y \in (t_n)_{n=0}^\infty$ *with* $x < y$. *Then there exists a constant* F_k *only depending on* k *such that*

$$N_0 := \operatorname{card}\{n : J_n \subseteq [x,y],\ |J_n| \ge |[x,y]|/2\} \le F_k,$$

where $\operatorname{card} E$ *denotes the cardinality of the set* E.

Similarly to [14] and [15], we need the following estimate involving characteristic intervals and orthonormal spline functions:

Lemma 4.6. *Let* (t_n) *be a* k*-admissible point sequence in* $[0,1]$ *and let* $(f_n)_{n \ge -k+2}$ *be the corresponding orthonormal spline system of order* k. *Then, for each interval* $V = [\alpha, \beta] \subset [0,1]$,

$$\sum_{n:\, J_n \subset V} |J_n|^{1/2} \int_{V^c} |f_n(t)|\, dt \lesssim_k |V|.$$

Once we know the estimates for orthonormal spline functions as in Lemma 4.4 and the basic combinatorial result for their characteristic intervals, i.e. Lemma 4.5, this result follows by the same argument that was used in the proof of Lemma 4.6 in [14], so we skip its proof.

Instead of Lemma 3.4 of [15], we will use the following:

LEMMA 4.7. *Let $(t_n)_{n=0}^\infty$ be a k-admissible knot sequence that is $(k-1)$-regular, and let $\Delta = D_{m,i}^{(k-1)}$ for some m and i. For $\ell \geq 0$, let*

$$N(\Delta) := \{n : \operatorname{card}(\Delta \cap \mathcal{T}_n) = k,\ J_n \subset \Delta\},$$
$$M(\Delta,\ell) := \{n : d_n(\Delta) = \ell,\ \operatorname{card}(\Delta \cap \mathcal{T}_n) \geq k,\ |J_n \cap \Delta| = 0\},$$

where in both definitions we count the points in $\Delta \cap \mathcal{T}_n$ including multiplicities. Then

$$\frac{1}{|\Delta|} \sum_{n \in N(\Delta)} |J_n| \lesssim_k 1, \qquad \sum_{n \in M(\Delta,\ell)} \frac{|J_n|}{\operatorname{dist}(J_n,\Delta) + |\Delta|} \lesssim_{k,\gamma} (\ell+1)^2. \tag{4.9}$$

Proof. For every $n \in N(\Delta)$, there are only the $k-1$ possibilities $D_{m,i}^{(1)}, \ldots, D_{m,i+k-2}^{(1)}$ for J_n and by Lemma 4.5, each interval $D_{m,j}^{(1)}$, $j = i, \ldots, i+k-2$, occurs at most F_k times as a characteristic interval. This implies the first inequality in (4.9).

To prove the second, assume that each J_n, $n \in M(\Delta,\ell)$, lies to the right of Δ, since the other case is handled similarly. The argument is split into two parts depending on the value of ℓ, beginning with $\ell \leq k$. In that case, for $n \in M(\Delta,\ell)$, let $J_n^{1/2}$ be the unique interval determined by the conditions

$$\sup J_n^{1/2} = \sup J_n, \qquad |J_n^{1/2}| = |J_n|/2.$$

Since $d_n(\Delta) = \ell$ is constant, we group the intervals J_n into packets, where all intervals in one packet have the same left endpoint and maximal intervals from different packets are disjoint (up to possibly one point). By Lemma 4.5, each $t \in [0,1]$ belongs to at most F_k intervals $J_n^{1/2}$. The $(k-1)$-regularity and $\ell \leq k$ now imply $|J_n| \lesssim_{k,\gamma} |\Delta|$ and $\operatorname{dist}(\Delta, J_n) \lesssim_{k,\gamma} |\Delta|$ for $n \in M(\Delta,\ell)$, and thus every interval J_n with $n \in M(\Delta,\ell)$ is a subset of a fixed interval whose length is comparable to $|\Delta|$. Putting these things together, we obtain

$$\sum_{n \in M(\Delta,\ell)} \frac{|J_n|}{\operatorname{dist}(J_n,\Delta) + |\Delta|} \leq \frac{1}{|\Delta|} \sum_{n \in M(\Delta,\ell)} |J_n| = \frac{2}{|\Delta|} \sum_{n \in M(\Delta,\ell)} \int_{J_n^{1/2}} dx \lesssim_{k,\gamma} 1,$$

which completes the case of $\ell \leq k$.

Next, assume $\ell \geq k+1$ and define $(L_j)_{j=1}^\infty$ as the strictly decreasing sequence of all sets L that satisfy

$$L = D_{n,i}^{(k-1)} \quad \text{and} \quad \sup L = \sup \Delta$$

for some n and i. Moreover, set

$$M_j(\varDelta,\ell) := \{n \in M(\varDelta,\ell) : \operatorname{card}(L_j \cap \mathcal{T}_n) = k\},$$

i.e., L_j is a union of $k-1$ grid point intervals in the grid $\mathcal{T}_n$. Then, since $|\varDelta| + \operatorname{dist}(J_n,\varDelta) \gtrsim_\gamma |\varDelta| + \operatorname{dist}(t,\varDelta)$ for $t \in J_n^{1/2}$ by $(k-1)$-regularity,

$$\sum_{n\in M_j(\varDelta,\ell)} \frac{|J_n|}{\operatorname{dist}(J_n,\varDelta)+|\varDelta|} \lesssim_{k,\gamma} \sum_{n\in M_j(\varDelta,\ell)} \int_{J_n^{1/2}} \frac{1}{\operatorname{dist}(t,\varDelta)+|\varDelta|}\,dt.$$

If $n \in M_j(\varDelta,\ell)$ we get, again due to $(k-1)$-regularity,

$$\inf J_n^{1/2} \ge \inf J_n \ge \gamma^{-k}|L_j| + \sup \varDelta,$$

and

$$\sup J_n^{1/2} \le \inf J_n + |J_n| \le C_k\gamma^\ell |L_j| + \sup \varDelta$$

for some constant C_k only depending on k. Combining this with Lemma 4.5, which implies that each point t belongs to at most F_k intervals $J_n^{1/2}$, we get

$$\sum_{n\in M_j(\varDelta,\ell)} \int_{J_n^{1/2}} \frac{1}{\operatorname{dist}(t,\varDelta)+|\varDelta|}\,dt \lesssim \int_{\gamma^{-k}|L_j|+|\varDelta|}^{C_k\gamma^\ell|L_j|+|\varDelta|} \frac{1}{s}\,ds. \tag{4.10}$$

Next we will show that the above integration intervals can intersect for $\lesssim \ell$ indices j. Let $j_2 \ge j_1$, so that $L_{j_1} \supset L_{j_2}$, and write $j_2 = j_1 + 2kr + t$ with $t \le 2k-1$. Then, by Lemma 3.1,

$$C_k\gamma^\ell|L_{j_2}| \le C_k\gamma^\ell|L_{j_1+2kr}| \le C_k\gamma^\ell\eta^r|L_{j_1}|,$$

where $\eta = \gamma^{k-1}/(1+\gamma^{k-1}) < 1$. If now $r \ge C_{k,\gamma}\ell$ for a suitable constant $C_{k,\gamma}$ depending only on k and γ, we have

$$C_k\gamma^\ell|L_{j_2}| \le \gamma^{-k}|L_{j_1}|.$$

Thus, each point s in the integral in (4.10) for some j belongs to at most $C_{k,\gamma}\ell$ intervals $[\gamma^{-k}|L_j| + |\varDelta|, C_k\gamma^\ell|L_j| + |\varDelta|]$ where j is varying. So by summing over j we conclude

$$\sum_{n\in M(\varDelta,\ell)} \frac{|J_n|}{\operatorname{dist}(J_n,\varDelta)+|\varDelta|} \le C_{k,\gamma}\ell \int_{|\varDelta|}^{(1+C_k\gamma^\ell)|\varDelta|} \frac{1}{s}\,ds \le C_{k,\gamma}\ell^2.$$

This completes the analysis of the case $\ell \ge k+1$, and the proof of the lemma. ∎

5. Four conditions on spline series and their relations. Let (t_n) be a k-admissible sequence of knots with the corresponding orthonormal spline system $(f_n)_{n\geq -k+2}$. For a sequence $(a_n)_{n\geq -k+2}$ of coefficients, let

$$P := \Big(\sum_{n=-k+2}^{\infty} a_n^2 f_n^2\Big)^{1/2} \quad \text{and} \quad S := \max_{m\geq -k+2}\Big|\sum_{n=-k+2}^{m} a_n f_n\Big|.$$

If $f \in L^1$, we denote by Pf and Sf the respective functions P and S corresponding to the coefficient sequence $a_n = \langle f, f_n\rangle$. Consider the following conditions:

(A) $P \in L^1$.
(B) The series $\sum_{n=-k+2}^{\infty} a_n f_n$ converges unconditionally in L^1.
(C) $S \in L^1$.
(D) There exists a function $f \in H^1$ such that $a_n = \langle f, f_n\rangle$.

We will discuss relations between those four conditions and prove the implications indicated in the diagram below; some results need regularity conditions on (t_n), which we also indicate.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(A) \arrow[r, Rightarrow, shift left, "{\text{Proposition 5.2, } \sup_\varepsilon \|\sum \varepsilon_n a_n f_n\|_1 \lesssim_k \|P\|_1}"] \arrow[dr, Rightarrow, "{\text{Proposition 5.2, } \|S\|_1 \lesssim_k \|P\|_1}"] & (B) \arrow[l, Rightarrow, shift left, "{\|P\|_1 \lesssim \sup_\varepsilon \|\sum \varepsilon_n a_n f_n\|_1, \text{ Proposition 5.1}}"] \\
(D) \arrow[u, Rightarrow, "{\text{Proposition 5.3, } (k-1)\text{-reg.} \Rightarrow \|Pf\|_1 \lesssim_{k,\gamma} \|f\|_{H^1}}"] & (C) \arrow[l, Rightarrow, "{k\text{-reg.} \Rightarrow \|f\|_{H^1} \lesssim_{k,\gamma} \|Sf\|_1, \text{ Proposition 5.4}}"]
\end{tikzcd}$$

For orthonormal spline systems with dyadic knots, relations (and equivalences) of these conditions have been studied by several authors, also in the case $p < 1$ (see e.g. [23, 1, 12]). For general Franklin systems corresponding to arbitrary sequences of knots, relations of these conditions were discussed in [15] (and earlier in [13], also for $p < 1$, but for a restricted class of point sequences). Below, we follow the approach of [15], adapted to the case of spline orthonormal systems of order k.

We begin with the implication (B)⇒(A), which is a consequence of Khinchin's inequality:

Proposition 5.1 ((B)⇒(A)). *Let (t_n) be a k-admissible sequence of knots with the corresponding general orthonormal spline system (f_n), and* *let* (a_n) *be a sequence of coefficients. If the series* $\sum_{n=-k+2}^{\infty} a_n f_n$ *converges unconditionally in* L^1*, then* $P \in L^1$*. Moreover,*

$$\|P\|_1 \lesssim \sup_{\varepsilon\in\{-1,1\}^{\mathbb{Z}}} \Big\| \sum_{n=-k+2}^{\infty} \varepsilon_n a_n f_n \Big\|_1.$$

Next, we investigate the implications (A)⇒(B) and (A)⇒(C). Once we know the estimates and combinatorial results of Sections 3 and 4, the proof is the same as in [15, proof of Proposition 4.3], so we just state the result.

Proposition 5.2 ((A)⇒(B) and (A)⇒(C)). *Let* (t_n) *be a* k*-admissible sequence of knots and let* (a_n) *be a sequence of coefficients such that* $P \in L^1$*. Then* $S \in L^1$ *and* $\sum a_n f_n$ *converges unconditionally in* L^1*; moreover,*

$$\sup_{\varepsilon\in\{-1,1\}^{\mathbb{Z}}} \Big\| \sum_{n=-k+2}^{\infty} \varepsilon_n a_n f_n \Big\| \lesssim_k \|P\|_1 \quad \text{and} \quad \|S\|_1 \lesssim_k \|P\|_1.$$

Next we discuss (D)⇒(A).

Proposition 5.3 ((D)⇒(A)). *Let* (t_n) *be a* k*-admissible point sequence that is* $(k-1)$*-regular with parameter* γ*. Then there exists a constant* $C_{k,\gamma}$*, depending only on* k *and* γ*, such that for each atom* ϕ*,*

$$\|P\phi\|_1 \le C_{k,\gamma}.$$

Consequently, if $f \in H^1$*, then*

$$\|Pf\|_1 \le C_{k,\gamma}\|f\|_{H^1}.$$

Before we proceed to the proof, let us remark that essentially the same arguments give a direct proof of (D)⇒(C), under the same assumption of $(k-1)$-regularity of (t_n), and moreover

$$\|Sf\|_1 \le C_{k,\gamma}\|f\|_{H^1}.$$

We do not present it here, since we have the implications (D)⇒(A) under the assumption of $(k-1)$-regularity and (A)⇒(C) under the assumption of k-admissibility only. Note that Proposition 6.1 below shows that without the assumption of $(k-1)$-regularity of the point sequence, the implications (D)⇒(A) and (D)⇒(C) need not be true.

Proof of Proposition 5.3. Let ϕ be an atom with $\int_0^1 \phi(u)\,du = 0$ and let $\varGamma = [\alpha,\beta]$ be an interval such that $\operatorname{supp}\phi \subset \varGamma$ and $\sup|\phi| \le |\varGamma|^{-1}$. Define $n_\varGamma := \max\{n : \operatorname{card}(\mathcal{T}_n \cap \varGamma) \le k-1\}$, where in the maximum, we also count multiplicities of knots. It will be shown that

$$\|P_1\phi\|_1, \|P_2\phi\|_1 \lesssim_{\gamma,k} 1,$$

where

$$P_1\phi = \Big(\sum_{n\le n_\varGamma} a_n^2 f_n^2\Big)^{1/2} \quad \text{and} \quad P_2\phi = \Big(\sum_{n> n_\varGamma} a_n^2 f_n^2\Big)^{1/2}.$$

First, we consider P_1 and prove the stronger inequality

$$\sum_{n\le n_\Gamma} |a_n|\,\|f_n\|_1 \lesssim_{k,\gamma} 1,$$

where $a_n = \langle \phi, f_n\rangle$. For each $n \le n_\Gamma$, we define $\Gamma_{n,\alpha}$ as the unique closed interval $D_{n,j}^{(k-1)}$ with minimal j such that

$$\alpha \le \min D_{n,j+1}^{(k-1)}.$$

We note that

$$\Gamma_{n_1,\alpha} \supseteq \Gamma_{n_2,\alpha} \qquad \text{for } n_1 \le n_2,$$

and, by $(k-1)$-regularity,

$$|\Gamma_{n,\alpha}| \gtrsim_{\gamma,k} |\Gamma|.$$

Let $g_n = \sum_{j=1}^{n+k-1} w_j N_{n,j}^{(k)}$ be the unnormalized orthogonal spline function as in (4.1) and with the coefficients (w_j) as in (4.3). For $\xi \in \Gamma$, we have (cf. (3.1))

$$|g_n'(\xi)| \lesssim_k \sum_j \frac{|w_j| + |w_{j-1}|}{|D_{n,j}^{(k-1)}|}, \tag{5.1}$$

where we sum over those j such that $\Gamma \cap \operatorname{supp} N_{n,j}^{(k-1)} = \Gamma \cap D_{n,j}^{(k-1)} \ne \emptyset$. By $(k-1)$-regularity, all lengths $|D_{n,j}^{(k-1)}|$ in this summation are comparable to $|\Gamma_{n,\alpha}|$. Moreover, by (4.6),

$$|w_j| \lesssim_k \frac{q^{d_n(\tau_{n,j})}}{|J_n| + \operatorname{dist}(D_{n,j}^{(k)}, J_n) + |D_{n,j}^{(k)}|}.$$

Again by $(k-1)$-regularity, for j in (5.1),

$$|D_{n,j}^{(k-1)}| \gtrsim_{k,\gamma} |\Gamma_{n,\alpha}|,$$

$$\operatorname{dist}(D_{n,j}^{(k)}, J_n) + |D_{n,j}^{(k)}| \gtrsim_{k,\gamma} \operatorname{dist}(J_n, \Gamma_{n,\alpha}) + |\Gamma_{n,\alpha}|.$$

Combining the above inequalities, we estimate the right hand side in (5.1) further and get, with the notation $\Gamma_n := \Gamma_{n,\alpha}$,

$$|g_n'(\xi)| \lesssim_{k,\gamma} \frac{1}{|\Gamma_n|}\,\frac{q^{d_n(\Gamma_n)}}{|J_n| + \operatorname{dist}(J_n,\Gamma_n) + |\Gamma_n|}. \tag{5.2}$$

As a consequence, for every $\tau \in \Gamma$,

$$\begin{aligned}|a_n| &= \Big|\int_\Gamma \phi(t)[f_n(t) - f_n(\tau)]\,dt\Big| \le \int_\Gamma \frac{1}{|\Gamma|}\sup_{\xi\in\Gamma}|f_n'(\xi)|\,|t-\tau|\,dt \\ &\lesssim_k |\Gamma|\,|J_n|^{1/2}\sup_{\xi\in\Gamma}|g_n'(\xi)| \lesssim_{k,\gamma} \frac{|\Gamma|}{|\Gamma_n|}\,\frac{|J_n|^{1/2} q^{d_n(\Gamma_n)}}{|J_n| + \operatorname{dist}(J_n,\Gamma_n) + |\Gamma_n|}.\end{aligned}$$

Let $\Delta_1 \supset \cdots \supset \Delta_s$ be the collection of all different intervals appearing as Γ_n for $n \le n_\Gamma$. By Lemma 3.1, we have some geometric decay in the measure of Δ_i. Now fix Δ_i and $\ell \geq 0$ and consider indices $n \leq n_\Gamma$ such that $\Gamma_n = \Delta_i$ and $d_n(\Gamma_n) = \ell$. By the last display and Lemma 4.3,

$$|a_n| \, \|f_n\|_1 \lesssim_{k,\gamma} \frac{|\Gamma|}{|\Delta_i|} \frac{|J_n| q^\ell}{|J_n| + \operatorname{dist}(J_n, \Delta_i) + |\Delta_i|},$$

and thus Lemma 4.7 implies

$$\sum_{n:\, \Gamma_n = \Delta_i,\, d_n(\Gamma_n) = \ell} |a_n| \, \|f_n\|_1 \lesssim_{k,\gamma} (\ell+1)^2 q^\ell \frac{|\Gamma|}{|\Delta_i|}.$$

Now, summing over ℓ and then over i (recall that $|\Delta_i|$ decays like a geometric progression by Lemma 3.1 and $|\Delta_i| \gtrsim_{k,\gamma} |\Gamma|$ since $n \leq n_\Gamma$) yields

$$\sum_{n \leq n_\Gamma} |a_n| \, \|f_n\|_1 \lesssim_{k,\gamma} 1.$$

This implies the desired inequality $\|P_1 \phi\|_1 \lesssim_{k,\gamma} 1$ for the first part of $P\phi$.

Next, we look at $P_2\phi$ and define V to be the smallest interval whose endpoints in $\mathcal{T}_{n_\Gamma+1}$ and which contains Γ. Moreover, $\widetilde{V}$ is defined to be the smallest interval with endpoints in $\mathcal{T}_{n_\Gamma+1}$ and such that $\widetilde{V}$ contains k points in $\mathcal{T}_{n_\Gamma+1}$ to the left of Γ and as well k points in $\mathcal{T}_{n_\Gamma+1}$ to the right of Γ. We observe that due to $(k-1)$-regularity and the fact that Γ contains at least k points from $\mathcal{T}_{n_\Gamma+1}$,

$$\begin{aligned} &|V| \sim_{k,\gamma} |\widetilde{V}| \sim_{k,\gamma} |\Gamma|, \\ &|(\widetilde{V} \setminus V) \cap [0, \inf \Gamma]| \sim_{k,\gamma} |(\widetilde{V} \setminus V) \cap [\sup \Gamma, 1]| \sim_{k,\gamma} |\widetilde{V}|. \end{aligned} \tag{5.3}$$

First, we consider the integral of $P_2\phi$ over $\widetilde{V}$ and obtain by the Cauchy–Schwarz inequality

$$\int_{\widetilde{V}} P_2\phi(t) \, dt \leq \|\mathbb{1}_{\widetilde{V}}\|_2 \|\phi\|_2 \leq \frac{|\widetilde{V}|^{1/2}}{|\Gamma|^{1/2}} \lesssim_{k,\gamma} 1.$$

It remains to estimate $\int_{\widetilde{V}^c} P_2\phi(t) \, dt$. Since for $n > n_\Gamma$, the endpoints of $\widetilde{V}$ are in $\mathcal{T}_n$, either we have $J_n \subset \widetilde{V}$, or J_n is to the right of $\tilde{V}$, or J_n is to the left of $\widetilde{V}$. If $J_n \subset \widetilde{V}$, then

$$|a_n| = \Big| \int_\Gamma \phi(t) f_n(t) \, dt \Big| \leq \frac{\|f_n\|_1}{|\Gamma|} \lesssim_k \frac{|J_n|^{1/2}}{|\Gamma|},$$

and therefore, by Lemma 4.6 and (5.3),

$$\begin{aligned} \sum_{n:\, J_n \subset \widetilde{V},\, n > n_\Gamma} |a_n| \int_{\widetilde{V}^c} |f_n(t)| \, dt &\lesssim_k \frac{1}{|\Gamma|} \sum_{n: J_n \subset \widetilde{V}} |J_n|^{1/2} \int_{\widetilde{V}^c} |f_n(t)| \, dt \\ &\lesssim_k \frac{|\widetilde{V}|}{|\Gamma|} \lesssim_{k,\gamma} 1. \end{aligned}$$

Now, let J_n be to the right of $\widetilde{V}$; the case of J_n to the left of $\widetilde{V}$ is considered similarly. By (4.7) for $p = \infty$,

$$|a_n| \leq \frac{1}{|\Gamma|} \int_{\Gamma} |f_n(t)|\,dt \leq \frac{1}{|\Gamma|} \int_{V} |f_n(t)|\,dt \lesssim_{k,\gamma} \frac{q^{d_n(V)}|J_n|^{1/2}}{\operatorname{dist}(V, J_n) + |J_n|}.$$

This inequality, Lemma 4.3 and the fact that $\operatorname{dist}(V, J_n) \gtrsim_{k,\gamma} \operatorname{dist}(V, J_n) + |V|$ (cf. (5.3)) allow us to deduce

$$\sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} |a_n|\,\|f_n\|_1 \lesssim_{k,\gamma} \sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} \frac{q^{d_n(V)}|J_n|}{\operatorname{dist}(V, J_n) + |V|}.$$

Note that V can be a union of $k-1$, k or $k+1$ intervals from $\mathcal{T}_{n_\Gamma+1}$; therefore, let V^+ be a union of $k-1$ grid intervals from $\mathcal{T}_{n_\Gamma+1}$, with right endpoint of V^+ coinciding with the right endpoint of V. As J_n is to the right of V, we have $d_n(V) = d_n(V^+)$, $\operatorname{dist}(V, J_n) = \operatorname{dist}(V^+, J_n)$ and—by $(k-1)$-regularity—$|V| \sim_{k,\gamma} |V^+|$, which implies

$$\sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} \frac{q^{d_n(V)}|J_n|}{\operatorname{dist}(V, J_n) + |V|} \lesssim_{k,\gamma} \sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} \frac{q^{d_n(V^+)}|J_n|}{\operatorname{dist}(V^+, J_n) + |V^+|}.$$

Finally, we employ Lemma 4.7 to conclude

$$\begin{aligned}\sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} |a_n|\,\|f_n\|_1 &\lesssim_{k,\gamma} \sum_{\ell=0}^{\infty} q^\ell \sum_{\substack{n>n_\Gamma \\ d_n(V^+)=\ell \\ J_n \text{ to the right of } \widetilde{V}}} \frac{|J_n|}{\operatorname{dist}(V^+, J_n) + |V^+|} \\ &\lesssim_{k,\gamma} \sum_{\ell=0}^{\infty} (\ell+1)^2 q^\ell \lesssim_k 1.\end{aligned}$$

To conclude the proof, note that if $f \in H^1$ and $f = \sum_{m=1}^\infty c_m \phi_m$ is an atomic decomposition of f, then $\langle f, f_n\rangle = \sum_{m=1}^\infty c_m \langle \phi_m, f_n\rangle$, and $Pf(t) \leq \sum_{m=1}^\infty |c_m| P\phi_m(t)$. ∎

Finally, we discuss (C)⇒(D).

Proposition 5.4 ((C)⇒(D)). *Let (t_n) be a k-admissible sequence of knots in $[0,1]$ which is k-regular with parameter γ and let (a_n) be a sequence of coefficients such that $S = \sup_m \big|\sum_{n\leq m} a_n f_n\big| \in L^1$. Then there exists a function $f \in H^1$ with $a_n = \langle f, f_n\rangle$ for each n. Moreover,*

$$\|f\|_{H^1} \lesssim_{k,\gamma} \|Sf\|_1.$$

Proof. As $S \in L^1$, there is $f \in L^1$ such that $f = \sum_{n\geq -k+2} a_n f_n$ with convergence in L^1. Indeed, this is a consequence of the relative weak compactness of uniformly integrable subsets in L^1 and the basis property of (f_n)

in L^1. Thus, we need only show that $f \in H^1$, and this is done by finding a suitable atomic decomposition of f.

We define $E_0 = B_0 = [0,1]$ and, for $r \geq 1$,

$$E_r = [S > 2^r], \qquad B_r = [\mathcal{M}\mathbb{1}_{E_r} > c_{k,\gamma}],$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function and $0 < c_{k,\gamma} \leq 1/2$ is a small constant only depending on k and γ which is chosen according to a few restrictions that will be given during the proof. We note that

$$\mathcal{M}\mathbb{1}_{E_r}(t) = \sup_{I \ni t} \frac{|I \cap E_r|}{|I|}, \qquad t \in [0,1],$$

where the supremum is taken over all intervals containing t. Since $\mathcal{M}$ is of weak type $(1,1)$, we have $|B_r| \lesssim_{k,\gamma} |E_r|$. As $S \in L^1$, it follows that $|E_r| \to 0$ and hence $|B_r| \to 0$ as $r \to \infty$. Now, decompose the open set B_r into a countable union of disjoint open intervals,

$$B_r = \bigcup_{\kappa} \Gamma_{r,\kappa},$$

where for fixed r, no two intervals $\Gamma_{r,\kappa}$ have a common endpoint and the above equality is up to a measure zero set (each open set of real numbers can be decomposed into a countable union of open intervals, but it can happen that two intervals have the same endpoint; in that case, we collect those two intervals into one $\Gamma_{r,\kappa}$). This can be achieved by taking as $\Gamma_{r,\kappa}$ the collection of level sets of positive measure of the function $t \mapsto |[0,t] \cap B_r^c|$.

Moreover, observe that if $\Gamma_{r+1,\xi}$ is one of the intervals in the decomposition of B_{r+1}, then there is an interval $\Gamma_{r,\kappa}$ in the decomposition of B_r such that $\Gamma_{r+1,\xi} \subset \Gamma_{r,\kappa}$.

Based on this decomposition, we define the following functions for $r \geq 0$:

$$g_r(t) := \begin{cases} f(t), & t \in B_r^c, \\ \dfrac{1}{|\Gamma_{r,\kappa}|} \displaystyle\int_{\Gamma_{r,\kappa}} f(t)\,dt, & t \in \Gamma_{r,\kappa}. \end{cases}$$

Observe that $f = g_0 + \sum_{r=0}^{\infty}(g_{r+1} - g_r)$ in L^1 and $g_{r+1} - g_r = 0$ on B_r^c. As a consequence,

$$\begin{aligned} \int_{\Gamma_{r,\kappa}} g_{r+1}(t)\,dt &= \int_{\Gamma_{r,\kappa} \cap B_{r+1}^c} g_{r+1}(t)\,dt + \int_{\Gamma_{r,\kappa} \cap B_{r+1}} g_{r+1}(t)\,dt \\ &= \int_{\Gamma_{r,\kappa} \cap B_{r+1}^c} f(t)\,dt + \sum_{\xi :\, \Gamma_{r+1,\xi} \subset \Gamma_{r,\kappa}} \int_{\Gamma_{r+1,\xi}} f(t)\,dt \\ &= \int_{\Gamma_{r,\kappa}} f(t)\,dt = \int_{\Gamma_{r,\kappa}} g_r(t)\,dt. \end{aligned}$$

The main step of the proof is to show that

$$|g_r(t)| \leq C_{k,\gamma} 2^r, \qquad \text{a.e. } t \in [0,1], \tag{5.4}$$

for some constant $C_{k,\gamma}$ only depending on k and γ. Once this inequality is proved, we take $\phi_0 \equiv 1$, $\eta_0 = \int_0^1 f(u)\,du$ and

$$\phi_{r,\kappa} := \frac{(g_{r+1} - g_r)\mathbb{1}_{\Gamma_{r,\kappa}}}{C_{k,\gamma} 2^r |\Gamma_{r,\kappa}|}, \qquad \eta_{r,\kappa} = C_{k,\gamma} 2^r |\Gamma_{r,\kappa}|$$

and observe that $f = \eta_0\phi_0 + \sum_{r,\kappa} \eta_{r,\kappa}\phi_{r,\kappa}$ is the desired atomic decomposition of f since

$$\begin{aligned}\sum_{r,\kappa} \eta_{r,\kappa} \leq C_{k,\gamma} \sum_{r,\kappa} 2^r |\Gamma_{r,\kappa}| &= C_{k,\gamma} \sum_r 2^r |B_r| \\ \lesssim_{k,\gamma} \sum_r 2^r |E_r| &\lesssim \|S\|_1.\end{aligned}$$

Thus it remains to prove inequality (5.4).

To do so, we first assume $t \in B_r^c$. Additionally, assume that t is such that the series $\sum_n a_n f_n(t)$ converges to $f(t)$ and t is not in (t_n). By Theorem 3.5, this holds for a.e. $\in [0,1]$. We fix m and let V_m be the maximal interval where the function $S_m := \sum_{n \leq m} a_n f_n$ is a polynomial of order k and that contains t. Then $V_m \not\subset B_r$ and since V_m is an interval containing t,

$$|V_m \cap E_r^c| \geq (1 - c_{k,\gamma})|V_m| \geq |V_m|/2.$$

Since $|S_m| \leq 2^r$ on E_r^c, the above display and Proposition 3.2 imply that $|S_m| \lesssim_k 2^r$ on V_m and in particular $|S_m(t)| \lesssim_k 2^r$. Now, $S_m(t) \to f(t)$ as $m \to \infty$ by the assumptions on t, and thus

$$|g_r(t)| = |f(t)| \lesssim_k 2^r.$$

This concludes the proof of (5.4) in the case of $t \in B_r^c$.

Next, we fix κ and consider g_r on $\Gamma := [\alpha, \beta] := \Gamma_{r,\kappa}$. Let n_Γ be the first index such that there are $k+1$ points from $\mathcal{T}_{n_\Gamma}$ contained in Γ, i.e., there exists a support $D_{n_\Gamma,i}^{(k)}$ of a B-spline function of order k in the grid $\mathcal{T}_{n_\Gamma}$ that is contained in Γ. Additionally, we define

$$U_0 := [\tau_{n_\Gamma,i-k}, \tau_{n_\Gamma,i}], \qquad W_0 := [\tau_{n_\Gamma,i+k}, \tau_{n_\Gamma,i+2k}].$$

Note that if $\alpha \in \mathcal{T}_{n_\Gamma}$, then α is a common endpoint of U_0 and Γ, otherwise α is an interior point of U_0. Similarly, if $\beta \in \mathcal{T}_{n_\Gamma}$, then β is a common endpoint of W_0 and Γ, otherwise β is an interior point of W_0. By k-regularity of (t_n), we have $\max(|U_0|, |W_0|) \lesssim_{k,\gamma} |\Gamma|$. We first estimate the part $S_\Gamma := \sum_{n \leq n_\Gamma} a_n f_n$ and show that $|S_\Gamma| \lesssim_{k,\gamma} 2^r$ on Γ. Observe that on $\Delta := U_0 \cup \Gamma \cup W_0$, S_Γ can be represented as a linear combination of B-splines (N_j) on the grid $\mathcal{T}_{n_\Gamma}$ of the form

$$S_\Gamma(t) = h(t) := \sum_{j=i-2k+1}^{i+2k-1} b_j N_j(t),$$

for some coefficients (b_j). For $j = i-2k+1, \dots, i+2k-1$, let J_j be a maximal interval of $\operatorname{supp} N_j$ and observe that due to k-regularity, $|J_j| \sim_{k,\gamma} |\Gamma| \sim_{k,\gamma} |\operatorname{supp} h|$.

If we assume that $\max_{J_j} |S_\Gamma| > C_k 2^r$, where C_k is the constant of Proposition 3.2 for $\rho = 1/2$, then Proposition 3.2 implies that $|S_\Gamma| > 2^r$ on a subset I_j of J_j with measure $\geq |J_j|/2$. Hence

$$|\operatorname{supp} h \cap E_r| \geq |J_j \cap E_r| \geq |J_j|/2 \gtrsim_{k,\gamma} |\operatorname{supp} h|.$$

We choose the constant $c_{k,\gamma}$ in the definition of B_r sufficiently small to guarantee that this last inequality implies $\operatorname{supp} h \subset B_r$. This contradicts the choice of Γ, which implies that our assumption $\max_{J_j} |S_\Gamma| > C_k 2^r$ is not true and thus

$$\max_{J_j} |S_\Gamma| \leq C_k 2^r, \qquad j = i-2k+1, \dots, i+2k-1.$$

By local stability of B-splines, i.e., inequality (3.2) in Proposition 3.3, this implies

$$|b_j| \lesssim_k 2^r, \qquad j = i-2k+1, \dots, i+2k-1,$$

and so $|S_\Gamma| \lesssim_k 2^r$ on Δ. This means

$$\int_\Gamma |S_\Gamma| \lesssim_k 2^r |\Gamma|, \tag{5.5}$$

which is inequality (5.4) for the part S_Γ.

In order to estimate the remaining part, we inductively define two sequences $(u_s, U_s)_{i\geq 0}$ and $(w_s, W_s)_{s\geq 0}$ consisting of integers and intervals. Set $u_0 = w_0 = n_\Gamma$ and inductively define u_{s+1} to be the first $n > u_s$ such that $t_n \in U_s$. Moreover, define U_{s+1} to be the B-spline support $D_{u_{s+1},i^{(k)}}$ in the grid $\mathcal{T}_{u_{s+1}}$, where i is minimal such that $D^{(k)}_{u_{s+1},i} \cap \Gamma \neq \emptyset$. Similarly, we define w_{s+1} to be the first $n > w_s$ such that $t_n \in W_s$ and W_{s+1} as the B-spline support $D^{(k)}_{w_{s+1},i}$ in the grid $\mathcal{T}_{w_{s+1}}$, where i is maximal such that $D^{(k)}_{w_{s+1},i} \cap \Gamma \neq \emptyset$. It can easily be seen that this construction implies $U_{s+1} \subset U_s$, $W_{s+1} \subset W_s$ and $\alpha \in U_s$, $\beta \in W_s$ for all $s \geq 0$, or more precisely: if $\alpha \in \mathcal{T}_{u_s}$, then α is either a common endpoint of U_s and Γ, or an inner point of U_s, and similarly if $\beta \in \mathcal{T}_{u_s}$, then β is either a common endpoint of W_s and Γ, or an inner point of W_s.

For a pair of indices ℓ, m, let

$$x_\ell := \sum_{\nu=0}^{k-1} N_{u_\ell, i+\nu} \mathbb{1}_{U_\ell}, \qquad y_m := \sum_{\nu=0}^{k-1} N_{w_m, j-\nu} \mathbb{1}_{W_m},$$

where $N_{u_\ell,i}$ is the B-spline function on the grid $\mathcal{T}_{u_\ell}$ with support U_ℓ, and $N_{w_m,j}$ is the B-spline function on $\mathcal{T}_{w_m}$ with support W_m. The function

$$\phi_{\ell,m} := x_\ell + \mathbb{1}_{\Gamma \setminus (U_\ell \cup W_m)} + y_m$$

is zero on $(U_\ell \cup \Gamma \cup W_m)^c$, one on $\Gamma \setminus (U_\ell \cup W_m)$ and a piecewise polynomial function of order k in between. For $\ell, m \geq 0$, consider the following subsets of $\{n : n > n_\Gamma\}$:

$$L(\ell) := \{n : u_\ell < n \leq u_{\ell+1}\}, \qquad R(m) := \{n : w_m < n \leq w_{m+1}\}.$$

If $n \in L(\ell) \cap R(m)$, we clearly have $\langle f_n, \phi_{\ell,m} \rangle = 0$ and thus

$$\int_\Gamma f_n(t)\,dt = \int_\Gamma f_n(t)\,dt - \int_0^1 f_n(t)\phi_{\ell,m}(t)\,dt = A_\ell(f_n) + B_m(f_n), \tag{5.6}$$

where

$$A_\ell(f_n) := \int_{\Gamma \cap U_\ell} f_n(t)\,dt - \int_{U_\ell} f_n(t) x_\ell(t)\,dt,$$

$$B_m(f_n) := \int_{\Gamma \cap W_m} f_n(t)\,dt - \int_{W_m} f_n(t) y_m(t)\,dt.$$

This implies

$$\begin{aligned} \Big| \int_\Gamma \sum_{n=n_\Gamma+1}^\infty a_n f_n(t)\,dt \Big| &= \Big| \sum_{\ell,m=0}^\infty \sum_{n \in L(\ell) \cap R(m)} a_n \big(A_\ell(f_n) + B_m(f_n) \big) \Big| \\ &\leq 2 \sum_{\ell=0}^\infty \int_{U_\ell} \Big| \sum_{n \in L(\ell)} a_n f_n(t) \Big|\,dt + 2 \sum_{m=0}^\infty \int_{W_m} \Big| \sum_{n \in R(m)} a_n f_n(t) \Big|\,dt. \end{aligned} \tag{5.7}$$

Consider the first sum on the right hand side. On $U_\ell = D^{(k)}_{u_\ell,i}$, the function $\sum_{n \in L(\ell)} a_n f_n$ can be represented as a linear combination of B-splines (N_j) on the grid $\mathcal{T}_{u_\ell}$ of the form

$$\sum_{n \in L(\ell)} a_n f_n = h_\ell := \sum_{j=i-k+1}^{i+k-1} b_j N_j,$$

for some coefficients (b_j). For $j = i-k+1, \ldots, i+k-1$, let J_j be a maximal grid interval of $\operatorname{supp} N_j$ and observe that due to k-regularity, $|J_j| \sim_{k,\gamma} |U_\ell| \sim_{k,\gamma} |\operatorname{supp} h_\ell|$. On J_j, the function $\sum_{n \in L(\ell)} a_n f_n$ is a polynomial of order k. If we assume $\max_{J_j} \big| \sum_{n \in L(\ell)} a_n f_n \big| > C_k 2^{r+1}$, where C_k is the constant of Proposition 3.2 for $\rho = 1/2$, then Proposition 3.2 implies that $\left|\sum_{n\in L(\ell)} a_n f_n\right| > 2^{r+1}$ on a set $J_j^* \subset J_j$ with $|J_j^*| = |J_j|/2$; but this means $\max(|\sum_{n\le u_\ell} a_n f_n|, |\sum_{n\le u_{\ell+1}} a_n f_n|) > 2^r$ on J_j^*. Hence

$$|E_r \cap \operatorname{supp} h_\ell| \ge |E_r \cap J_j| \ge |J_j^*| \ge |J_j|/2 \gtrsim_k |\operatorname{supp} h_\ell|.$$

We choose the constant $c_{k,\gamma}$ in the definition of B_r sufficiently small to guarantee that this last inequality implies $\operatorname{supp} h_\ell \subset B_r$. This contradicts the choice of Γ, which implies that our assumption $\max_{J_j} \left|\sum_{n\in L(\ell)} a_n f_n\right| > C_k 2^r$ is not true and thus

$$\max_{J_j} \Big|\sum_{n\in L(\ell)} a_n f_n\Big| \le C_k 2^r, \qquad j = i-k+1,\dots,i+k-1.$$

By local stability of B-splines, i.e., inequality (3.2), this implies

$$|b_j| \lesssim_k 2^r, \qquad j = i-k+1,\dots,i+k-1,$$

and so $\left|\sum_{n\in L(\ell)} a_n f_n\right| \lesssim_k 2^r$ on U_ℓ, which gives

$$\int_{U_\ell} \Big|\sum_{n\in L(\ell)} a_n f_n\Big| \lesssim_k 2^r |U_\ell|.$$

Combining Lemma 3.1, the inclusions $U_{\ell+1} \subset U_\ell$ and the inequality $|U_0| \lesssim_{k,\gamma} |\Gamma|$, we see that $\sum_{\ell=0}^{\infty} |U_\ell| \lesssim_{k,\gamma} |\Gamma|$. Thus we get

$$\sum_{\ell=0}^{\infty} \int_{U_\ell} \Big|\sum_{n\in L(\ell)} a_n f_n\Big| \lesssim_{k,\gamma} 2^r |\Gamma|.$$

The second sum on the right hand side of (5.7) is estimated similarly, which gives

$$\sum_{m=0}^{\infty} \int_{W_m} \Big|\sum_{n\in R(m)} a_n f_n\Big| \lesssim_{k,\gamma} 2^r |\Gamma|.$$

Combining these estimates with (5.7) and (5.5), we find

$$\Big|\int_{\Gamma} f(t)\,dt\Big| = \Big|\int_{\Gamma} \sum_n a_n f_n(t)\,dt\Big| \lesssim_{k,\gamma} 2^r |\Gamma|,$$

which implies (5.4) on Γ, and thus the proof is complete. ∎

6. Proof of the main theorem. For the proof of the necessity part of Theorem 2.4, we will use the following:

PROPOSITION 6.1. *Let (t_n) be a k-admissible sequence of knots that is k-regular with parameter γ, but not $(k-1)$-regular. Then*

$$\sup \Big\| \sup_n |a_n(\phi) f_n| \Big\|_1 = \infty,$$

where the first sup *is taken over all atoms ϕ, and $a_n(\phi) := \langle \phi, f_n\rangle$.*

Proposition 6.1 implies in particular that Proposition 5.3 cannot be extended to arbitrary partitions. For the proof of Proposition 6.1 we need the following technical lemma.

LEMMA 6.2. *Let (t_n) be a k-admissible sequence of knots that is k-regular with parameter $\gamma \geq 1$, but not $(k-1)$-regular. Let ℓ be an arbitrary positive integer. Then, for all $A \geq 2$, there exists a finite increasing sequence $(n_j)_{j=0}^{\ell-1}$ such that if τ_{n_j,i_j} is the new point in $\mathcal{T}_{n_j}$ not present in $\mathcal{T}_{n_j-1}$ and*

$$\Lambda_j := [\tau_{n_j,i_j-k}, \tau_{n_j,i_j-1}), \qquad L_j := [\tau_{n_j,i_j-1}, \tau_{n_j,i_j}), \qquad R_j := [\tau_{n_j,i_j}, \tau_{n_j,i_j+1}),$$

then for all i, j with $0 \leq i < j \leq \ell - 1$ we have:

(1) $R_i \cap R_j = \emptyset$,
(2) $\Lambda_i = \Lambda_j$,
(3) $(2\gamma - 1)|L_j| \geq |[\tau_{n_j,i_j-k-1}, \tau_{n_j,i_j-k}]| \geq |L_j|/(2\gamma)$,
(4) $|R_j| \leq (2\gamma - 1)|L_j|$,
(5) $|L_j| \leq 2(\gamma + 1)k|R_j|$,
(6) $\min(|L_j|, |R_j|) \geq A|\Lambda_j|$.

Proof. First, we choose a sequence $(n_j)_{j=0}^{lk}$ so that (1)–(4) hold. Next, we choose a subsequence $(n_{m_j})_{j=0}^{l-1}$ so that (5) and (6) hold as well.

Since (t_n) is not $(k-1)$-regular, for all C_0 there exist n_0 and i_0 such that

$$\text{(6.1)} \qquad \text{either } C_0|D^{(k-1)}_{n_0,i_0-k}| \leq |D^{(k-1)}_{n_0,i_0-k+1}| \text{ or } |D^{(k-1)}_{n_0,i_0-k}| \geq C_0|D^{(k-1)}_{n_0,i_0-k+1}|.$$

We choose C_0 sufficiently large such that with $C_j := C_{j-1}/\gamma - 1$ for $j \geq 1$ we have $C_{k\ell} \geq 2\gamma$. We will make an additional restriction on C_0 at the end of the proof. Without loss of generality, we can assume that the first inequality in (6.1) holds. Taking $\Lambda_0 = [\tau_{n_0,i_0-k}, \tau_{n_0,i_0-1})$ and $L_0 = [\tau_{n_0,i_0-1}, \tau_{n_0,i_0})$, $R_0 = [\tau_{n_0,i_0}, \tau_{n_0,i_0+1})$, we have

$$\text{(6.2)} \qquad |[\tau_{n_0,i_0-k+1}, \tau_{n_0,i_0}]| \geq C_0|\Lambda_0|.$$

A direct consequence of (6.2) is

$$\text{(6.3)} \qquad |L_0| \geq (C_0 - 1)|\Lambda_0|.$$

By k-regularity we have

$$|D^{(k)}_{n_0,i_0-k-1}| \geq \frac{|D^{(k)}_{n_0,i_0-k}|}{\gamma} = \frac{|\Lambda_0| + |L_0|}{\gamma},$$

which implies

$$
\begin{aligned}
(6.4)\qquad |[\tau_{n_0,i_0-k-1},\tau_{n_0,i_0-k}]| &= |D^{(k)}_{n_0,i_0-k-1}| - |\varLambda_0| \ge \frac{|\varLambda_0|+|L_0|}{\gamma} - |\varLambda_0| \\
&\ge \frac{|L_0|}{2\gamma} + \frac{|\varLambda_0|}{\gamma} + \frac{C_0-1}{2\gamma}|\varLambda_0| - |\varLambda_0| \\
&= \frac{|L_0|}{2\gamma} + \Big(\frac{C_0+1}{2\gamma} - 1\Big)|\varLambda_0| \ge \frac{|L_0|}{2\gamma},
\end{aligned}
$$

i.e., the right hand inequality of (3) for $j = 0$. To get the upper estimate, note that by k-regularity,

$$|\varLambda_0| + |[\tau_{n_0,i_0-k-1},\tau_{n_0,i_0-k}]| \le \gamma(|\varLambda_0| + |L_0|),$$

hence by (6.3),

$$(6.5)\qquad |[\tau_{n_0,i_0-k-1},\tau_{n_0,i_0-k}]| \le \gamma|L_0| + (\gamma-1)|\varLambda_0| \le (2\gamma-1)|L_0|.$$

This and the previous calculation give (3) for $j = 0$. Therefore, the construction can be continued either to the right or to the left of $\varLambda_0$.

We continue the construction to the right of $\varLambda_0$ by induction. Having defined n_j, $\varLambda_j$, L_j and R_j, we take

$$n_{j+1} := \min\{n > n_j : t_n \in \varLambda_j \cup L_j\}, \qquad j \ge 0.$$

By definition of R_j and n_{j+1}, property (1) is satisfied for all $j \ge 0$. We identify $t_{n_{j+1}} = \tau_{n_{j+1},i_{j+1}}$. Thus, by construction, $t_{n_j} = \tau_{n_j,i_j}$ is a common endpoint of L_j and R_j for $j \ge 1$.

In order to prove (2), we will show by induction that

$$(6.6)\qquad |[\tau_{n_j,i_j-k+1},\tau_{n_j,i_j}]| \ge C_j|\varLambda_j| \quad\text{and}\quad \varLambda_{j+1} = \varLambda_j$$

for all $j = 0,\ldots,k\ell$. We remark that the equality $\varLambda_{j+1} = \varLambda_j$ is equivalent to the condition $\tau_{n_{j+1},i_{j+1}} \in L_j$.

The inequality of (6.6) for $j = 0$ is exactly (6.2). If the identity in (6.6) were not satisfied for $j = 0$, i.e., $\tau_{n_1,i_1} \in \varLambda_0$, by k-regularity of (t_n), applied to the partition $\mathcal{T}_{n_1}$, we would have

$$|\varLambda_0| \ge \frac{1}{\gamma}|L_0|,$$

which contradicts (6.3) for our choice of C_0. This means $\varLambda_1 = \varLambda_0$, and so (6.6) is true for $j = 0$. Next, assume that (6.6) is satisfied for $j - 1$, where $1 \le j \le k\ell - 1$. By k-regularity, applied to $\mathcal{T}_{n_j}$, and employing (6.6) for $j-1$ repeatedly, we obtain

$$
\begin{aligned}
|\varLambda_j| + |L_j| = |\varLambda_j \cup L_j| &\ge \frac{1}{\gamma}(\tau_{n_j,i_j+1} - \tau_{n_j,i_j-k+1}) \\
&= \frac{1}{\gamma}(\tau_{n_{j-1},i_{j-1}} - \tau_{n_{j-1},i_{j-1}-k+1}) \\
&\ge \frac{C_{j-1}}{\gamma}|\varLambda_{j-1}| = \frac{C_{j-1}}{\gamma}|\varLambda_j|.
\end{aligned}
$$

This means, by the recursive definition of C_j, that

$$|L_j| \geq C_j|\Lambda_j|, \tag{6.7}$$

and in particular the first identity in (6.6) is true for j. If the identity in (6.6) were not satisfied for j, i.e., $\tau_{n_{j+1},i_{j+1}} \in \Lambda_j$, by k-regularity of (t_n), applied to $\mathcal{T}_{n_{j+1}}$, we would have

$$|\Lambda_j| \geq \frac{1}{\gamma}|L_j|,$$

which contradicts (6.7) and our choice of C_0. This proves (6.6) for j, and thus property (2) is true for all $j = 0, \ldots, k\ell$.

Moreover, choosing C_0 sufficiently large, namely such that $C_{kl} \geq 2(\gamma+1)kA$, (6.7) implies

$$|L_j| \geq 2(\gamma+1)kA|\Lambda_j|, \tag{6.8}$$

which is a part of (6).

The lower estimate in (3) is proved by repeating the argument giving (6.4) and using (6.7) instead of (6.4). The upper estimate uses the same arguments as the proof of (6.5), but now we have to use (6.7) as well.

Next, we look at (4). By k-regularity and (6.7), as $C_j > 1$, we have

$$|R_j| + |L_j| \leq \gamma(|L_j| + |\Lambda_j|) \leq 2\gamma|L_j|,$$

which is exactly (4).

We prove (5) by choosing a suitable subsequence of $(n_j)_{j=0}^{k\ell}$. First, assume that (5) fails for k consecutive indices, i.e., for some s,

$$|R_{s+r}| < \alpha|L_{s+r}| \leq \alpha|L_s|, \qquad r = 1, \ldots, k, \tag{6.9}$$

where $\alpha := (2(\gamma+1)k)^{-1}$. We have $L_j = L_{j+1} \cup R_{j+1}$ for $0 \leq j \leq k\ell - 1$. Thus, on the one hand,

$$|L_s \setminus L_{s+k}| = \sum_{r=1}^{k} |R_{s+r}| \leq \alpha k|L_s| \tag{6.10}$$

by (6.9); on the other hand, by k-regularity of $\mathcal{T}_{n_{s+k}}$,

$$|L_s \setminus L_{s+k}| \geq \frac{1}{\gamma}|L_{s+k}| = \frac{1}{\gamma}\Big(|L_s| - \sum_{r=1}^{k} |R_{s+r}|\Big) \geq \frac{1-\alpha k}{\gamma}|L_s|. \tag{6.11}$$

Now, (6.10) contradicts (6.11) for our choice of α. We have thus proved that there is at least one index $s+r$, $1 \leq r \leq k$, such that (5) is satisfied for $s+r$. Hence we can extract a sequence of length ℓ from $(n_j)_{j=1}^{k\ell}$ satisfying (5). For simplicity, this subsequence is called $(n_j)_{j=0}^{\ell-1}$ again.

Property (6) for R_j is now a simple consequence of (6.8), property (5) and the choice of $(n_j)_{j=0}^{\ell-1}$. Thus, the proof of the lemma is complete. ∎

Now, we are ready to proceed to the proof of Proposition 6.1.

Proof of Proposition 6.1. Let ℓ be an arbitrary positive integer and $A \geq 2$ a number to be chosen later. Lemma 6.2 gives a sequence $(n_j)_{j=0}^{\ell-1}$ such that conditions (1)–(6) in Lemma 6.2 are satisfied. We assume that $|\Lambda_0| > 0$. Let $\tau := \tau_{n_0,i_0-1}$, $x := \tau - 2|\Lambda_0|$ and $y := \tau + 2|\Lambda_0|$. Then we define an atom ϕ by

$$\phi \equiv \frac{1}{4|\Lambda_0|}(\mathbb{1}_{[x,\tau]} - \mathbb{1}_{[\tau,y]})$$

and let j be an arbitrary integer with $0 \leq j \leq \ell - 1$. By partial integration, the expression $a_{n_j}(\phi) = \langle \phi, f_{n_j} \rangle$ can be written as

$$\begin{aligned} 4|\Lambda_0| a_{n_j}(\phi) &= \int_x^\tau f_{n_j}(t)\,dt - \int_\tau^y f_{n_j}(t)\,dt \\ &= \int_x^\tau f_{n_j}(t) - f_{n_j}(\tau)\,dt - \int_\tau^y f_{n_j}(t) - f_{n_j}(\tau)\,dt \\ &= \int_x^\tau (x-t) f'_{n_j}(t)\,dt - \int_\tau^y (y-t) f'_{n_j}(t)\,dt. \end{aligned}$$

In order to estimate $|a_{n_j}(\phi)|$ from below, we estimate the absolute values of $I_1 := \int_x^\tau (x-t) f'_{n_j}(t)\,dt$ from below and of $I_2 := \int_\tau^y (y-t) f'_{n_j}(t)\,dt$ from above. We begin with I_2.

Consider the function g_{n_j}, connected with f_{n_j} via $f_{n_j} = g_{n_j}/\|g_{n_j}\|_2$ and $\|g_{n_j}\|_2 \sim_k |J_{n_j}|^{-1/2}$ (cf. (4.1) and Lemma 4.3). In the notation of Lemma 6.2, g_{n_j} is obtained by inserting the point $t_{n_j} = \tau_{n_j,i_j}$ in $\mathcal{T}_{n_j-1}$, and it is a common endpoint of intervals L_i and R_i. By construction of the characteristic interval J_{n_j}, properties (4)–(6) of Lemma 6.2, and the k-regularity of (t_n), we have

$$|J_{n_j}| \sim_{k,\gamma} |L_j| \sim_{k,\gamma} |R_j|. \tag{6.12}$$

By property (6), we have $[\tau, y] \subset L_j$, and therefore on $[\tau, y]$, the derivative of g_{n_j} has the representation (cf. (3.1))

$$g'_{n_j}(u) = (k-1) \sum_{i=i_j-k+1}^{i_j-1} \xi_i N_{n_j,i}^{(k-1)}(u), \qquad u \in [\tau, y],$$

where $\xi_i = (w_i - w_{i-1})/|D_{n_j,i}^{(k-1)}|$ and the coefficients w_i are given by (4.3) associated to the partition $\mathcal{T}_{n_j}$. For $i = i_j - k + 1, \ldots i_j - 1$ we have $L_j \subset D_{n_j,i}^{(k-1)}$, which together with the k-regularity of (t_n) and property (6) implies

$$|J_{n_j}| \sim_k |L_j| \sim_{k,\gamma} |D_{n_j,i}^{(k-1)}|, \qquad i = i_j - k + 1, \ldots, i_j - 1. \tag{6.13}$$

Moreover, by Lemma 4.4,

$$|w_i| \lesssim_k \frac{1}{|J_{n_j}|}, \qquad 1 \le i \le n_j + k - 1.$$

Therefore

$$|f'_{n_j}(t)| \sim_k |J_{n_j}|^{1/2} |g'_{n_j}(t)| \lesssim_{k,\gamma} |L_j|^{-3/2} \qquad \text{for } t \in [\tau, y].$$

Consequently, putting the above facts together,

$$|I_2| \lesssim_{k,\gamma} |\Lambda_0|^2 \cdot |L_j|^{-3/2}. \tag{6.14}$$

We now estimate I_1. By properties (3) and (6) of Lemma 6.2 (with $A \ge 2\gamma$), we have $[x, \tau] \subset [\tau_{n_j, i_j-k-1}, \tau_{n_j, i_j-1}]$, and therefore on $[x,\tau]$, g'_{n_j} has the representation (cf. (3.1))

$$g'_{n_j}(u) = (k-1) \sum_{i=i_j-2k+1}^{i_j-2} \xi_i N^{(k-1)}_{n_j,i}(u), \qquad u \in [x, \tau].$$

We split $I_1 = I_{1,1} + I_{1,2}$ according to whether $i \ne i_j - k$ or $i = i_j - k$ in the above representation of g'_{n_j} on $[x,\tau]$.

Note that $[\tau_{n_j,i_j-k-1}, \tau_{n_j,i_j-k}] \subset D^{(k-1)}_{n_j,i}$ for $i_j - 2k + 1 \le i < i_j - k$ and $L_j \subset D^{(k-1)}_{n_j,i}$ for $i_j - k < i \le i_j - 2$. Therefore, by properties (3) and (6) of Lemma 6.2 and the k-regularity of the sequence of knots we have

$$|D^{(k-1)}_{n_j,i}| \sim_{k,\gamma} |L_j| \quad \text{for } i_j - 2k + 1 \le i \le i_j - 2,\ i \ne i_j - k.$$

So, by arguments analogous to the proof of (6.14) we get

$$|I_{1,1}| \sim_k |J_{n_j}|^{1/2} \Big| \int_x^\tau (t-x) \sum_{\substack{i=i_j-2k+1 \\ i \ne i_j - k}}^{i_j-2} \xi_i N^{(k-1)}_{n_j,i}(t)\, dt \Big| \lesssim_{k,\gamma} |\Lambda_0|^2 \cdot |L_j|^{-3/2}. \tag{6.15}$$

Moreover, for $i = i_j - k$, we have $D^{(k-1)}_{n_j,i_j-k} = \Lambda_0$, so

$$\begin{aligned} |I_{1,2}| &\sim_k |J_{n_j}|^{1/2} \Big| \int_x^\tau (t-x) \xi_{i_j-k} N^{(k-1)}_{n_j,i_j-k}(t)\, dt \Big| \\ &\ge |\xi_{i_j-k}|\, |J_{n_j}|^{1/2} |\Lambda_0| \int_x^\tau N^{(k-1)}_{n_j,i_j-k}(t)\, dt \\ &= |\xi_{i_j-k}|\, |\Lambda_0|\, |J_{n_j}|^{1/2} \frac{|D^{(k-1)}_{n_j,i_j-k}|}{k-1} = |\xi_{i_j-k}|\, |J_{n_j}|^{1/2} \frac{|\Lambda_0|^2}{k-1}, \end{aligned} \tag{6.16}$$

because $t - x \ge |\Lambda_0|$ for $t \in \operatorname{supp} N^{(k-1)}_{n_j,i_j-k}$. Since the sequence w_j is checkerboard (cf. (4.4)),

$$|\xi_{i_j-k}| = \frac{|w_{i_j-k}| + |w_{i_j-k-1}|}{|D^{(k-1)}_{n_j,i_j-k}|} \geq \frac{|w_{i_j-k}|}{|D^{(k-1)}_{n_j,i_j-k}|}.$$

By definition of w_{i_j-k},

$$|w_{i_j-k}| \geq |\alpha_{i_j-k}|\,|b_{i_j-k,i_j-k}|,$$

where α_{i_j-k} is the factor from formula (4.2) and b_{i_j-k,i_j-k} is an entry of the inverse of the B-spline Gram matrix, both corresponding to the partition $\mathcal{T}_{n_j}$. Formulas (4.2) and (6.12) imply that α_{i_j-k} is bounded from below by a positive constant that only depends on k and γ ([1]). Moreover, $|b_{i_j-k,i_j-k}| \geq \|N^{(k)}_{n_j,i_j-k}\|_2^{-2} \gtrsim_k |D^{(k)}_{n_j,i_j-k}|^{-1}$ (cf. (3.6)). Note that $D^{(k)}_{n_j,i_j-k} = \Lambda_0 \cup L_j$, so $|D^{(k)}_{n_j,i_j-k}| \sim_{k,\gamma} |L_j|$. Thus, $|\xi_{i_j-k}| \gtrsim_{k,\gamma} |\Lambda_0|^{-1}|L_j|^{-1}$. Inserting the above calculations in (6.16), we find

$$|I_{1,2}| \gtrsim_{k,\gamma} |J_{n_j}|^{1/2}\frac{|\Lambda_0|}{|L_j|} \sim_{k,\gamma} |\Lambda_0|\,|L_j|^{-1/2}. \tag{6.17}$$

We now impose conditions on the constant $A \geq 2\gamma$ from the beginning of the proof and property (6) in Lemma 6.2. It follows from (6.17), (6.15) and (6.14) that there are $C_{k,\gamma} > 0$ and $c_{k,\gamma} > 0$, depending only on k and γ, such that

$$\begin{aligned} 4|\Lambda_0|\,|a_{n_j}(\phi)| &\geq |I_{1,2}| - |I_{1,1}| - |I_2| \geq C_{k,\gamma}|\Lambda_0|\,|L_j|^{-1/2} - c_{k,\gamma}|\Lambda_0|^2|L_j|^{-3/2} \\ &= |\Lambda_0|\,|L_j|^{-1/2}(C_{k,\gamma} - c_{k,\gamma}|\Lambda_0|\,|L_j|^{-1}). \end{aligned}$$

By property (6) in Lemma 6.2 we have $|\Lambda_0|\,|L_j|^{-1} \leq 1/A$. Choosing A sufficiently large to guarantee

$$C_{k,\gamma} - \frac{c_{k,\gamma}}{A} \geq \frac{C_{k,\gamma}}{2},$$

we get a constant $m_{k,\gamma}$, depending only on k and γ, such that

$$m_{k,\gamma}|L_j|^{-1/2} \leq |a_{n_j}(\phi)|, \qquad j = 0,\ldots,\ell-1. \tag{6.18}$$

Next, we estimate $\int_{R_j} |g_{n_j}(t)|\,dt$ from below. First, Proposition 3.3, property (6) of Lemma 6.2 and the k-regularity of (t_n) yield

$$\int_{R_j} |g_{n_j}(t)|\,dt \gtrsim_{k,\gamma} |R_j|\,|w_{i_j}|,$$

([1]) Formula (4.2) is applied with $\mathcal{T}_n = \mathcal{T}_{n_j}$ and corresponding to $\tau_{i_0} = \tau_{n_j,i_j}$. Then $[\tau_{i_0-1}, \tau_{i_0}] = L_j$ and $[\tau_{i_0}, \tau_{i_0+1}] = R_j$. By k-regularity and $|\Lambda_0 \cup L_j| \sim_{k,\gamma} |L_j|$, each denominator in (4.2) is $\sim_{k,\gamma} |L_j|$. Each numerator in (4.2) is greater than either L_j or R_j, so by (6.12) and k-regularity it is $\sim_{k,\gamma} |L_j|$ as well.

where w_{i_j} corresponds to the partition $\mathcal{T}_{n_j}$. By definition of w_{i_j},

$$\int_{R_j} |g_{n_j}(t)|\, dt \gtrsim_{k,\gamma} |R_j|\, |\alpha_{i_j}|\, |b_{i_j,i_j}|.$$

By arguments similar to those above, $|\alpha_{i_j}|$ is bounded from below by a constant only depending on k and γ, and $|b_{i_j,i_j}| \gtrsim_k |D^{(k)}_{n_j,i_j}|^{-1}$. Since by k-regularity, $|R_j| \sim_{k,\gamma} |D^{(k)}_{n_j,i_j}|$, we finally get

$$\int_{R_j} |g_{n_j}(t)|\, dt \gtrsim_{k,\gamma} 1,$$

which means for f_{n_j} that

$$\int_{R_j} |f_{n_j}(t)|\, dt \gtrsim_{k,\gamma} |J_{n_j}|^{1/2} \gtrsim_{k,\gamma} |L_j|^{1/2}.$$

Combining this last estimate with (6.18) and (1) of Lemma 6.2 gives

$$\int_0^1 \sup_n |a_n(\phi) f_n(t)|\, dt \ge \sum_{j=1}^{\ell} \int_{R_j} |a_{n_j}(\phi) f_{n_j}(t)|\, dt \gtrsim_{k,\gamma} \ell.$$

This construction applies to every positive integer ℓ, proving the assertion of the proposition for $|\Lambda_0| > 0$.

The case $|\Lambda_0| = 0$ is handled similarly, with the difference that the atom ϕ is defined to be centered at τ_{n_0,i_0-1} and the length of the support is sufficiently small, depending on ℓ and $|L_0|$. ∎

With Proposition 6.1 and the results of Section 5 at hand, the proof of Theorem 2.4 follows the proof of Theorem 2.2 in [15], but we present it here for the sake of completeness.

Proof of Theorem 2.4. We start by proving the unconditional basis property of $(f_n) = (f_n^{(k)})$ assuming the $(k-1)$-regularity of (t_n). If (t_n) is $(k-1)$-regular, it is not difficult to check that it is also k-regular. As a consequence, Theorem 2.3 implies that (f_n) is a basis in H^1. Let $f \in H^1$ with $f = \sum a_n f_n$ and $\varepsilon \in \{-1,1\}^{\mathbb{Z}}$. We need to prove the convergence of $\sum \varepsilon_n a_n f_n$ in H^1. Let $m_1 \le m_2$. Then

$$\begin{aligned}\Big\| \sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n \Big\|_{H^1} &\lesssim_{k,\gamma} \Big\| S\Big(\sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n \Big) \Big\|_1 \lesssim_k \Big\| P\Big(\sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n \Big) \Big\|_1 \\ &= \Big\| P\Big(\sum_{n=m_1}^{m_2} a_n f_n \Big) \Big\|_1 \lesssim_{k,\gamma} \Big\| \sum_{n=m_1}^{m_2} a_n f_n \Big\|_{H^1}\end{aligned}$$

where we have used Propositions 5.4, 5.2 and 5.3 (cf. also the diagram on page 135). So, since $\sum a_n f_n$ converges in H^1, so does $f_\varepsilon := \sum \varepsilon_n a_n f_n$, and the same calculation as above shows

$$\|f_\varepsilon\|_{H^1} \lesssim_{k,\gamma} \|f\|_{H^1}.$$

This implies that (f_n) is an unconditional basis in H^1.

We now prove the converse: (f_n) being an unconditional basis in H^1 implies $(k-1)$-regularity. First, if (t_n) is not k-regular, (f_n) is not a basis in H^1 by Theorem 2.3. Thus, it remains to consider the case when (t_n) is k-regular, but not $(k-1)$-regular. By Theorem 2.3 again, (f_n) is then a basis in H^1. Suppose that (f_n) is an unconditional basis in H^1. Then, for $f = \sum a_n f_n$ and $\varepsilon \in \{-1,1\}^{\mathbb{Z}}$, the function $f_\varepsilon := \sum \varepsilon_n a_n f_n$ is also in H^1. Since $\|\cdot\|_1 \le \|\cdot\|_{H^1}$, the series $\sum a_n f_n$ also converges unconditionally in L^1, and thus Proposition 5.1 (i.e., Khinchin's inequality) implies

$$\|Pf\|_1 \lesssim \sup_\varepsilon \|f_\varepsilon\|_1 \le \sup_\varepsilon \|f_\varepsilon\|_{H^1} \lesssim \|f\|_{H^1},$$

which is impossible due to Proposition 6.1, even for atoms. This concludes the proof of Theorem 2.4. ∎

As an immediate consequence of Theorem 2.4, a fifth condition equivalent to (A)–(D) is the unconditional convergence of $\sum_n a_n f_n$ in H^1:

Corollary 6.3. *Let (t_n) be a k-admissible and $(k-1)$-regular sequence of points, with (f_n) the corresponding orthonormal spline system of order k. Let (a_n) be a sequence of coefficients. Then conditions* (A)–(D) *from Section* 5 *are equivalent. Moreover, they are equivalent to*

(E) *The series $\sum_n a_n f_n$ converges unconditionally in H^1.*

In addition, for $f \in H^1$, $f = \sum_n a_n f_n$, we have

$$\|f\|_{H^1} \sim \|Sf\|_1 \sim \|Pf\|_1 \sim \sup_{\varepsilon \in \{-1,1\}^{\mathbb{Z}}} \Big\| \sum_n \varepsilon_n a_n f_n \Big\|_1,$$

with the implied constants depending only on k and the parameter of $(k-1)$-regularity of (t_n).

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CHAPTER 5

Orthogonal projectors onto spaces of periodic splines

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Orthogonal projectors onto spaces of periodic splines[☆]

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ABSTRACT

The main result of this paper is a proof that for any integrable function f on the torus, any sequence of its orthogonal projections $(\widetilde{P}_n f)$ onto periodic spline spaces with arbitrary knots $\widetilde{\Delta}_n$ and arbitrary polynomial degree converges to f almost everywhere with respect to the Lebesgue measure, provided the mesh diameter $|\widetilde{\Delta}_n|$ tends to zero. We also give a new and simpler proof of the fact that the operators $\widetilde{P}_n$ are bounded on L^∞ independently of the knots $\widetilde{\Delta}_n$.

1. Introduction

1.1. Splines on an interval

In this article we prove some results about the periodic spline orthoprojector. In order to achieve this, we rely on existing results for the non-periodic spline orthoprojector on a compact interval, so we first describe some of those results for the latter operator. Let $k \in \mathbb{N}$ and $\Delta = (t_i)_{i=\ell}^{r+k}$ a knot sequence satisfying

$$\begin{aligned} &t_i \le t_{i+1}, \qquad t_i < t_{i+k}, \\ &t_\ell = \cdots = t_{\ell+k-1}, \qquad t_{r+1} = \cdots = t_{r+k}. \end{aligned}$$

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Associated to this knot sequence, we define $(N_i)_{i=\ell}^r$ as the sequence of L^∞-normalized B-spline functions of order k on Δ that have the properties

$$\operatorname{supp} N_i = [t_i, t_{i+k}],\ N_i \geq 0, \quad \sum_{i=\ell}^r N_i \equiv 1.$$

We write $|\Delta| = \max_{\ell \leq j \leq r}(t_{j+1} - t_j)$ for the maximal mesh width of the partition Δ. Then, define the space $\mathcal{S}_k(\Delta)$ as the set of polynomial splines of order k (or at most degree $k-1$) with knots Δ, which is the linear span of the B-spline functions $(N_i)_{i=\ell}^r$. Moreover, let P_Δ be the orthogonal projection operator onto the space $\mathcal{S}_k(\Delta)$ with respect to the ordinary (real) inner product $\langle f, g\rangle = \int_{t_\ell}^{t_{r+1}} f(x)g(x)\,\mathrm{d}x$, i.e.,

$$\langle P_\Delta f, s\rangle = \langle f, s\rangle \quad \text{for all } s \in \mathcal{S}_k(\Delta).$$

The operator P_Δ is also given by the formula

$$P_\Delta f = \sum_{i=\ell}^r \langle f, N_i\rangle N_i^*, \tag{1.1}$$

where $(N_i^*)_{i=\ell}^r$ denotes the dual basis to (N_i) defined by the relations $\langle N_i^*, N_j\rangle = 0$ when $j \neq i$ and $\langle N_i^*, N_i\rangle = 1$ for all $i = \ell, \ldots, r$. A famous theorem by A. Shadrin states that the L^∞-norm of this projection operator is bounded independently of the knot sequence Δ:

Theorem 1.1 ([8]). *There exists a constant c_k depending only on the spline order k such that for all knot sequences $\Delta = (t_i)_{i=\ell}^{r+k}$ as above,*

$$\|P_\Delta : L^\infty[t_\ell, t_{r+1}] \to L^\infty[t_\ell, t_{r+1}]\| \leq c_k.$$

We are also interested in the following equivalent formulation of this theorem, which is proved in [1]: for a knot sequence Δ, let (a_{ij}) be the matrix $(\langle N_i^*, N_j^*\rangle)$, which is the inverse of the banded matrix $(\langle N_i, N_j\rangle)$. Then, the assertion of Theorem 1.1 is equivalent to the existence of two constants $K_0 > 0$ and $\gamma_0 \in (0, 1)$ depending only on the spline order k such that

$$|a_{ij}| \leq \frac{K_0 \gamma_0^{|i-j|}}{\max\{\kappa_i, \kappa_j\}}, \quad \ell \leq i, j \leq r, \tag{1.2}$$

where κ_i denotes the length of $\operatorname{supp} N_i$. The proof of this equivalence uses Demko's theorem [4] on the geometric decay of inverses of band matrices and de Boor's stability (see [2] or [5, Chapter 5, Theorem 4.2]) which states that for $0 < p \leq \infty$, the L^p-norm of a B-spline series is equivalent to a weighted ℓ^p-norm of its coefficients, i.e. there exists a constant D_k depending only on the spline order k such that:

$$D_k k^{-1/p}\Big(\sum_j |c_j|^p \kappa_j\Big)^{1/p} \leq \Big\|\sum_j c_j N_j\Big\|_{L^p} \leq \Big(\sum_j |c_j|^p \kappa_j\Big)^{1/p}.$$

In fact, for a_{ij}, we actually have the following improvement of (1.2) (see [6]): There exist two constants $K > 0$ and $\gamma \in (0, 1)$ that depend only on the spline order k such that

$$|a_{ij}| \leq \frac{K\gamma^{|i-j|}}{h_{ij}}, \quad \ell \leq i, j \leq r, \tag{1.3}$$

where h_{ij} denotes the length of the convex hull of $\operatorname{supp} N_i \cup \operatorname{supp} N_j$. This inequality can be used to obtain almost everywhere convergence for spline projections of L^1-functions:

Theorem 1.2 ([6]). *For all $f \in L^1[t_\ell, t_{r+1}]$ there exists a subset $A \subset [t_\ell, t_{r+1}]$ of full Lebesgue measure such that for all sequences (Δ_n) of partitions of $[t_\ell, t_{r+1}]$ such that $|\Delta_n| \to 0$, we have*

$$\lim_{n\to\infty} P_{\Delta_n} f(x) = f(x), \quad x \in A.$$

Our aim in this article is to prove an analogue of Theorem 1.2 for orthoprojectors on periodic spline spaces. In this case, we do not have a version of (1.3) at our disposal, since the proof of this inequality does not carry over to the periodic setting. However, by comparing orthogonal projections onto periodic spline spaces to suitable non-periodic projections, we are able to obtain a periodic version of Theorem 1.2.

In the course of the proof of the periodic version of Theorem 1.2, we also need a periodic version of Theorem 1.1, which can be proved by first establishing the same assertion for infinite point sequences and then by viewing periodic functions as defined on the whole real line [9]. The proof of Theorem 1.1 for infinite point sequences is announced in [8] and carried out [3]. In this article we give a different proof of the periodic version of Shadrin's theorem by employing a similar comparison of periodic and non-periodic projection operators as in the proof of the periodic version of Theorem 1.2. This proof directly passes from the interval case to the periodic result without recourse to infinite point sequences.

1.2. Periodic splines

Let $n \geq k$ be a natural number and $\widetilde{\Delta} = (s_j)_{j=0}^{n-1}$ be a sequence of distinct points on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ identified canonically with $[0, 1)$, such that for all j we have

$$s_j \leq s_{j+1}, \qquad s_j < s_{j+k},$$

and we extend $(s_j)_{j=0}^{n-1}$ periodically by

$$s_{rn+j} = r + s_j$$

for $r \in \mathbb{Z} \setminus \{0\}$ and $0 \leq j \leq n-1$.

Now, the main result of this article reads as follows:

Theorem 1.3. *For all functions $f \in L^1(\mathbb{T})$ there exists a set $\widetilde{A}$ of full Lebesgue measure such that for all sequences of partitions $(\widetilde{\Delta}_n)$ on $\mathbb{T}$ as above with $|\widetilde{\Delta}_n| \to 0$, we have*

$$\lim_{n\to\infty} \widetilde{P}_n f(x) = f(x), \quad x \in \widetilde{A},$$

where $\widetilde{P}_n$ denotes the orthogonal projection operator onto the periodic spline space of order k with knots $\widetilde{\Delta}_n$.

In order to prove this result, we also need a periodic version of Theorem 1.1:

Theorem 1.4. *There exists a constant c_k depending only on the spline order k such that for all knot sequences $\widetilde{\Delta} = (s_j)_{j=0}^{n-1}$ on $\mathbb{T}$, the associated orthogonal projection operator $\widetilde{P}$ satisfies the inequality*

$$\|\widetilde{P} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\| \leq c_k.$$

The idea of the proofs of Theorems 1.3 and 1.4 is to estimate the difference between the periodic projection operator $\widetilde{P}$ and the non-periodic projection operator P for certain non-periodic point sequences associated to $\widetilde{\Delta} = (s_i)_{i=0}^{n-1}$.

The article is organized as follows. In Section 2, we prove a simple lemma on the growth behaviour of linear combinations of non-periodic B-spline functions which is needed frequently later in the proofs of both Theorems 1.3 and 1.4. Section 3 is devoted to the proof of Theorem 1.4, which is needed for the proof of Theorem 1.3 in Section 4. Finally, in Section 5, we also apply our method of proof to recover Shadrin's theorem for infinite point sequences (see [3,8]).

2. A simple upper estimate for B-spline sums

Let A be a subset of $[t_\ell, t_{r+1}]$. Then, define the set of indices $\mathfrak{i}(A)$ whose B-splines are not identically zero on A as

$$\mathfrak{i}(A) := \{i : A \cap \operatorname{int}(\operatorname{supp} N_i) \neq \emptyset\},$$

where int U denotes the interior of the set U. We also write $\mathrm{i}(x)$ for $\mathrm{i}(\{x\})$. If we have two subsets U, V of indices, we write $d(U, V)$ for the distance between U and V induced by the metric $d(i, j) = |i - j|$.

We will use the notation $A(t) \lesssim B(t)$ to indicate the existence of a constant C that depends only on the spline order k such that for all t we have $A(t) \le CB(t)$, where t denotes all explicit or implicit dependencies that the expressions A and B might have.

The fact that B-spline functions are localized, so a fortiori the set $\mathrm{i}(x)$ is localized for any $x \in [t_\ell, t_{r+1}]$, can be used to derive the following lemma:

Lemma 2.1. *Let J be a subset of the index set $\{\ell, \ell+1, \dots, r-1, r\}$, $f = \sum_{j\in J} \langle h, N_j\rangle N_j^*$ and $p \in [1, \infty]$. Then, for all $x \in [t_\ell, t_{r+1}]$, we have the estimates*

$$|f(x)| \lesssim \gamma^{d(\mathrm{i}(x),J)} \|h\|_p \max_{m\in \mathrm{i}(x), j\in J} \frac{\kappa_j^{1/p'}}{h_{jm}} \tag{2.1}$$

$$\le \gamma^{d(\mathrm{i}(x),J)} \|h\|_p \max_{m\in \mathrm{i}(x), j\in J} \big(\max\{\kappa_m, \kappa_j\}\big)^{-1/p} \tag{2.2}$$

$$\le \gamma^{d(\mathrm{i}(x),J)} \|h\|_p \cdot |I(x)|^{-1/p}, \quad 1 \le p \le \infty, \tag{2.3}$$

where $\gamma \in (0, 1)$ is the constant appearing in (1.3), $I(x)$ is the interval $I = [t_i, t_{i+1})$ containing the point x and the exponent p' is such that $1/p + 1/p' = 1$.

Proof. Since $N_j^* = \sum_m a_{jm} N_m$,

$$f(x) = \sum_{j\in J} \sum_{m\in \mathrm{i}(x)} a_{jm} \langle h, N_j\rangle N_m(x).$$

This implies

$$|f(x)| \lesssim \max_{m\in \mathrm{i}(x)} \Big(\sum_{j\in J} \frac{\gamma^{|j-m|}}{h_{jm}} \|h\|_p \|N_j\|_{p'}\Big),$$

where we used inequality (1.3) for a_{jm}, Hölder's inequality with the conjugate exponent $p' = p/(p-1)$ to p and the fact that the B-spline functions N_m form a partition of unity. Using again the uniform boundedness of N_j, we obtain

$$|f(x)| \lesssim \|h\|_p \max_{m\in \mathrm{i}(x)} \Big(\sum_{j\in J} \frac{\gamma^{|j-m|}}{h_{jm}} \kappa_j^{1/p'}\Big).$$

Estimating this last sum by $\sum_{j\in J} \gamma^{|j-m|} \cdot \max_{j\in J} \kappa_j^{1/p'}/h_{jm}$ and summing the resulting geometric series now yields (2.1). In order to get (2.2) from (2.1), observe that $\max\{\kappa_j, \kappa_m\} \le h_{jm}$ and $1/p' + 1/p = 1$. For the deduction of (2.3) from (2.2), we note that for all $m \in \mathrm{i}(x)$, by definition, $|I(x)| \le \kappa_m$, which directly implies (2.3). □

Remark 2.2. We note that we directly obtain the second estimate in the above lemma if we use the weaker inequality (1.2) instead of (1.3). We also observe that the form of f in the above lemma means that $\langle f, N_j\rangle = 0$ for $j \notin J$.

3. The periodic spline orthoprojector is uniformly bounded on L^∞

In this section, we give a direct proof of Theorem 1.4 on the boundedness of periodic spline projectors without recourse to infinite knot sequences. Here, we will only use the geometric decay of the matrix (a_{jm}) defined above for splines on an interval.

A vital tool in the proofs of both Theorems 1.1 and 1.2 are B-spline functions. We will also use them extensively and introduce their periodic version, cf. [7, Chapter 8.1, pp. 297–308]. Associated to the periodic point sequence $(s_j)_{j=0}^{n-1}$ and its periodic extension as in Section 1.2 we define the non-periodic point sequence

$$t_j = s_j, \quad \text{for } j = -k+1, \dots, n+k-1$$

and denote the corresponding non-periodic B-spline functions by $(N_j)_{j=-k+1}^{n-1}$ with $\operatorname{supp} N_j = [t_j, t_{j+k}]$. Then we define for $x \in [0, 1)$

$$\widetilde{N}_j(x) = N_j(x), \quad j = 0, \dots, n-k,$$

if we canonically identify $\mathbb{T}$ with $[0, 1)$. Moreover, for $j = n-k+1, \dots, n-1$,

$$\widetilde{N}_j(x) = \begin{cases} N_{j-n}(x), & \text{if } x \in [0, s_j], \\ N_j(x), & \text{if } x \in (s_j, 1). \end{cases}$$

We denote by $\widetilde{P}$ the orthogonal projection operator onto the space of periodic splines of order k with knots $(s_j)_{j=0}^{n-1}$, which is the linear span of the B-spline functions $(\widetilde{N}_j)_{j=0}^{n-1}$ and similarly to the non-periodic case we define

$$\imath(A) = \{0 \le j \le n-1 : A \cap \operatorname{int}(\operatorname{supp} \widetilde{N}_j) \neq \emptyset\}, \quad A \subset \mathbb{T}.$$

Lemma 3.1. *Let f_i be a function on $\mathbb{T}$ with $\operatorname{supp} f_i \subset [s_i, s_{i+1}]$ for some index i in the range $0 \le i \le n-1$. Then, for any $x \in \mathbb{T}$,*

$$|\widetilde{P} f_i(x)| \lesssim \gamma^{\widetilde{d}(\imath(x), \imath(\operatorname{supp} f_i))} \|f_i\|_\infty,$$

where $\widetilde{d}$ is the distance function induced by the canonical metric in $\mathbb{Z}/n\mathbb{Z}$ and $\gamma \in (0, 1)$ is the constant appearing in inequality (1.3).

Proof. We assume that the index i is chosen such that $s_i < s_{i+1}$, since if $s_i = s_{i+1}$, the function f_i is identically zero in L^∞. Also, without loss of generality, we can assume that $i = 0$, since otherwise we could just shift the point sequence.

Given a function f on $\mathbb{T}$, we associate a non-periodic function Tf defined on $[s_0, s_{n+1}]$ given by

$$Tf(t) = f(\pi(t)), \quad t \in [s_0, s_{n+1}],$$

where $\pi(t)$ is the quotient mapping from $\mathbb{R}$ to $\mathbb{T}$. We observe that T is a linear operator, $\|T : L^2(\mathbb{T}) \to L^2([s_0, s_{n+1}])\| = \sqrt{2}$ and $\|T : L^\infty(\mathbb{T}) \to L^\infty([s_0, s_{n+1}])\| = 1$. Moreover, for $x \in \mathbb{T}$, let $r(x)$ be the representative of x in the interval $[s_0, s_n)$. We want to estimate $\widetilde{P} f_0(x)$. In order to do this, we first decompose

$$\widetilde{P} f_0(x) = T\widetilde{P} f_0(r(x)) = PTf_0(r(x)) + (T\widetilde{P} f_0 - PTf_0)(r(x)), \tag{3.1}$$

where P is the orthogonal projection operator onto the space of splines of order k corresponding to the point sequence $\Delta = (t_j)_{j=-k+1}^{n+k}$ associated to the non-periodic grid points in the interval $[s_0, s_{n+1}]$, i.e.,

$$t_j = s_j, \quad j = 0, \dots, n+1,$$
$$t_{-k+1} = \dots = t_{-1} = s_0, \qquad t_{n+2} = \dots = t_{n+k} = s_{n+1}.$$

Also, let $(N_j)_{j=-k+1}^{n}$ be the L^∞-normalized B-spline basis corresponding to this point sequence.

We estimate the first term $PTf_0(r(x))$ from the decomposition in (3.1) of $\widetilde{P} f_0(x)$. Since P is a projection operator onto splines on an interval, we use representation (1.1) to get

$$PTf_0(r(x)) = \sum_{j=-k+1}^{n} \langle Tf_0, N_j \rangle N_j^*(r(x)),$$

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and, since $\operatorname{supp} Tf_0 \subset [s_0, s_1] \cup [s_n, s_{n+1}] = [t_0, t_1] \cup [t_n, t_{n+1}]$ by definition of f_0 and T and $\operatorname{supp} N_j \subset [t_j, t_{j+k}]$ for all $j = -k+1, \ldots, n$,

$$PTf_0(r(x)) = \sum_{j \in J_1} \langle Tf_0, N_j \rangle N_j^*(r(x)),$$

with $J_1 = \{-k+1, \ldots, 0\} \cup \{n-k+1, \ldots, n\}$. Employing now inequality (2.3) of Lemma 2.1 with $p = \infty$ to this sum, we obtain

$$|PTf_i(r(x))| \lesssim \gamma^{d(\mathrm{i}(r(x)), J_1)} \|Tf_i\|_\infty \lesssim \gamma^{\widetilde{d}(\mathrm{i}(x), \mathrm{i}(\operatorname{supp} f_i))} \|f_i\|_\infty. \tag{3.2}$$

Now we turn to the second term on the right hand side of (3.1). Let $g := (T\widetilde{P} - PT)f_0 = (T\widetilde{P} - T)f_0 + (T - PT)f_0$. Observe that $g \in \mathcal{S}_k(\Delta)$ since the range of both $T\widetilde{P}$ and P is contained in $\mathcal{S}_k(\Delta)$. Moreover,

$$\langle (T\widetilde{P} - T)f_0, N_j \rangle = \langle \widetilde{P}f_0 - f_0, \widetilde{N}_j \rangle, \quad j = 0, \ldots, n-k+1.$$

This equation is true in the given range of the parameter j, since in this case, the functions N_j and $\widetilde{N}_j$ coincide. The fact that $\widetilde{P}$ is an orthogonal projection onto the span of the functions $(\widetilde{N}_j)_{j=0}^{n-1}$ then implies

$$\langle T\widetilde{P}f_0 - Tf_0, N_j \rangle = \langle \widetilde{P}f_0 - f_0, \widetilde{N}_j \rangle = 0, \quad j = 0, \ldots, n-k+1.$$

Combining this with the fact

$$\langle Tf_0 - PTf_0, N_j \rangle = 0, \quad j = -k+1, \ldots, n,$$

since P is an orthogonal projection onto a spline space as well, we obtain that

$$\langle g, N_j \rangle = 0, \quad j = 0, \ldots, n-k+1.$$

Therefore, we can expand g as a B-spline sum

$$g = \sum_{j \in J_2} \langle g, N_j \rangle N_j^*,$$

with $J_2 = \{-k+1, \ldots, -1\} \cup \{n-k+2, \ldots, n\}$. Now, we employ inequality (2.2) of Lemma 2.1 on the function g with the parameter $p = 2$ to get for the point $y = r(x)$

$$|g(y)| \lesssim \gamma^{d(\mathrm{i}(y), J_2)} \|g\|_2 \max_{j \in J_2} |\operatorname{supp} N_j|^{-1/2}.$$

Since $g = (T\widetilde{P} - PT)f_0$ and the operator $T\widetilde{P} - PT$ has norm $\leq 2\sqrt{2}$ on L^2, we get

$$|g(y)| \lesssim \gamma^{d(\mathrm{i}(y), J_2)} \|f_0\|_2 |\operatorname{supp} f_0|^{-1/2},$$

where we also used the fact that $\operatorname{supp} N_j \supset [s_0, s_1] = [t_0, t_1]$ or $\operatorname{supp} N_j \supset [s_n, s_{n+1}] = [t_n, t_{n+1}]$ for $j \in J_2$. Since $d(\mathrm{i}(y), J_2) \geq \widetilde{d}(\mathrm{i}(x), \mathrm{i}(\operatorname{supp} f_0))$ and $\|f_0\|_2 \leq \|f_0\|_\infty |\operatorname{supp} f_0|^{1/2}$, we finally get

$$|g(y)| \lesssim \gamma^{\widetilde{d}(\mathrm{i}(x), \mathrm{i}(\operatorname{supp} f_0))} \|f_0\|_\infty.$$

Looking at (3.1) and combining the latter estimate with (3.2), the proof is completed. □

This lemma can be used directly to prove Theorem 1.4 on the uniform boundedness of periodic orthogonal spline projection operators on L^∞:

Proof of Theorem 1.4. We just decompose the function f as $f = \sum_{i=0}^{n-1} f \cdot \mathbb{1}_{[s_i, s_{i+1})}$ and apply Lemma 3.1 to each summand and the assertion $\|\widetilde{P}f\|_\infty \lesssim \|f\|_\infty$ follows after summation of a geometric series. □

Remark 3.2. (i) Since $\widetilde{P}$ is a selfadjoint operator, Theorem 1.4 also implies that $\widetilde{P}$ is bounded as an operator from $L^1(\mathbb{T})$ to $L^1(\mathbb{T})$ by the same constant c_k as in the above theorem. Moreover, by interpolation, $\widetilde{P}$ is also bounded by c_k as an operator from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ for any $p \in [1, \infty]$.

(ii) In the proof of Lemma 3.1, we only use inequality (2.2) of Lemma 2.1 which follows from inequality (1.2) on the inverse of the B-spline Gram matrix and does not need its stronger form (1.3). Similarly to the equivalence of Shadrin's theorem and (1.2) in the non-periodic case, we can derive the equivalence of Theorem 1.4 and the estimate

$$|\widetilde{a}_{ij}| \leq \frac{K\gamma^{\widetilde{d}(i,j)}}{\max(\widetilde{\kappa}_i, \widetilde{\kappa}_j)}, \quad 0 \leq i, j \leq n-1,$$

where $(\widetilde{a}_{ij})$ denotes the inverse of the Gram matrix $(\langle \widetilde{N}_i, \widetilde{N}_j\rangle)$, $K > 0$ and $\gamma \in (0,1)$ are constants depending only on the spline order k, $\widetilde{\kappa}_i$ denotes the length of the support of $\widetilde{N}_i$ and $\widetilde{d}$ is the canonical distance in $\mathbb{Z}/n\mathbb{Z}$. The proof of this equivalence uses the same tools as the proof in the non-periodic case: a periodic version of both Demko's theorem and de Boor's stability.

4. Almost everywhere convergence

In this section we prove Theorem 1.3 on the a.e. convergence of periodic spline projections.

Proof of Theorem 1.3. Without loss of generality, we assume that $\widetilde{\Delta}_n$ has n points. Let $\widetilde{\Delta}_n = (s_j^{(n)})_{j=0}^{n-1}$ and $(\widetilde{N}_j^{(n)})_{j=0}^{n-1}$ be the corresponding periodic B-spline functions. Associated to it, define the non-periodic point sequence $\Delta_n = (t_j^{(n)})_{j=-m}^{n+k-1}$ with the boundary points 0 and 1 as

$$t_j^{(n)} = s_j^{(n)}, \quad j = 0, \dots, n-1,$$
$$t_{-m}^{(n)} = \cdots = t_{-1}^{(n)} = 0, \qquad t_n^{(n)} = \cdots = t_{n+k-1}^{(n)} = 1.$$

We choose the integer m such that the multiplicity of the point 0 in Δ_n is k and denote by $(N_j^{(n)})_{j=-m}^{n-1}$ the non-periodic B-spline functions corresponding to this point sequence and by P_n the orthogonal projection operator onto the span of $(N_j^{(n)})_{j=-m}^{n-1}$.

We will show that $\widetilde{P}_n f(x) \to f(x)$ for all x in the set A from Theorem 1.2 of full Lebesgue measure such that $\lim P_n Tf(x) = Tf(x)$ for all $x \in A$, where T is just the operator that canonically identifies a function defined on $\mathbb{T}$ with the corresponding function defined on $[0,1)$ and we write x for a point in $\mathbb{T}$ as well as for its representative in the interval $[0,1)$. Observe that this operator T is different from the operator T in the proof of Lemma 3.1.

So, choose an arbitrary (non-zero) point $x \in A$ and decompose $\widetilde{P}_n f(x)$:

$$\widetilde{P}_n f(x) = T\widetilde{P}_n f(x) = P_n Tf(x) + \big(T\widetilde{P}_n f(x) - P_n Tf(x)\big). \tag{4.1}$$

For the first term of (4.1), $P_n Tf(x)$, we have that $\lim_{n\to\infty} P_n Tf(x) = Tf(x) = f(x)$ since $x \in A$.

It remains to estimate the second term $g_n(x) = T\widetilde{P}_n f(x) - P_n Tf(x) = T\widetilde{P}_n f(x) - Tf(x) + Tf(x) - P_n Tf(x)$ of (4.1). In order to do this, we write $g_n \in \mathcal{S}_k(\Delta_n)$ like the function g in the proof of Lemma 3.1:

$$g_n = \sum_{j\in J_n} \langle g_n, N_j^{(n)}\rangle N_j^{(n)*},$$

with $J_n = \{-m, \dots, -1\} \cup \{n-k, \dots, n-1\}$ and $(N_j^{(n)*})$ being the dual basis to $(N_j^{(n)})$. We now apply inequality (2.1) of Lemma 2.1 with $p = 1$ to g_n and get

$$|g_n(x)| \lesssim \gamma^{d(\mathfrak{i}_n(x), J_n)} \|g_n\|_1 \max_{\ell \in \mathfrak{i}_n(x), j\in J_n} \frac{1}{h_{\ell j}^{(n)}},$$

where $h_{\ell j}^{(n)}$ denotes the length of the convex hull of $\operatorname{supp} N_\ell^{(n)} \cup \operatorname{supp} N_j^{(n)}$ and $\mathfrak{i}_n(x)$ is the set of indices i such that x is contained in the support of $N_i^{(n)}$. Since for $\ell \in \mathfrak{i}_n(x)$, the point x is contained in $\operatorname{supp} N_\ell^{(n)}$ and for $j \in J_n$ either the point 0 or the point 1 is contained in $\operatorname{supp} N_j^{(n)}$, we can further estimate

$$|g_n(x)| \lesssim \gamma^{d(\mathfrak{i}_n(x), J_n)} \|g_n\|_1 \frac{1}{\min(x, 1-x)}.$$

Now, $\|g_n\|_1 = \|(T\widetilde{P}_n - P_nT)f\|_1 \lesssim \|f\|_1$, since the operator T has norm one on L^1 and $\widetilde{P}_n$ and P_n are both bounded on L^1 uniformly in n by Theorem 1.4 (cf. Remark 3.2) and Theorem 1.1, respectively. Since $|\widetilde{\Delta}_n|$ tends to zero, and a fortiori the same is true for $|\Delta_n|$, we have that $d(\mathfrak{i}_n(x), J_n)$ tends to infinity as $n \to \infty$. This implies $\lim_{n\to\infty} g_n(x) = 0$, and therefore, by the choice of the point x and decomposition (4.1), $\lim \widetilde{P}_n f(x) = f(x)$. Since $x \in A$ was arbitrary and A is a set of full Lebesgue measure, we obtain

$$\lim_{n\to\infty} \widetilde{P}_n f(y) = 0, \quad \text{for a.e. } y \in \mathbb{T},$$

and the proof is completed. □

5. The case of infinite point sequences

In this last section, we use the methods introduced in the previous sections to recover Shadrin's theorem for infinite point sequences (see [8,3]).

Let $(s_i)_{i\in\mathbb{Z}}$ be a biinfinite point sequence in $\mathbb{R}$ satisfying

$$s_i \le s_{i+1}, \qquad s_i < s_{i+k},$$

with the corresponding B-spline functions $(\widetilde{N}_i)_{i\in\mathbb{Z}}$ satisfying $\operatorname{supp} \widetilde{N}_i = [s_i, s_{i+k}]$. Furthermore, we denote by $\widetilde{P}$ the orthogonal projection operator onto the closed linear span of the functions $(\widetilde{N}_i)_{i\in\mathbb{Z}}$.

Lemma 5.1. *Let f be a function on $(\inf s_i, \sup s_i)$ with compact support. Then, for any $x \in (\inf s_i, \sup s_i)$,*

$$|\widetilde{P}f(x)| \lesssim \gamma^{d(\mathfrak{i}(x), \mathfrak{i}(\operatorname{supp} f))} \|f\|_\infty,$$

where $\gamma \in (0, 1)$ is the constant appearing in inequality (1.3).

Proof. For notational simplicity, we assume in this proof that the sequence (s_i) is strictly increasing. Let $x \in (\inf s_i, \sup s_i)$ and let $I(x)$ be the interval $I = [s_i, s_{i+1})$ containing x. Since f has compact support and the sequence (s_i) is biinfinite, we can choose the indices ℓ and r such that $\{x\} \cup \operatorname{supp} f \subset [s_\ell, s_{r+1})$ and with $J = \{\ell - k + 1, \dots, \ell - 1\} \cup \{r - k + 2, \dots, r\}$, the inequality

$$\gamma^{d(\mathfrak{i}(x), J)} |\operatorname{supp} f|^{1/2} |I(x)|^{-1/2} \le \gamma^{d(\mathfrak{i}(x), \mathfrak{i}(\operatorname{supp} f))}$$

is true.

Next, define the point sequence $\Delta = (t_i)_{i=\ell-k+1}^{r+k}$ by

$$t_i = s_i, \quad i = \ell, \dots, r+1,$$
$$a = t_{\ell-k+1} = \dots = t_\ell = s_\ell, \qquad b = t_{r+k} = \dots = t_{r+1} = s_{r+1},$$

and let the collection $(N_i)_{i=\ell-k+1}^{r}$ be the corresponding B-spline functions and P the associated orthogonal projector. Let T be the operator that restricts a function defined on $(\inf s_i, \sup s_i)$ to the interval $[a, b]$. Note that this operator T is different from those in the proofs of Lemma 3.1 and Theorem 1.3. In order to estimate $\widetilde{P}f(x)$, we decompose

$$\widetilde{P}f(x) = T\widetilde{P}f(x) = PTf(x) + \big(T\widetilde{P}f(x) - PTf(x)\big). \tag{5.1}$$

Observe that $PTf = \sum_{n\in F} \langle f, N_n\rangle N_n^*$, where $F = \mathfrak{i}(\operatorname{supp} f)$. Applying inequality (2.3) of Lemma 2.1 with the exponent $p = \infty$, we obtain

$$|PTf(x)| \lesssim \gamma^{d(\mathfrak{i}(x), F)} \|f\|_\infty.$$

We now consider the second part of the decomposition (5.1), the function $g = (T\widetilde{P} - PT)f = (T\widetilde{P} - T + T - PT)f$. Again, as we did for the function g in the proof of Lemma 3.1, we can write $g \in \mathcal{S}_k(\Delta)$ as

$$g = \sum_{j\in J} \langle g, N_j\rangle N_j^*$$

with $J = \{\ell - k + 1, \dots, \ell - 1\} \cup \{r - k + 2, \dots, r\}$ as defined above. Now, by inequality (2.3) of Lemma 2.1 with the exponent $p = 2$, we get

$$\begin{aligned}|g(x)| &\lesssim \gamma^{d(\mathrm{i}(x),J)} \|g\|_2 \cdot |I(x)|^{-1/2} \lesssim \gamma^{d(\mathrm{i}(x),J)} \|f\|_2 \cdot |I(x)|^{-1/2} \\ &\leq \gamma^{d(\mathrm{i}(x),J)} |\operatorname{supp} f|^{1/2} |I(x)|^{-1/2} \|f\|_\infty.\end{aligned}$$

Finally, due to the choice of ℓ and r,

$$\gamma^{d(\mathrm{i}(x),J)} |\operatorname{supp} f|^{1/2} |I(x)|^{-1/2} \leq \gamma^{d(\mathrm{i}(x),\mathrm{i}(\operatorname{supp} f))},$$

which proves the lemma. □

We can now use this lemma to *define* $\widetilde{P}f$ for functions $f \in L^\infty(\inf s_i, \sup s_i)$ that are not necessarily in $L^2(\inf s_i, \sup s_i)$ if $\inf s_i = -\infty$ or $\sup s_i = +\infty$. If we let $f_i := f \mathbb{1}_{[s_i, s_{i+1})}$, then f_i has compact support and the above lemma implies that the pointwise series

$$\widetilde{P}f(x) := \sum_{i \in \mathbb{Z}} \widetilde{P}f_i(x), \quad x \in (\inf s_i, \sup s_i),$$

is absolutely convergent and, moreover, there exists a constant C depending only on the spline order k such that

$$\|\widetilde{P}f\|_\infty \leq C \|f\|_\infty.$$

This operator enjoys the characteristic property of an orthogonal projection:

$$\langle \widetilde{P}f - f, \widetilde{N}_i \rangle = 0, \quad i \in \mathbb{Z}.$$

Remark 5.2. One can combine the proofs of Lemmas 5.1 and 3.1 to also obtain the uniform boundedness of the spline orthoprojector on L^∞ for one-sided infinite point sequences.

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CHAPTER 6

Unconditionality of periodic orthonormal spline systems in L^p

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Unconditionality of periodic orthonormal spline systems in L^p

by

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Abstract. Given any natural number k and any dense point sequence (t_n) on the torus $\mathbb{T}$, we prove that the corresponding periodic orthonormal spline system of order k is an unconditional basis in L^p for $1 < p < \infty$.

1. Introduction. In this work, we are concerned with periodic orthonormal spline systems of arbitrary order k with arbitrary partitions. We let $(s_n)_{n=1}^{\infty}$ be a dense sequence of points in the torus $\mathbb{T}$ such that each point occurs at most k times. Such point sequences are called *admissible*.

For $n \geq k$, we define $\hat{\mathcal{S}}_n$ to be the space of polynomial splines of order k with grid points $(s_j)_{j=1}^{n}$. For each $n \geq k+1$, the space $\hat{\mathcal{S}}_{n-1}$ has codimension 1 in $\hat{\mathcal{S}}_n$, and therefore there exists a function $\hat{f}_n \in \hat{\mathcal{S}}_n$ with $\|\hat{f}_n\|_2 = 1$ that is orthogonal to $\hat{\mathcal{S}}_{n-1}$. Observe that this function $\hat{f}_n$ is unique up to sign. In addition, let $(\hat{f}_n)_{n=1}^{k}$ be an orthonormal basis for $\hat{\mathcal{S}}_k$. The system of functions $(\hat{f}_n)_{n=1}^{\infty}$ is called a *periodic* orthonormal spline system of order k corresponding to the sequence $(s_n)_{n=1}^{\infty}$. We remark that if a point x occurs m times in the sequence $(s_n)_{n=1}^{\infty}$ before index N, the space $\hat{\mathcal{S}}_N$ consists of splines that are in particular $k-1-m$ times continuously differentiable at x, where for $k-1-m \leq -1$ we mean that no restrictions at the point x are imposed. This means that if $m = k$ and also $s_N = x$, we have $\hat{\mathcal{S}}_{N-1} = \hat{\mathcal{S}}_N$ and therefore it makes no sense to consider non-admissible point sequences.

The main result of this article is the following

Theorem 1.1. *Let $k \in \mathbb{N}$ and $(s_n)_{n\geq 1}$ be an admissible sequence of knots in $\mathbb{T}$. Then the corresponding periodic orthonormal spline system of order k is an unconditional basis in $L^p(\mathbb{T})$ for every $1 < p < \infty$.*

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This is the periodic version of the main result in [13]. We now give a few comments on the history of this result. We can similarly define the spaces $\mathcal{S}_n$ corresponding to an admissible point sequence (t_n) on the interval $[0, 1]$. A celebrated result of A. Shadrin [16] states that the orthogonal projection operator onto $\mathcal{S}_n$ is bounded on $L^\infty[0, 1]$ by a constant that depends only on the spline order k. As a consequence, $(f_n)_n$ (also similarly defined to $\hat{f}_n$) is a Schauder basis in $L^p[0, 1]$, $1 \le p < \infty$, and in the space $C[0, 1]$ of continuous functions. There are various results on the unconditionality of spline systems restricting either the spline order k or the partition $(t_n)_{n\ge0}$. The first result in this direction is [1], where it is proved that the classical Franklin system—that is, the orthonormal spline systems of order 2 corresponding to the dyadic knot sequence $(1/2, 1/4, 3/4, 1/8, 3/8, \ldots)$—is an unconditional basis in $L^p[0, 1]$, $1 < p < \infty$. This argument was extended in [3] to prove unconditionality of orthonormal spline systems of arbitrary order, but still restricted to dyadic knots. Considerable effort has been made in the past to weaken the restriction to dyadic knot sequences. In a series of papers [9, 11, 10] this restriction was removed step-by-step for general Franklin systems, with the final result that for each admissible point sequence $(t_n)_{n\ge0}$ with parameter $k = 2$, the associated general Franklin system forms an unconditional basis in $L^p[0, 1]$, $1 < p < \infty$. By combining the methods used in [11, 10] with some new inequalities from [15] it was proved in [13] that non-periodic orthonormal spline systems are unconditional bases in $L^p[0, 1]$, $1 < p < \infty$, for any spline order k and any admissible point sequence (t_n).

The periodic analogue of Shadrin's theorem can be obtained from Shadrin's result [16] using [5]. Alternatively, [14] gives a direct proof. In the case of dyadic knots, J. Domsta [8] obtained exponential decay for the inverse of the Gram matrix of periodic B-splines, which were exploited to prove the unconditionality of the periodic orthonormal spline systems with dyadic knots in L^p for $1 < p < \infty$. In [12] it was proved that for any admissible point sequence the corresponding periodic Franklin system (i.e. the case $k = 2$) forms an unconditional basis in $L^p[0, 1]$, $1 < p < \infty$. Here we obtain an estimate for general periodic orthonormal spline functions, which combined with the methods developed in [10] yields the unconditionality of periodic orthonormal spline systems in $L^p(\mathbb{T})$.

The main idea of the proofs of (f_n) or $(\hat{f}_n)$ being an unconditional basis in L^p, $p \in (1, \infty)$, in [10, 12, 13] is that to a single function f_n, a grid point interval is associated on which most of the mass of f_n is concentrated. In the case of Haar functions h_n, its support splits into two intervals I and J, where the function h_n is positive on I and negative on J. As the associated interval, we could just use the smaller of I and J.

The organization of the present article is as follows. In Section 2, we give some preliminary results concerning polynomials, splines and non-periodic orthonormal spline functions. Section 3 develops crucial estimates for the periodic orthonormal spline functions $\hat{f}_n$ and gives several relations between $\hat{f}_n$ and its non-periodic counterpart. In Section 4 we prove a few technical lemmata used in the proof of Theorem 1.1, and Section 5 finally proves Theorem 1.1.

We remark that the results and most of the proofs in Sections 4 and 5 closely follow [10]. However, the proof of the crucial Lemma 4.4 is new and much shorter than in [10].

2. Preliminaries. Let k be a positive integer. The parameter k will always be used for the order of the underlying polynomials or splines. We use the notation $A(t) \sim B(t)$ to indicate the existence of two constants $c_1, c_2 > 0$ that depend only on k, such that $c_1 B(t) \leq A(t) \leq c_2 B(t)$ for all t, where t denotes all implicit and explicit dependences that the expressions A and B might have. If the constants c_1, c_2 depend on an additional parameter p, we write this as $A(t) \sim_p B(t)$. Correspondingly, we use the symbols $\lesssim$, $\gtrsim$, $\lesssim_p$, $\gtrsim_p$. For a subset E of the real line, we denote by $|E|$ the Lebesgue measure of E and by $\mathbb{1}_E$ the characteristic function of E.

We will need the classical Remez inequality:

Theorem 2.1 (Remez). *Let $V \subset \mathbb{R}$ be a compact interval and $E \subset V$ a measurable subset. Then, for all polynomials p of order k on V,*

$$\|p\|_{L_\infty(V)} \leq \left(4\frac{|V|}{|E|}\right)^{k-1} \|p\|_{L_\infty(E)}.$$

This immediately yields the following corollary:

Corollary 2.2. *Let p be a polynomial of order k on a compact interval $V \subset \mathbb{R}$. Then*

$$\left|\left\{x \in V : |p(x)| \geq 8^{-k+1}\|p\|_{L_\infty(V)}\right\}\right| \geq |V|/2.$$

Proof. This is a direct application of Theorem 2.1 with $E := \{x \in V : |p(x)| \leq 8^{-k+1}\|p\|_{L^\infty(V)}\}$. ∎

Let

(2.1)
$$\mathcal{T} = (0 = \tau_{-k} = \cdots = \tau_{-1} < \tau_0 \leq \cdots \leq \tau_{n-1} < \tau_n = \cdots = \tau_{n+k-1} = 1)$$

be a partition of $[0,1]$ consisting of knots of multiplicity at most k, that is,

$\tau_i < \tau_{i+k}$ for all $0 \le i \le n-1$. Let $\mathcal{S}_\mathcal{T}$ be the space of polynomial splines of order k with knots $\mathcal{T}$. The basis of L^∞-normalized B-spline functions in $\mathcal{S}_\mathcal{T}$ is denoted by $(N_{i,k})_{i=-k}^{n-1}$ or for short $(N_i)_{i=-k}^{n-1}$. Corresponding to this basis, there exists a biorthogonal basis of $\mathcal{S}_\mathcal{T}$, which is denoted by $(N_{i,k}^*)_{i=-k}^{n-1}$ or $(N_i^*)_{i=-k}^{n-1}$. Moreover, we write $\nu_i = \tau_{i+k} - \tau_i = |\operatorname{supp} N_i|$.

We now recall a few important results for B-splines N_i and their dual functions N_i^*.

THEOREM 2.3 (Shadrin [16]). *Let P be the orthogonal projection operator onto $\mathcal{S}_\mathcal{T}$ with respect to the canonical inner product in $L^2[0,1]$. Then there exists a constant C_k depending only on the spline order k such that*

$$\|P : L^\infty[0,1] \to L^\infty[0,1]\| \le C_k.$$

PROPOSITION 2.4 (B-spline stability). *Let $1 \le p \le \infty$ and $g = \sum_{j=-k}^{n-1} a_j N_j$ be a linear combination of B-splines. Then*

$$|a_j| \lesssim |L_j|^{-1/p} \|g\|_{L^p(L_j)}, \qquad -k \le j \le n-1, \tag{2.2}$$

where L_j is a subinterval $[\tau_i, \tau_{i+1}]$ of $[\tau_j, \tau_{j+k}]$ of maximal length. Additionally,

$$\|g\|_p \sim \Big(\sum_{j=-k}^{n-1} |a_j|^p \nu_j\Big)^{1/p} = \|(a_j \nu_j^{1/p})_{j=-k}^{n-1}\|_{\ell^p}. \tag{2.3}$$

Moreover, if $h = \sum_{j=-k}^{n-1} b_j N_j^*$, then

$$\|h\|_p \sim \Big(\sum_{j=-k}^{n-1} |b_j|^p \nu_j^{1-p}\Big)^{1/p} = \|(b_j \nu_j^{1/p-1})_{j=-k}^{n-1}\|_{\ell^p}. \tag{2.4}$$

Inequalites (2.2) and (2.3) are respectively Lemmas 4.1 and 4.2 in [7, Chapter 5]. Inequality (2.4) is a consequence of Theorem 2.3. For a deduction of the lower estimate in (2.4) from this result, see [4, Property P.7]. The proof of the upper estimate uses a simple duality argument which we shall present here:

Proof of the upper estimate in (2.4). We only consider the case $p < \infty$ and we assume without loss of generality that $b_j \ge 0$. Let $N_{j,p} = \nu_j^{-1/p} N_j$ be the p-normalized B-spline function and $N_{j,p}^* = \nu_j^{1/p} N_j^*$ be the corresponding p-normalized dual B-spline function. By definition, the system $N_{j,p}^*$ forms a dual basis to the system of functions $N_{j,p}$. By choosing $p' = p/(p-1)$ and $\alpha = 2/p'$ (so $2 - \alpha = 2/p$) we obtain, by (2.3),

$$\begin{aligned}\sum_j b_j^2 &= \Big\langle \sum_j b_j^\alpha N_{j,p}^*, \sum_j b_j^{2-\alpha} N_{j,p}\Big\rangle \le \Big\| \sum_j b_j^{2-\alpha} N_{j,p}\Big\|_p \Big\| \sum_j b_j^\alpha N_{j,p}^*\Big\|_{p'} \\ &= \Big\| \sum_j b_j^{2-\alpha} \nu_j^{-1/p} N_j\Big\|_p \Big\| \sum_j b_j^\alpha \nu_j^{1/p} N_j^*\Big\|_{p'} \\ &\lesssim \Big(\sum_j b_j^2\Big)^{1/p} \Big\| \sum_j b_j^\alpha \nu_j^{1/p} N_j^*\Big\|_{p'}.\end{aligned}$$

So we get

$$\Big(\sum_j b_j^2\Big)^{1/p'} \lesssim \Big\| \sum_j b_j^\alpha \nu_j^{1/p} N_j^*\Big\|_{p'}. \tag{2.5}$$

Setting $a_j = b_j^\alpha \nu_j^{1/p}$, we see that $b_j^2 = (a_j \nu_j^{-1/p})^{2/\alpha} = a_j^{p'} \nu_j^{-p'/p} = a_j^{p'} \nu_j^{1-p'}$, and therefore we may write (2.5) as

$$\Big(\sum_j a_j^{p'} \nu_j^{1-p'}\Big)^{1/p'} \lesssim \Big\| \sum_j a_j N_j^*\Big\|_{p'},$$

which is the upper estimate in (2.4). ∎

It can be shown that Shadrin's theorem actually implies the following estimate on the B-spline Gram matrix inverse:

Theorem 2.5 ([15]). *Let $k \in \mathbb{N}$, let the partition $\mathcal{T}$ be defined as in (2.1) and let (a_{ij}) be the inverse of the Gram matrix $(\langle N_i, N_j\rangle)$ of B-spline functions. Then*

$$|a_{ij}| \le C \frac{q^{|i-j|}}{|\operatorname{conv}(\operatorname{supp} N_i \cup \operatorname{supp} N_j)|}, \qquad -k \le i, j \le n-1,$$

where the constants $C > 0$ and $0 < q < 1$ depend only on the spline order k and where by $\operatorname{conv}(U)$ for $U \subset [0,1]$ we denote the smallest subinterval of $[0,1]$ that contains U.

Let $f \in L^p[0,1]$ for some $1 \le p < \infty$. Since the orthonormal spline system $(f_n)_{n \ge -k+2}$ is a basis in $L^p[0,1]$, we can write $f = \sum_{n=-k+2}^\infty a_n f_n$. In terms of this expansion, we define the *maximal function* $Mf := \sup_m \big|\sum_{n \le m} a_n f_n\big|$. Given a measurable function g, we denote by $\mathcal{M}g$ the *Hardy–Littlewood maximal function* of g defined as

$$\mathcal{M}g(x) := \sup_{I \ni x} |I|^{-1} \int_I |g(t)| \,\mathrm{d}t,$$

where the supremum is taken over all intervals I containing the point x.

A corollary of Theorem 2.5 is the following relation between M and $\mathcal{M}$:

Theorem 2.6 ([15]). *If $f \in L^1[0,1]$, then*

$$Mf(t) \lesssim \mathcal{M}f(t), \qquad t \in [0,1].$$

2.1. Orthonormal spline functions, non-periodic case. This section recalls some facts about orthonormal spline functions $f_n = f_n^{(k)}$ for fixed $k \in \mathbb{N}$ and $n \geq 2$ induced by the admissible sequence (t_n).

We again consider the mesh $\mathcal{T}$ as before:

$$\mathcal{T} = (0 = \tau_{-k} = \cdots = \tau_{-1} < \tau_0 \leq \cdots \leq \tau_{i_0} \\ \leq \cdots \leq \tau_{n-1} < \tau_n = \cdots = \tau_{n+k-1} = 1),$$

where we singled out the point τ_{i_0}; and the partition $\widetilde{\mathcal{T}}$ is defined to be the same as $\mathcal{T}$, but with τ_{i_0} removed. In the same way we denote by $(N_i : -k \leq i \leq n-1)$ the B-spline functions corresponding to $\mathcal{T}$ and by $(\widetilde{N}_i : -k \leq i \leq n-2)$ the B-spline functions corresponding to $\widetilde{\mathcal{T}}$. Böhm's formula [2] gives us the following relationship between N_i and $\widetilde{N}_i$:

(2.6)
$$\widetilde{N}_i(t) = \begin{cases} N_i(t) & \text{if } -k \leq i \leq i_0 - k - 1, \\ \dfrac{\tau_{i_0} - \tau_i}{\tau_{i+k} - \tau_i} N_i(t) + \dfrac{\tau_{i+k+1} - \tau_{i_0}}{\tau_{i+k+1} - \tau_{i+1}} N_{i+1}(t) & \text{if } i_0 - k \leq i \leq i_0 - 1, \\ N_{i+1}(t) & \text{if } i_0 \leq i \leq n-2. \end{cases}$$

In order to calculate the orthonormal spline function corresponding to the partitions $\widetilde{\mathcal{T}}$ and $\mathcal{T}$, we first determine a function $g \in \operatorname{span}\{N_i : -k \leq i \leq n-1\}$ such that $g \perp \widetilde{N}_j$ for all $-k \leq j \leq n-2$. Up to a multiplicative constant, the function g is of the form

$$g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^*, \tag{2.7}$$

where $(N_j^* : -k \leq j \leq n-1)$ is the system biorthogonal to the functions $(N_i : -k \leq i \leq n-1)$ and

(2.8)
$$\alpha_j = (-1)^{j-i_0+k} \Big(\prod_{\ell=i_0-k+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_\ell} \Big) \Big(\prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell} \Big), \quad i_0 - k \leq j \leq i_0.$$

Alternatively, the coefficients α_j can be described by the recursion

$$\alpha_{i+1} \frac{\tau_{i+k+1} - \tau_{i_0}}{\tau_{i+k+1} - \tau_{i+1}} + \alpha_i \frac{\tau_{i_0} - \tau_i}{\tau_{i+k} - \tau_i} = 0. \tag{2.9}$$

In order to give estimates for g, and a fortiori for the normalized function $f = g/\|g\|_2$, we assign to each g a characteristic interval that is a grid point interval $[\tau_i, \tau_{i+1}]$ and lies in the proximity of the newly inserted point τ_{i_0}:

Definition 2.7 ([13], Characteristic interval for non-periodic sequences). Let $\mathcal{T}, \widetilde{\mathcal{T}}$ be as above and τ_{i_0} be the new point in $\mathcal{T}$ that is not present in $\widetilde{\mathcal{T}}$. We define the *characteristic interval J corresponding to τ_{i_0}* as follows.

(1) Let

$$\Lambda^{(0)} := \Big\{ i_0 - k \le j \le i_0 : |[\tau_j, \tau_{j+k}]| \le 2 \min_{i_0 - k \le \ell \le i_0} |[\tau_\ell, \tau_{\ell+k}]| \Big\}$$

be the set of all indices j for which the corresponding support of the B-spline function N_j is approximately minimal. Observe that $\Lambda^{(0)}$ is non-empty.

(2) Define

$$\Lambda^{(1)} := \Big\{ j \in \Lambda^{(0)} : |\alpha_j| = \max_{\ell \in \Lambda^{(0)}} |\alpha_\ell| \Big\}.$$

For an arbitrary but fixed index $j^{(0)} \in \Lambda^{(1)}$, set $J^{(0)} := [\tau_{j^{(0)}}, \tau_{j^{(0)}+k}]$.

(3) The interval $J^{(0)}$ can now be written as the union of k grid intervals

$$J^{(0)} = \bigcup_{\ell=0}^{k-1} [\tau_{j^{(0)}+\ell}, \tau_{j^{(0)}+\ell+1}] \quad \text{with } j^{(0)} \text{ as above.}$$

We define the *characteristic interval* $J = J(\tau_{i_0})$ to be one of the above k intervals that has maximal length.

Using this definition of J, we recall the following estimates for g:

LEMMA 2.8 ([13]). *Let $\mathcal{T}$, $\widetilde{\mathcal{T}}$ be as above and let $g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^* = \sum_{j=-k}^{n-1} w_j N_j$ be the function from (2.7), where the coefficients (w_j) are defined by this equation. Moreover, let $f = g/\|g\|_2$ be the L^2-normalized orthogonal spline function corresponding to the mesh point τ_{i_0}. Then*

$$\|g\|_{L^p(J)} \sim \|g\|_p \sim |J|^{1/p-1}, \qquad 1 \le p \le \infty,$$

and therefore

$$\|f\|_{L^p(J)} \sim \|f\|_p \sim |J|^{1/p-1/2}, \qquad 1 \le p \le \infty,$$

where J is the characteristic interval associated to the point τ_{i_0} given in Definition 2.7.

Additionally, if $d_{\mathcal{T}}(z)$ denotes the number of grid points from $\mathcal{T}$ that lie between J and z including z and the endpoints of J, then there exists a $q \in (0,1)$ depending only on k such that

$$|w_j| \lesssim \frac{q^{d_{\mathcal{T}}(\tau_j)}}{|J| + \operatorname{dist}(\operatorname{supp} N_j, J) + \nu_j} \quad \textit{for all } -k \le j \le n-1. \tag{2.10}$$

Moreover, if $x < \inf J$, then

$$\|f\|_{L^p(0,x)} \lesssim \frac{q^{d_{\mathcal{T}}(x)} |J|^{1/2}}{(|J| + \operatorname{dist}(x,J))^{1-1/p}}, \qquad 1 \le p \le \infty. \tag{2.11}$$

Similarly, for $x > \sup J$,

$$\|f\|_{L^p(x,1)} \lesssim \frac{q^{d_{\mathcal{T}}(x)} |J|^{1/2}}{(|J| + \operatorname{dist}(x,J))^{1-1/p}}, \qquad 1 \le p \le \infty. \tag{2.12}$$

2.2. Combinatorics of characteristic intervals. We additionally have a combinatorial lemma concerning the collection of characteristic intervals corresponding to all grid points of an admissible sequence (t_n) of points and the corresponding orthonormal spline functions $(f_n)_{n=-k+2}^{\infty}$ of order k. For $n \geq 2$, the partitions $\mathcal{T}_n$ associated to f_n are defined to consist of the grid points $(t_j)_{j=-1}^{n}$, the knots $t_{-1} = 0$ and $t_0 = 1$ having both multiplicity k in $\mathcal{T}_n$ and we enumerate them as

$$\begin{aligned}\mathcal{T}_n = (0 = \tau_{n,-k} = \cdots = \tau_{n,-1} < \tau_{n,0} \\ \leq \cdots \leq \tau_{n,n-1} < \tau_{n,n} = \cdots = \tau_{n,n+k-1} = 1).\end{aligned}$$

If $n \geq 2$, we denote by $J_n^{(0)}$ and J_n the characteristic intervals $J^{(0)}$ and J from Definition 2.7 associated to the new grid point t_n, which is defined to be the characteristic interval associated to $(\mathcal{T}_{n-1}, \mathcal{T}_n)$. If n is in the range $-k+2 \leq n \leq 1$, we additionally set $J_n := [0,1]$.

LEMMA 2.9 ([13]). *Let V be an arbitrary subinterval of $[0,1]$ and let $\beta > 0$. Then there exists a constant $F_{k,\beta}$ only depending on k and β such that*

$$\operatorname{card}\{n : J_n \subseteq V,\ |J_n| \geq \beta|V|\} \leq F_{k,\beta},$$

where $\operatorname{card} E$ *denotes the cardinality of the set E.*

3. Periodic splines. In this section, we give estimates for periodic orthonormal spline functions $(\hat{f}_n)$ similar to the ones in Lemma 2.8 for non-periodic orthonormal splines. The main difficulty in proving such estimates is that we do not have a periodic version of Theorem 2.5 at our disposal. Instead, we estimate the differences between $\hat{f}_n$ and two suitable non-periodic orthonormal spline functions f_n.

Let $n \geq k$ and $(\hat{N}_i)_{i=0}^{n-1}$ be periodic B-spline functions of order k with an arbitrary admissible grid $(\sigma_j)_{j=0}^{n-1}$ on $\mathbb{T}$ canonically identified with $[0,1)$:

$$\hat{\mathcal{T}} = (0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-1} < 1).$$

Moreover, let $(\hat{N}_i^*)_{i=0}^{n-1}$ be the dual basis to $(\hat{N}_i)_{i=0}^{n-1}$ and $\hat{\mathcal{S}}_{\hat{\mathcal{T}}}$ be the linear span of $(\hat{N}_i)_{i=0}^{n-1}$. First, we recall a periodic version of Shadrin's theorem:

THEOREM 3.1. *Let $\hat{P}$ be the $L^2(\mathbb{T})$-orthogonal projection operator onto $\hat{\mathcal{S}}_{\hat{\mathcal{T}}}$. Then there exists a constant C_k depending only on the spline order k such that*

$$\|\hat{P} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\| \leq C_k.$$

We refer to the articles [16, 5] for a proof of this result for infinite knot sequences on the real line, which can then be carried over to $\mathbb{T}$. Alternatively, we refer to [14] for a direct proof.

Next, note that B-spline stability carries over to the periodic setting:

Proposition 3.2. *Let $n \geq 2k$ and $1 \leq p \leq \infty$. Then, for $g = \sum_{j=0}^{n-1} a_j \hat{N}_j$, we have*

$$\|g\|_p \sim \Big(\sum_{j=0}^{n-1} |a_j|^p |\operatorname{supp} \hat{N}_j| \Big)^{1/p} = \|(a_j \cdot |\operatorname{supp} \hat{N}_j|^{1/p})_{j=0}^{n-1}\|_{\ell^p}.$$

The matrix $(\hat{a}_{ij})_{i,j=0}^{n-1} := (\langle \hat{N}_i^*, \hat{N}_j^* \rangle)_{i,j=0}^{n-1}$ satisfies the following geometric decay inequality, which is a consequence of Theorem 3.1 on the uniform boundedness of the periodic orthogonal spline projection operator:

Proposition 3.3. *Let $n \geq 2k$. Then there exists a constant $q \in (0,1)$ depending only on the spline order k such that*

$$|\hat{a}_{ij}| \lesssim \frac{q^{\hat{d}(i,j)}}{\max(|\operatorname{supp} \hat{N}_i|, |\operatorname{supp} \hat{N}_j|)}, \qquad 0 \leq i, j \leq n-1,$$

where $\hat{d}$ is the periodic distance function on $\{0, \ldots, n-1\}$.

The proof of this proposition runs along the same lines as in the non-periodic case, where B-spline stability and Demko's theorem [6] on the geometric decay of inverses of band matrices is used. The proof in the non-periodic case can be found in [4].

Observe that the estimate contained in this proposition for periodic splines is not as good as the one from Theorem 2.5 for non-periodic splines due to the different term in the denominator. Next, we also get stability of the periodic dual B-spline functions (N_i^*):

Proposition 3.4. *Let $n \geq 2k$, $1 \leq p \leq \infty$ and $h = \sum_{j=0}^{n-1} b_j \hat{N}_j^*$. Then*

$$\|h\|_p \sim \Big(\sum_{j=0}^{n-1} |b_j|^p |\operatorname{supp} \hat{N}_j|^{1-p} \Big)^{1/p} = \|(b_j \cdot |\operatorname{supp} \hat{N}_j|^{1/p-1})_{j=0}^{n-1}\|_{\ell^p}.$$

Proof. We only prove the assertion for $p \in (1, \infty)$. The boundary cases follow by obvious modifications. By Propositions 3.3, 3.2, and Hölder's inequality,

$$\begin{aligned}
\Big\| \sum_j a_j \nu_j^{1/p'} \hat{N}_j^* \Big\|_p^p &= \Big\| \sum_j a_j \nu_j^{1/p'} \sum_i \hat{a}_{ij} \hat{N}_i \Big\|_p^p = \Big\| \sum_i \Big(\sum_j a_j \nu_j^{1/p'} \hat{a}_{ij} \Big) \hat{N}_i \Big\|_p^p \\
&\leq \sum_i \Big| \sum_j a_j \nu_j^{1/p'} \hat{a}_{ij} \Big|^p \nu_i \lesssim \sum_i \Big(\sum_j |a_j| \nu_j^{1/p'} \nu_i^{1/p} \frac{q^{\hat{d}(i,j)}}{\max(\nu_i, \nu_j)} \Big)^p \\
&\leq \sum_i \Big(\sum_j |a_j| q^{\hat{d}(i,j)} \Big)^p \leq \sum_i \Big(\sum_j |a_j|^p q^{\hat{d}(i,j) \frac{p}{2}} \Big) \cdot \Big(\sum_j q^{\hat{d}(i,j) \frac{p}{2(p-1)}} \Big)^{p-1} \\
&\lesssim \sum_i \sum_j |a_j|^p q^{\hat{d}(i,j) \frac{p}{2}} \lesssim \|a\|_p^p.
\end{aligned}$$

Setting $b_j = a_j \nu_j^{1/p'}$ yields the first inequality of dual B-spline stability. The other inequality is proved similarly to the result for the non-periodic case in Proposition 2.4. ■

3.1. Periodic orthonormal spline functions. We now consider the same situation as for the non-periodic case: Let

$$\hat{\mathcal{T}} = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{i_0} \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$$

be a partition of $\mathbb{T}$ canonically identified with $[0,1)$, and $\tilde{\hat{\mathcal{T}}}$ be the same partition, but with σ_{i_0} removed. Similarly, we denote by $(\hat{N}_j)_{j=0}^{n-1}$ the periodic B-spline functions of order k with respect to $\hat{\mathcal{T}}$ and by $(\tilde{\hat{N}}_j)_{j=0}^{n-2}$ the periodic B-spline functions of order k with respect to $\tilde{\hat{\mathcal{T}}}$. Here, we use the periodic extension of the sequence $(\sigma_j)_{j=0}^{n-1}$, i.e. $\sigma_{rn+j} = r + \sigma_j$ for $j \in \{0, \ldots, n-1\}$ and $r \in \mathbb{Z}$, and the indices of B-spline functions are taken modulo n.

To calculate the periodic orthonormal spline functions corresponding to the above grids, we determine a function $\hat{g} \in \operatorname{span}\{\hat{N}_i : 0 \le i \le n-1\}$ such that $\hat{g} \perp \tilde{\hat{N}}_j$ for all $0 \le j \le n-2$. That is, we assume that

$$\hat{g} = \sum_{j=0}^{n-1} \hat{\alpha}_j \hat{N}_j^*,$$

where $(\hat{N}_j^* : 0 \le j \le n-1)$ is the system biorthogonal to the functions $(\hat{N}_i : 0 \le i \le n-1)$ and $\hat{\alpha}_j = \langle g, \hat{N}_j \rangle$. For $\hat{g}$ to be orthogonal to $\tilde{\hat{N}}_j$ for $0 \le j \le n-2$, it has to satisfy the identities

$$0 = \langle \hat{g}, \tilde{\hat{N}}_i \rangle = \sum_{j=0}^{n-1} \hat{\alpha}_j \langle \hat{N}_j^*, \tilde{\hat{N}}_i \rangle, \qquad 0 \le i \le n-2.$$

We can look at the indices j here periodically, meaning that $\hat{\alpha}_j \ne 0$ only for $j \in \{i_0 - k, \ldots, i_0\}$. Observe that formula (2.6) extends to the periodic setting, which implies the following recursion for the coefficients $(\hat{\alpha}_j)$:

$$\hat{\alpha}_{i+1} \frac{\sigma_{i+k+1} - \sigma_{i_0}}{\sigma_{i+k+1} - \sigma_{i+1}} + \hat{\alpha}_i \frac{\sigma_{i_0} - \sigma_i}{\sigma_{i+k} - \sigma_i} = 0, \qquad i_0 - k \le i \le i_0 - 1. \tag{3.1}$$

With the starting value

$$\hat{\alpha}_{i_0-k} = \prod_{\ell=i_0-k+1}^{i_0-1} \frac{\sigma_{\ell+k} - \sigma_{i_0}}{\sigma_{\ell+k} - \sigma_\ell},$$

we get the explicit formula

(3.2)
$$\hat{\alpha}_j = (-1)^{j-i_0+k}\Big(\prod_{\ell=i_0-k+1}^{j-1}\frac{\sigma_{i_0}-\sigma_\ell}{\sigma_{\ell+k}-\sigma_\ell}\Big)\cdot\Big(\prod_{\ell=j+1}^{i_0-1}\frac{\sigma_{\ell+k}-\sigma_{i_0}}{\sigma_{\ell+k}-\sigma_\ell}\Big), \qquad i_0-k\le j\le i_0.$$

Now, similarly to Definition 2.7, we are able to define characteristic intervals for periodic grids as follows:

Definition 3.5 (Characteristic interval for periodic sequences). Let $\hat{\mathcal{T}}, \widetilde{\hat{\mathcal{T}}}$ be as above and σ_{i_0} be the new point in $\hat{\mathcal{T}}$ that is not present in $\widetilde{\hat{\mathcal{T}}}$. Under the restriction $n \ge 2k$, we define the (*periodic*) *characteristic interval $\hat{J}$ corresponding to* σ_{i_0} as follows:

(1) Let
$$\Lambda^{(0)} := \Big\{ i_0 - k \le j \le i_0 : |[\sigma_j, \sigma_{j+k}]| \le 2 \min_{i_0-k\le\ell\le i_0} |[\sigma_\ell, \sigma_{\ell+k}]| \Big\}$$
be the set of all indices j in the vicinity of the index i_0 for which the corresponding support of the periodic B-spline function $\hat{N}_j$ is approximately minimal. Observe that $\Lambda^{(0)}$ is non-empty.

(2) Define
$$\Lambda^{(1)} := \Big\{ j \in \Lambda^{(0)} : |\hat{\alpha}_j| = \max_{\ell\in\Lambda^{(0)}} |\hat{\alpha}_\ell| \Big\}.$$
For an arbitrary but fixed index $j^{(0)} \in \Lambda^{(1)}$, set $\hat{J}^{(0)} := [\sigma_{j^{(0)}}, \sigma_{j^{(0)}+k}]$.

(3) The interval $\hat{J}^{(0)}$ can now be written as the union of k grid intervals
$$\hat{J}^{(0)} = \bigcup_{\ell=0}^{k-1} [\sigma_{j^{(0)}+\ell}, \sigma_{j^{(0)}+\ell+1}] \qquad \text{with } j^{(0)} \text{ as above.}$$
Define the (*periodic*) *characteristic interval* $\hat{J} = \hat{J}(\sigma_{i_0})$ to be one of the above k intervals that has maximal length.

3.2. L^p norms of $\hat{g}$

Proposition 3.6. *Let $n \ge 2k+2$. Then*
$$\|\hat{g}\|_p \sim |\hat{J}|^{1/p-1}, \qquad 1 \le p \le \infty.$$

Proof. We can arrange the periodic point sequence $(\sigma_j)_{j=0}^{n-1}$ so that $\sigma_0 > 0$ and $i_0 = \lfloor n/2 \rfloor$. Corresponding to this sequence, we define a non-periodic sequence $(\tau_j)_{j=-k}^{n+k-1}$ by $\tau_j = \sigma_j$ for $j \in \{0, \dots, n-1\}$, $\tau_{-k} = \dots = \tau_{-1} = 0$ and $\tau_n = \dots = \tau_{n+k-1} = 1$. With this choice and the assumption $n \ge 2k+2$, the conditions $i_0 \ge k$ and $i_0 \le n-k-1$ are satisfied. Therefore, by comparing (2.8) with (3.2), we get $\alpha_j = \hat{\alpha}_j$ for $i_0 - k \le j \le i_0$, which yields
$$\hat{g} = \sum_{j=i_0-k}^{i_0} \hat{\alpha}_j \hat{N}_j^*, \qquad g = \sum_{j=i_0-k}^{i_0} \hat{\alpha}_j N_j^*.$$

Also, comparing the two definitions of J and $\hat{J}$, in the present case we see that $|J| = |\hat{J}|$, and thus we use B-spline stability to get

$$\|\hat{g}\|_p^p \sim \sum_{j=i_0-k}^{i_0} |\hat{\alpha}_j|^p |\operatorname{supp} N_j|^{1-p} \sim \|g\|_p^p \sim |\hat{J}|^{p-1},$$

where the last equivalence follows from Lemma 2.8. ∎

Lemma 3.7. *Let $n \geq 2k+2$. If $\hat{g} = \sum_{i=0}^{n-1} \hat{w}_i \hat{N}_i$, then*

$$|\hat{w}_i| \lesssim q^{\hat{d}(i,i_0)} \max_{i_0-k\leq j\leq i_0} \frac{1}{\max(|\operatorname{supp} \hat{N}_i|, |\operatorname{supp} \hat{N}_j|)}$$

where we take the index j modulo n and $\hat{d}$ is the periodic distance function on $\{0,\ldots,n-1\}$.

Proof. By looking at formula (3.2), we see that $|\hat{\alpha}_j| \leq 1$ for all j, and therefore, by Proposition 3.3,

$$|w_i| = \Big|\sum_{j=i_0-k}^{i_0} \hat{\alpha}_j \hat{a}_{ij}\Big| \lesssim \sum_{j=i_0-k}^{i_0} |\hat{a}_{ij}| \lesssim \sum_{j=i_0-k}^{i_0} \frac{q^{\hat{d}(i,j)}}{\max(|\operatorname{supp} \hat{N}_i|, |\operatorname{supp} \hat{N}_j|)}.$$

This readily implies the assertion. ∎

Proposition 3.8. *There exists an index $N(k)$ that depends only on k such that for all partitions $\hat{\mathcal{T}}$ with $n \geq N(k)$, we have*

$$\|\hat{g}\|_{L^p(\hat{J})} \gtrsim |\hat{J}|^{1/p-1}, \qquad p \in [1,\infty].$$

Proof. Assuming again that $i_0 = \lfloor n/2 \rfloor$ and $n \geq 2k+2$, we begin by considering the difference between the periodic function $\hat{g}$ and the non-periodic function g corresponding to the partition $\mathcal{T} = (\tau_j)_{j=-k}^{n+k-1}$ with $\tau_j = \sigma_j$ for $j \in \{0,\ldots,n-1\}$, $\tau_{-k} = \cdots = \tau_{-1} = 0$ and $\tau_n = \cdots = \tau_{n+k-1} = 1$:

$$u := g - \hat{g} = \sum_{j=-k}^{n-1} \beta_j N_j^*,$$

where the coefficients β_j are so chosen that this equation is true. This is possible since both g and $\hat{g}$ are contained in the linear span of the functions N_j^*. By defining the set of boundary indices B in $\mathcal{T}$ by

$$B = \{-k,\ldots,-1\} \cup \{n-k,\ldots,n-1\} \subset \{-k,\ldots,n-1\},$$

we see that for $j \in B^c$,

$$\beta_j = \langle u, N_j\rangle = \langle g - \hat{g}, N_j\rangle = \langle g, N_j\rangle - \langle \hat{g}, \hat{N}_j\rangle = \alpha_j - \hat{\alpha}_j = 0,$$

where the last equality follows from the fact that $\alpha_j = \hat{\alpha}_j$ for all indices j in our current definition of $\mathcal{T}$. Therefore, $u = g - \hat{g}$ can be expressed as

$$u = \sum_{j\in B} \beta_j N_j^*. \tag{3.3}$$

Now, we estimate the coefficients β_j for $j \in B$ by Lemma 3.7:

$$\begin{aligned}
|\beta_j| &= |\langle g - \hat{g}, N_j\rangle| = |\langle \hat{g}, N_j\rangle| \\
&= \Big| \sum_{i=0}^{n-1} \hat{w}_i \langle \hat{N}_i, N_j\rangle \Big| \lesssim \sum_{i=0}^{n-1} |\hat{w}_i| \cdot |\operatorname{supp} \hat{N}_i \cap \operatorname{supp} N_j| \\
&\lesssim \sum_{i=0}^{n-1} q^{\hat{d}(i,i_0)} \max_{m=i_0-k}^{i_0} \frac{1}{\max(|\operatorname{supp} \hat{N}_i|, |\operatorname{supp} \hat{N}_m|)} \cdot |\operatorname{supp} \hat{N}_i \cap \operatorname{supp} N_j| \\
&\le \sum_{i:\, |\operatorname{supp} \hat{N}_i \cap \operatorname{supp} N_j| > 0} q^{\hat{d}(i,i_0)},
\end{aligned}$$

and since $j \in B = \{-k, \dots, -1\} \cup \{n-k, \dots, n-1\}$, we have

$$|\beta_j| \lesssim q^{\hat{d}(0,i_0)} \lesssim q^{n/2}, \qquad j \in B. \tag{3.4}$$

So, we estimate for $x \in \hat{J}$:

$$\begin{aligned}
|u(x)| &= \Big| \sum_{j \in B} \beta_j N_j^*(x) \Big| = \Big| \sum_{j \in B} \beta_j \sum_{i=-k}^{n-1} a_{ij} N_i(x) \Big| \\
&= \Big| \sum_{j \in B} \beta_j \sum_{i:\, \hat{J} \subset \operatorname{supp} N_i} a_{ij} N_i(x) \Big| \\
&\lesssim \sum_{j \in B} |\beta_j| \max_{i:\, \hat{J} \subset \operatorname{supp} N_i} |a_{ij}|.
\end{aligned}$$

Hence, by (3.4) and the estimate in Theorem 2.5 for the *non-periodic* matrix (a_{ij}),

$$|u(x)| \lesssim q^{n/2} \max_{i:\, \hat{J} \subset \operatorname{supp} N_i} \max_{j \in B} \frac{q^{|i-j|}}{h_{ij}},$$

where $h_{ij} = |\operatorname{conv}(\operatorname{supp} N_i \cup \operatorname{supp} N_j)|$. Since $\hat{J} \subset \operatorname{supp} N_i$ for the above indices i, we have $h_{ij} \ge |\hat{J}| = |J|$, and therefore

$$|u(x)| = |(g - \hat{g})(x)| \lesssim q^n |J|^{-1}.$$

This means that on J, we can estimate $\hat{g}$ from below: if $x \in J$ is such that $|g(x)| \ge \|g\|_{L^\infty(J)}/2$, then $|g(x)| \gtrsim |J|^{-1}$ by Lemma 2.8 and we get

$$\begin{aligned}
|\hat{g}(x)| &= |g(x) - (g(x) - \hat{g}(x))| \ge |g(x)| - |g(x) - \hat{g}(x)| \\
&\ge C_1 |J|^{-1} - C_2 |J|^{-1} q^n,
\end{aligned}$$

where C_1 and C_2 are constants that only depend on k and $q < 1$. So there exists an index $N(k)$ such that for all $n \ge N(k)$,

$$\|\hat{g}\|_{L^\infty(\hat{J})} \gtrsim |\hat{J}|^{-1}.$$

Since $\hat{g}$ is a polynomial on $\hat{J}$, by Corollary 2.2 we now get, for any $p \in [1, \infty]$,

$$\|\hat{g}\|_{L^p(\hat{J})} \gtrsim |\hat{J}|^{1/p-1},$$

which is the assertion. ■

3.3. More estimates for $\hat{g}$. We now change our point of view slightly and compare the function $\hat{g}$ with a non-periodic function g where we shift the sequence $\hat{\mathcal{T}} = (\sigma_j)_{j=0}^{n-1}$ in such a way that we split a largest grid point interval in the middle:

$$\sigma_0 = 1 - \sigma_{n-1} = \frac{1}{2} \max_{0 \le j \le n-1} (\sigma_j - \sigma_{j-1}),$$

and, as before, choose $\mathcal{T} = (\tau_j)_{j=-k}^{n+k-1}$ such that $\tau_j = \sigma_j$ for $j \in \{0, \dots, n-1\}$ so that

$$\tau_0 - \tau_{-1} = \tau_n - \tau_{n-1} = \frac{1}{2} \max_{0 \le j \le n-1} (\sigma_j - \sigma_{j-1}).$$

We refer to this choice of $\mathcal{T}$ as the *maximal splitting* of $\hat{\mathcal{T}}$. Similar to the above, we define $\widetilde{\hat{\mathcal{T}}}$ and $\widetilde{\mathcal{T}}$ to be the partitions $\hat{\mathcal{T}}$ and $\mathcal{T}$ respectively with the grid points σ_{i_0} and τ_{i_0} removed.

If we work under this assumption, it is not necessarily the case that $|J| = |\hat{J}|$ as it can happen that J lies near τ_0 or τ_n, but we have

PROPOSITION 3.9. *Let J be the characteristic interval corresponding to the point sequences $(\mathcal{T}, \widetilde{\mathcal{T}})$ and let $\hat{J}$ be the periodic characteristic interval corresponding to $(\hat{\mathcal{T}}, \widetilde{\hat{\mathcal{T}}})$ with the above maximal splitting. Then*

$$|J| \sim |\hat{J}|.$$

Proof. Definitions 2.7 and 3.5 yield

$$|J| \sim \min_{i_0-k \le j \le i_0} |\operatorname{supp} N_j|, \qquad |\hat{J}| \sim \min_{i_0-k \le j \le i_0} |\operatorname{supp} \hat{N}_j|, \tag{3.5}$$

where the periodic indices are interpreted in the sense of the usual periodic continuation of subindices. Then the very definition of the point sequence $\mathcal{T}$ in terms of $\hat{\mathcal{T}}$ implies

$$|\operatorname{supp} N_j| \le |\operatorname{supp} \hat{N}_j|, \qquad -k \le j \le n-1,$$

so, in combination with (3.5), we get $|J| \lesssim |\hat{J}|$. To show the converse inequality, we show

$$\min_{i_0-k \le j \le i_0} |\operatorname{supp} \hat{N}_j| \lesssim \min_{i_0-k \le j \le i_0} |\operatorname{supp} N_j|. \tag{3.6}$$

We assume that j_0 is an index such that $|\operatorname{supp} N_{j_0}| = \min_{i_0-k \le j \le i_0} |\operatorname{supp} N_j|$. If $j_0 \notin B = \{-k, \dots, -1\} \cup \{n-k, \dots, n-1\}$, we even have $|\operatorname{supp} N_{j_0}| =$

$|\operatorname{supp} \hat{N}_{j_0}|$. If $j_0 \in B$, then due to the choice of the maximal splitting,

$$|\operatorname{supp} N_{j_0}| \geq \frac{1}{2} \max_{0 \leq j \leq n-1} (\sigma_{j+1} - \sigma_j) \geq \frac{1}{2k} |\operatorname{supp} \hat{N}_j|$$

for all indices j. So, in particular, (3.6) holds. Thus we have shown the converse inequality $|\hat{J}| \lesssim |J|$ as well and the proof is complete. ∎

We also have the following relation between the dual B-spline coefficients of g and $\hat{g}$:

PROPOSITION 3.10. *For the maximal splitting, there exists a constant* $c \sim 1$ *such that for all* $j \notin B$,

$$\alpha_j = c \cdot \hat{\alpha}_j.$$

Proof. Comparing the recursion formulas (2.9) for α_j and (3.1) for $\hat{\alpha}_j$, we see that for $j \in \{i_0 - k, \ldots, i_0 - 1\}$,

$$\frac{\hat{\alpha}_{j+1}}{\hat{\alpha}_j} = \frac{\alpha_{j+1}}{\alpha_j}, \qquad \{j, j+1\} \subset B^c, \tag{3.7}$$

since by definition $\tau_i = \sigma_i$ for $0 \leq i \leq n-1$. So, now take an arbitrary $j \in B^c$. Looking at the formulas for α_j and $\hat{\alpha}_j$ we write

$$\begin{aligned} \frac{\hat{\alpha}_j}{\alpha_j} = &\Bigg(\prod_{\ell = i_0 - k + 1}^{j-1} \frac{\sigma_{i_0} - \sigma_\ell}{\tau_{i_0} - \tau_\ell} \Bigg) \Bigg(\prod_{\ell = i_0 - k + 1}^{j-1} \frac{\tau_{\ell+k} - \tau_\ell}{\sigma_{\ell+k} - \sigma_\ell} \Bigg) \\ &\cdot \Bigg(\prod_{\ell = j+1}^{i_0 - 1} \frac{\sigma_{\ell+k} - \sigma_{i_0}}{\tau_{\ell+k} - \tau_{i_0}} \Bigg) \Bigg(\prod_{\ell = j+1}^{i_0 - 1} \frac{\tau_{\ell+k} - \tau_\ell}{\sigma_{\ell+k} - \sigma_\ell} \Bigg). \end{aligned}$$

Note that for every $s, t \in \{i_0 - k + 1, \ldots, i_0 + k - 1\}$ such that $0 < s - t \leq k$ either $\sigma_s - \sigma_t = \tau_s - \tau_t$ or $\sigma_s - \sigma_t > \tau_s - \tau_t$, and the latter can only happen when $[\tau_{-1}, \tau_0]$ or $[\tau_{n-1}, \tau_n]$ is a subset of $[\tau_t, \tau_s]$, so

$$\sigma_s - \sigma_t \geq \tau_s - \tau_t \geq \frac{1}{2} \max_{0 \leq j \leq n-1} (\sigma_{j+1} - \sigma_j) \geq \frac{1}{2k} (\sigma_s - \sigma_t).$$

Hence we obtain $\sigma_s - \sigma_t \sim \tau_s - \tau_t$. Therefore $\alpha_j \sim \hat{\alpha}_j$. This in combination with (3.7) proves the proposition. ∎

PROPOSITION 3.11. *Let* $x \in [\sigma_\ell, \sigma_{\ell+1}]$. *Then there exists an interval* $C = C(x) \subset \mathbb{T}$ *which is minimal (with respect to inclusion) with*

$$\hat{J} \cup [\sigma_\ell, \sigma_{\ell+1}] \subset C$$

such that if $K(C)$ *is the number of grid points of* $\hat{\mathcal{T}}$ *contained in* C, *then*

$$|\hat{g}(x)| \lesssim \frac{\hat{q}^{K(C)}}{|C|},$$

where $\hat{q} \in (0,1)$ *depends only on* k.

Proof. In order to estimate $\hat{g}$, we consider the difference $u := g - c \cdot \hat{g}$ and g separately, where g is the orthogonal spline function corresponding to $(\widetilde{\mathcal{T}}, \mathcal{T})$ that arises from the maximal splitting and $c \sim 1$ denotes the constant from Proposition 3.10. We can write

$$u = \sum_{j=-k}^{n-1} \beta_j N_j^*$$

for some coefficients β_j. This is possible since g and $\hat{g}$ are in the linear span of $(N_j^*)_{j=-k}^{n-1}$. For $j \notin B = \{-k, \ldots, -1\} \cup \{n-k, \ldots, n-1\}$, we calculate

$$\beta_j = \langle g - c \cdot \hat{g}, N_j \rangle = \langle g, N_j \rangle - c \cdot \langle \hat{g}, \hat{N}_j \rangle = \alpha_j - c \cdot \hat{\alpha}_j = 0,$$

where the last equality follows from Proposition 3.10. Therefore, the function $u = g - c \cdot \hat{g}$ can be expressed as

$$u = \sum_{j \in B} \beta_j N_j^* \tag{3.8}$$

and its coefficients β_j can be estimated by

$$\begin{aligned}
|\beta_j| &= |\langle g - c\hat{g}, N_j \rangle| \lesssim |\langle g, N_j \rangle| + |\langle \hat{g}, N_j \rangle| \\
&= \Big| \sum_{i=-k}^{n-1} w_i \langle N_i, N_j \rangle \Big| + \Big| \sum_{i=0}^{n-1} \hat{w}_i \langle \hat{N}_i, N_j \rangle \Big| =: \varSigma_1 + \varSigma_2.
\end{aligned}$$

Now, by using (2.10) and the fact that $j \in B$,

$$\begin{aligned}
\varSigma_1 &\le \sum_{i=-k}^{n-1} |w_i| \, |\operatorname{supp} N_i \cap \operatorname{supp} N_j| \\
&\lesssim \sum_{i=-k}^{n-1} \frac{q^{d_{\mathcal{T}}(\tau_i)}}{|J| + \operatorname{dist}(\operatorname{supp} N_i, J) + |\operatorname{supp} N_i|} |\operatorname{supp} N_i \cap \operatorname{supp} N_j| \\
&\le \sum_{i \colon |\operatorname{supp} N_i \cap \operatorname{supp} N_j| > 0} q^{d(i,i_0)} \le q^{\hat{d}(0,i_0)}.
\end{aligned}$$

The term $\varSigma_2$ is estimated similarly by using Lemma 3.7:

$$\begin{aligned}
\varSigma_2 &\le \sum_{i=0}^{n-1} |\hat{w}_i| \cdot |\operatorname{supp} \hat{N}_i \cap \operatorname{supp} N_j| \\
&\lesssim \sum_{i=0}^{n-1} q^{\hat{d}(i,i_0)} \max_{i_0 - k \le m \le i_0} \frac{1}{\max(|\operatorname{supp} \hat{N}_i|, |\operatorname{supp} \hat{N}_m|)} \cdot |\operatorname{supp} \hat{N}_i \cap \operatorname{supp} N_j| \\
&\le \sum_{i \colon |\operatorname{supp} \hat{N}_i \cap \operatorname{supp} N_j| > 0} q^{\hat{d}(i,i_0)} \lesssim q^{\hat{d}(0,i_0)}.
\end{aligned}$$

Combining the estimates for Σ_1 and Σ_2, we get $|\beta_j| \lesssim q^{\hat{d}(0,i_0)}$. Consequently, for $x \in [\tau_\ell, \tau_{\ell+1})$,

$$\begin{aligned}|u(x)| &= \Big|\sum_{j\in B} \beta_j N_j^*(x)\Big| = \Big|\sum_{j\in B} \beta_j \sum_{i=-k}^{n-1} a_{ij} N_i(x)\Big| = \Big|\sum_{j\in B} \beta_j \sum_{i=\ell-k+1}^{\ell} a_{ij} N_i(x)\Big| \\ &\le \sum_{j\in B} |\beta_j| \max_{i=\ell-k+1}^{\ell} |a_{ij}|.\end{aligned}$$

By the above calculation and the estimate for the *non-periodic* Gram matrix inverse in Theorem 2.5, we get

$$|u(x)| \lesssim q^{\hat{d}(0,i_0)} \max_{i=\ell-k+1}^{\ell} \max_{j\in B} \frac{q^{|i-j|}}{h_{ij}},$$

where $h_{ij} = |\mathrm{conv}(\operatorname{supp} N_i \cup \operatorname{supp} N_j)|$. Since for $j \in B$, either $h_{ij} \ge \tau_0 - \tau_{-1}$ or $h_{ij} \ge \tau_n - \tau_{n-1}$, the defining property of the maximal splitting yields $h_{ij} \ge \frac{1}{2}\max_{0\le m\le n-1}(\sigma_m - \sigma_{m-1})$, and therefore

$$|u(x)| \lesssim \frac{q^{\hat{d}(i_0,\ell)}}{\max_m(\sigma_m - \sigma_{m-1})}, \tag{3.9}$$

since also $\hat{d}(i_0,\ell) \le \hat{d}(i_0,0) + \hat{d}(0,\ell) \le \hat{d}(i_0,0) + 2k + \min_{j\in B, \ell-k+1\le i\le \ell} |i-j|$. Thus, by Lemma 2.8 and (3.9),

$$\begin{aligned}|\hat{g}(x)| &\le c^{-1}|g(x)| + |\hat{g}(x) - c^{-1}g(x)| \\ &\lesssim \frac{q^{d(i_0,\ell)}}{|\mathrm{conv}([\tau_\ell, \tau_{\ell+1}] \cup J)|} + \frac{q^{\hat{d}(i_0,\ell)}}{\max_m(\sigma_m - \sigma_{m-1})},\end{aligned}$$

which, with the use of Proposition 3.9 and the definitions of the characteristic intervals J and $\hat{J}$, finishes the proof. ■

So, by defining the normalized orthonormal spline function $\hat{f} = \hat{g}/\|\hat{g}\|_2$, we immediately obtain

COROLLARY 3.12. *Let U be an arbitrary subset of $\mathbb{T}$. Then*

$$\int_U |\hat{f}(x)|^p \,\mathrm{d}x \lesssim |\hat{J}|^{p/2} \sum_{\ell:\, [\sigma_\ell, \sigma_{\ell+1}]\cap U \ne \emptyset} \frac{\hat{q}^{pK(C(\sigma_\ell))}}{|C(\sigma_\ell)|^p} |U \cap [\sigma_\ell, \sigma_{\ell+1}]|$$

where $\hat{q} \in (0,1)$ depends only on k.

We will also need the pointwise estimate of the maximal spline projection operator by the Hardy–Littlewood maximal function in the periodic case, which is true in (the non-periodic case is Theorem 2.6):

THEOREM 3.13. *Let $\hat{P}$ be the orthogonal projection onto $\hat{\mathcal{S}}_{\hat{\mathcal{T}}}$. Then*

$$|\hat{P}h(t)| \lesssim \hat{\mathcal{M}}h(t), \qquad h \in L^1(\mathbb{T}),$$

where $\hat{\mathcal{M}}h(t) = \sup_{I\ni t}|I|^{-1}\int_I |h(y)|\,\mathrm{d}y$ *is the periodic Hardy–Littlewood maximal function operator and the* sup *is taken over all intervals* $I \subset \mathbb{T}$ *containing the point* t.

Proof. Let h be such that $\operatorname{supp} h \subset [\sigma_\ell, \sigma_{\ell+1}]$ for some $\ell \in \{0,\dots,n-1\}$. The first thing we show is that for any index r,

$$\|\hat{P}h\|_{L^1[\sigma_r,\sigma_{r+1}]} \lesssim q^{\hat{d}(r,\ell)}\|h\|_{L^1}.$$

For this we write, when $t \in [\sigma_r, \sigma_{r+1}]$,

$$\hat{P}h(t) = \sum_{j:\operatorname{supp}\hat{N}_j\ni t}\ \sum_{i:\operatorname{supp}\hat{N}_i\supset[\sigma_\ell,\sigma_{\ell+1}]} \hat{a}_{ij}\langle h, \hat{N}_i\rangle \hat{N}_j(t).$$

After using Proposition 3.3 and a simple Hölder, this is less than

$$\|h\|_{L^1}\cdot \sum_{j:\operatorname{supp}\hat{N}_j\ni t}\ \sum_{i:\operatorname{supp}\hat{N}_i\supset[\sigma_\ell,\sigma_{\ell+1}]} \frac{q^{\hat{d}(i,j)}}{\max(|\operatorname{supp}\hat{N}_i|, |\operatorname{supp}\hat{N}_j|)}\hat{N}_j(t).$$

Integrating this estimate over $[\sigma_r, \sigma_{r+1}]$, we get

$$\|\hat{P}h\|_{L^1[\sigma_r,\sigma_{r+1}]} \lesssim \|h\|_{L^1} q^{\hat{d}(\ell,r)}. \tag{3.10}$$

The same can be proved for the non-periodic projection operator P, since we can use the same estimates.

Now, we take an arbitrary function h and localize it by setting

$$h_\ell = h \cdot \mathbb{1}_{[\sigma_\ell,\sigma_{\ell+1}]}.$$

We fix a point $t \in [\sigma_m, \sigma_{m+1}]$ and associate to $\hat{P}$ the non-periodic projection operator P corresponding to the maximal splitting. Then

$$\hat{P}h(t) = Ph(t) + \big(\hat{P}h(t) - Ph(t)\big). \tag{3.11}$$

In order to show $\hat{P}h(t) \lesssim \hat{\mathcal{M}}h(t)$, we first recall that Theorem 2.6 yields $|Ph(t)| \lesssim \mathcal{M}h(t) \le \hat{\mathcal{M}}h(t)$. For the second term $(\hat{P} - P)h$, we write

$$(\hat{P} - P)h = \sum_{\ell=0}^{n-1}(\hat{P} - P)h_\ell$$

and prove an estimate for $g_\ell(t) := (\hat{P} - P)h_\ell$. Observe that

$$g_\ell(t) = \sum_{i\in B}\langle g_\ell, N_i\rangle N_i^*(t) = \sum_{j=m-k+1}^{m}\sum_{i\in B} a_{ij}\langle g_\ell, N_i\rangle N_j(t),$$

since the range of both $\hat{P}$ and P is contained in the linear span of the functions N_i^* and $h_\ell - \hat{P}h_\ell$ and $h_\ell - Ph_\ell$ are both orthogonal to the span of N_i, $i \notin B$. Therefore, by using Theorem 2.5 for a_{ij},

$$|g_\ell(t)| \lesssim \sum_{j=m-k+1}^{m} \sum_{i\in B} \frac{q^{|i-j|}}{h_{ij}} \|g_\ell\|_{L^1(\operatorname{supp} N_i)}.$$

Consequently, by (3.10) and its non-periodic counterpart,

$$|g_\ell(t)| \lesssim \sum_{j=m-k+1}^{m} \sum_{i\in B} \frac{q^{|i-j|}}{h_{ij}} q^{\hat{d}(i,\ell)} \|h_\ell\|_{L^1}. \tag{3.12}$$

Since we have performed the maximal splitting for our periodic partition, we get

$$h_{ij} \geq \frac{1}{2} \max_\nu(\sigma_\nu - \sigma_{\nu-1}), \qquad i \in B.$$

Denoting by $C_{\ell m}$ the convex set that contains $[\sigma_\ell, \sigma_{\ell+1}] \cup [\sigma_m, \sigma_{m+1}]$ and has the minimal number of grid points, we get

$$h_{ij} \gtrsim \frac{|C_{\ell m}|}{\hat{d}(\ell, m)}, \qquad i \in B.$$

Thus, we estimate (3.12) by

$$\sum_{j=m-k+1}^{m} \sum_{i\in B} q^{|i-j|+\hat{d}(i,\ell)} \hat{d}(\ell, m) \frac{\|h\|_{L^1[\sigma_\ell,\sigma_{\ell+1}]}}{|C_{m\ell}|}$$
$$\lesssim \hat{d}(\ell, m) \max_{i\in B}(q^{|i-m|+\hat{d}(i,\ell)}) \cdot \hat{\mathcal{M}}h(t)$$

for all $t \in [\sigma_m, \sigma_{m+1}]$. By the triangle inequality, $\hat{d}(\ell, m) \leq \hat{d}(i, m) + \hat{d}(i, \ell) \leq |i - m| + \hat{d}(i, \ell)$ and thus we can estimate further

$$|g_\ell(t)| \lesssim \max_{i\in B} \alpha^{|i-m|+\hat{d}(i,\ell)} \hat{\mathcal{M}}h(t),$$

where α can be chosen as $q^{1/2}$. Summing this over ℓ, we finally obtain

$$|(\hat{P} - P)h(t)| \lesssim \alpha^{\hat{d}(0,m)} \hat{\mathcal{M}}h(t) \leq \hat{\mathcal{M}}h(t),$$

which in combination with (3.11) and the result for $Ph(t)$ yields the assertion of the theorem. ■

3.4. Combinatorics of characteristic intervals. Similarly to the non-periodic case we can analyze the combinatorics of subsequent characteristic intervals. Let $(s_n)_{n=1}^\infty$ be an admissible sequence of points in $\mathbb{T}$ and $(\hat{f}_n)_{n=1}^\infty$ be the corresponding periodic orthonormal spline functions of order k. For $n \geq 1$, the partitions $\hat{\mathcal{T}}_n$ associated to $\hat{f}_n$ are defined to consist of the grid points $(s_j)_{j=1}^n$ and we enumerate them as

$$\hat{\mathcal{T}}_n = (0 \leq \sigma_{n,0} \leq \cdots \leq \sigma_{n,n-1} < 1).$$

If $n \geq 2k$, we denote by $\hat{J}_n^{(0)}$ and $\hat{J}_n$ the characteristic intervals $\hat{J}^{(0)}$ and $\hat{J}$ from Definition 3.5 associated to the new grid point s_n. For any $x \in \mathbb{T}$, let $C_n(x)$ be the interval from Proposition 3.11 associated to $\hat{J}_n$. We define $\hat{d}_n(x)$ to be the number of grid points in $\hat{\mathcal{T}}_n$ between x and $\hat{J}_n$ contained in $C_n(x)$ counting x and the endpoints of $\hat{J}_n$. Moreover, for a subinterval V of $\mathbb{T}$, we denote $\hat{d}_n(V) = \min_{x\in V} \hat{d}_n(x)$.

An immediate consequence of the definition of $\hat{J}_n$ is that the sequence $(\hat{J}_n)$ of characteristic intervals forms a nested collection of sets, i.e., two sets in it are either disjoint or one is contained in the other.

Since the definition of $\hat{J}_n$ only involves local properties of the point sequence (s_j), and the definition of $\hat{J}_n$ is the same as the definition of J_n for any identification of $\mathbb{T}$ with $[0, 1)$ such that between the newly inserted point s_{i_0} and 0 or 1 there are more than k grid points of $\mathcal{T}_n$, we also get the periodic version of Lemma 2.9.

LEMMA 3.14. *Let V be an arbitrary subinterval of $\mathbb{T}$ and let $\beta > 0$. Then there exists a constant $F_{k,\beta}$ only depending on k and β such that*

$$\operatorname{card}\{n \geq 2k : \hat{J}_n \subseteq V,\ |\hat{J}_n| \geq \beta|V|\} \leq F_{k,\beta}.$$

Additionally, Lemma 3.14 has the following corollary:

COROLLARY 3.15. *Let $(\hat{J}_{n_i})_{i=1}^{\infty}$ be a decreasing sequence of characteristic intervals, i.e. $\hat{J}_{n_{i+1}} \subseteq \hat{J}_{n_i}$. Then there exists a number $\kappa \in (0, 1)$ and a constant C_k, both depending only on k, such that*

$$|\hat{J}_{n_i}| \leq C_k \kappa^i |\hat{J}_{n_1}|, \qquad i \in \mathbb{N}.$$

4. Technical estimates. The lemmas proved in this section are similar to the corresponding results in [10] or [13], and also the proofs are more or less the same. The exception is Lemma 4.4 for which we give a new, shorter proof.

LEMMA 4.1. *Let $N(k)$ be given by Proposition 3.8, $f = \sum_{n\geq N(k)}^{\infty} a_n \hat{f}_n$ and V be a subinterval of $\mathbb{T}$. Then*

$$\int_{V^c} \sum_{j\in\Gamma} |a_j \hat{f}_j(t)|\, \mathrm{d}t \lesssim \int_V \Big(\sum_{j\in\Gamma} |a_j \hat{f}_j(t)|^2\Big)^{1/2} \mathrm{d}t, \tag{4.1}$$

where

$$\Gamma := \{j : \hat{J}_j \subset V \text{ and } N(k) \leq j < \infty\}.$$

Proof. If $|V| = 1$, then (4.1) holds trivially, so assume that $|V| < 1$. For fixed $n \in \Gamma$, Corollary 3.12 and Proposition 3.8 imply

$$\int_{V^c} |\hat{f}_n(t)|\, \mathrm{d}t \lesssim \hat{q}^{\hat{d}_n(V^c)} |\hat{J}_n|^{1/2} \lesssim \hat{q}^{\hat{d}_n(V^c)} \int_{\hat{J}_n} |\hat{f}_n(t)|\, \mathrm{d}t. \tag{4.2}$$

Now choose $\beta = 1/4$ and let $\hat{J}_n^\beta$ be the unique closed interval that satisfies

$$|\hat{J}_n^\beta| = \beta|\hat{J}_n| \quad \text{and} \quad \operatorname{center}(\hat{J}_n^\beta) = \operatorname{center}(\hat{J}_n).$$

Since f_n is a polynomial of order k on the interval J_n, we apply Corollary 2.2 to (4.2) and estimate further

(4.3)

$$\int_{V^c} |a_n \hat{f}_n(t)|\,\mathrm{d}t \lesssim \hat{q}^{\hat{d}_n(V^c)} \int_{\hat{J}_n^\beta} |a_n \hat{f}_n(t)|\,\mathrm{d}t \le \hat{q}^{\hat{d}_n(V^c)} \int_{\hat{J}_n^\beta} \Big(\sum_{j\in\Gamma} |a_j \hat{f}_j(t)|^2\Big)^{1/2} \mathrm{d}t.$$

Define $\Gamma_s := \{j \in \Gamma : \hat{d}_j(V^c) = s\}$ for $s \ge 0$. If $(\hat{J}_{n_j})_{j=1}^N$ is a decreasing sequence of characteristic intervals with $n_j \in \Gamma_s$, we can split $(\hat{J}_{n_j})$ into at most two groups so that the intervals in each group have one endpoint in common.

Lemma 3.14 implies that there exists a constant F_k, only depending on k, such that each point $t \in V$ belongs to at most F_k intervals $\hat{J}_j^\beta$, $j \in \Gamma_s$. Thus, summing over $j \in \Gamma_s$, we see from (4.3) that

$$\begin{aligned}\sum_{j\in\Gamma_s} \int_{V^c} |a_j \hat{f}_j(t)|\,\mathrm{d}t &\lesssim \sum_{j\in\Gamma_s} \hat{q}^s \int_{\hat{J}_j^\beta} \Big(\sum_{\ell\in\Gamma} |a_\ell \hat{f}_\ell(t)|^2\Big)^{1/2} \mathrm{d}t \\ &\lesssim \hat{q}^s \int_V \Big(\sum_{\ell\in\Gamma} |a_\ell \hat{f}_\ell(t)|^2\Big)^{1/2} \mathrm{d}t.\end{aligned}$$

Finally, we sum over $s \ge 0$ to obtain inequality (4.1). ∎

Let g be a real-valued function defined on the torus $\mathbb{T}$. In the following, we denote by $[g > \lambda]$ the set $\{x \in \mathbb{T} : g(x) > \lambda\}$ for any $\lambda > 0$.

Lemma 4.2. *Let* $f = \sum_{n=1}^\infty a_n \hat{f}_n$ *with only finitely many non-zero coefficients* a_n, $\lambda > 0$, $r < 1$ *and*

$$E_\lambda = [Sf > \lambda], \quad B_{\lambda,r} = [\hat{\mathcal{M}} \mathbb{1}_{E_\lambda} > r],$$

where $Sf(t)^2 = \sum_{n=1}^\infty a_n^2 \hat{f}_n(t)^2$ *is the spline square function. Then*

$$E_\lambda \subset B_{\lambda,r}.$$

Proof. Fix $t \in E_\lambda$. The square function $Sf = (\sum_{n=1}^\infty |a_n \hat{f}_n|^2)^{1/2}$ is continuous except possibly at finitely many grid points, where Sf is at least continuous from one side. As a consequence, for $t \in E_\lambda$, there exists an interval $I \subset E_\lambda$ such that $t \in I$. This implies

$$\begin{aligned}(\hat{\mathcal{M}} \mathbb{1}_{E_\lambda})(t) &= \sup_{t\ni U} |U|^{-1} \int_U \mathbb{1}_{E_\lambda}(x)\,\mathrm{d}x \\ &= \sup_{t\ni U} \frac{|E_\lambda \cap U|}{|U|} \ge \frac{|E_\lambda \cap I|}{|I|} = \frac{|I|}{|I|} = 1 > r,\end{aligned}$$

so $t \in B_{\lambda,r}$, proving the lemma. ∎

LEMMA 4.3. *Let $f = \sum_{n \geq N(k)} a_n \hat{f}_n$ with only finitely many non-zero coefficients a_n, $\lambda > 0$ and $r < 1$, where $N(k)$ is given by Proposition 3.8. Define*

$$E_\lambda := [Sf > \lambda], \qquad B_{\lambda,r} := [\hat{\mathcal{M}} \mathbb{1}_{E_\lambda} > r],$$

where $Sf(t)^2 = \sum_{n \geq N(k)} a_n^2 \hat{f}_n(t)^2$ is the spline square function. If

$$\Lambda = \{n : \hat{J}_n \not\subset B_{\lambda,r} \text{ and } N(k) \leq n < \infty\} \quad \text{and} \quad g = \sum_{n \in \Lambda} a_n \hat{f}_n,$$

then

$$\int_{E_\lambda} Sg(t)^2 \,\mathrm{d}t \lesssim_r \int_{E_\lambda^c} Sg(t)^2 \,\mathrm{d}t. \tag{4.4}$$

Proof. First, we observe that if $B_{\lambda,r} = \mathbb{T}$ then Λ is empty and (4.4) holds trivially. So assume $B_{\lambda,r} \neq \mathbb{T}$. By Propositions 3.6 and 3.8,

$$\int_{E_\lambda} Sg(t)^2 \,\mathrm{d}t = \sum_{n \in \Lambda} \int_{E_\lambda} |a_n \hat{f}_n(t)|^2 \,\mathrm{d}t \lesssim \sum_{n \in \Lambda} \int_{\hat{J}_n} |a_n \hat{f}_n(t)|^2 \,\mathrm{d}t.$$

We split the latter expression into

$$I_1 := \sum_{n \in \Lambda} \int_{\hat{J}_n \cap E_\lambda^c} |a_n \hat{f}_n(t)|^2 \,\mathrm{d}t, \quad I_2 := \sum_{n \in \Lambda} \int_{\hat{J}_n \cap E_\lambda} |a_n \hat{f}_n(t)|^2 \,\mathrm{d}t.$$

Clearly,

$$I_1 \leq \sum_{n \in \Lambda} \int_{E_\lambda^c} |a_n \hat{f}_n(t)|^2 \,\mathrm{d}t = \int_{E_\lambda^c} Sg(t)^2 \,\mathrm{d}t. \tag{4.5}$$

To estimate I_2, we first observe that $E_\lambda \subset B_{\lambda,r}$ by Lemma 4.2. Since the set $B_{\lambda,r} = [\hat{\mathcal{M}} \mathbb{1}_{E_\lambda} > r]$ is open in $\mathbb{T}$, we decompose it into a countable collection $(V_j)_{j=1}^\infty$ of disjoint open subintervals of $\mathbb{T}$. Utilizing this decomposition, we estimate

$$I_2 \leq \sum_{n \in \Lambda} \sum_{j : |\hat{J}_n \cap V_j| > 0} \int_{\hat{J}_n \cap V_j} |a_n \hat{f}_n(t)|^2 \,\mathrm{d}t. \tag{4.6}$$

If $n \in \Lambda$ and $|\hat{J}_n \cap V_j| > 0$, then, by definition of Λ, $\hat{J}_n$ is an interval containing at least one endpoint x of V_j for which

$$\hat{\mathcal{M}} \mathbb{1}_{E_\lambda}(x) \leq r.$$

This implies

$$|E_\lambda \cap \hat{J}_n \cap V_j| \leq r \cdot |\hat{J}_n \cap V_j| \quad \text{or equivalently} \quad |E_\lambda^c \cap \hat{J}_n \cap V_j| \geq (1-r) \cdot |\hat{J}_n \cap V_j|.$$

Using this inequality and the fact that $|\hat{f}_n|^2$ is a polynomial of order $2k - 1$

on $\hat{J}_n$ allows us to use Corollary 2.2 to deduce from (4.6) that

$$\begin{aligned} I_2 &\lesssim_r \sum_{n\in\Lambda} \sum_{j:\,|\hat{J}_n\cap V_j|>0} \int_{E_\lambda^c\cap\hat{J}_n\cap V_j} |a_n\hat{f}_n(t)|^2\,\mathrm{d}t \\ &\le \sum_{n\in\Lambda} \int_{E_\lambda^c\cap\hat{J}_n\cap B_{\lambda,r}} |a_n\hat{f}_n(t)|^2\,\mathrm{d}t \\ &\le \sum_{n\in\Lambda} \int_{E_\lambda^c} |a_n\hat{f}_n(t)|^2\,\mathrm{d}t = \int_{E_\lambda^c} Sg(t)^2\,\mathrm{d}t. \end{aligned}$$

The latter inequality combined with (4.5) completes the proof. ∎

Lemma 4.4. *Let V be an open subinterval of $\mathbb{T}$ and $f = \sum_n \hat{a}_n\hat{f}_n \in L^p(\mathbb{T})$ for $p \in (1,\infty)$ with $\operatorname{supp} f \subset V$. Then there exists a number $R > 1$ depending only on k such that*

$$\sum_n R^{p\hat{d}_n(V)}|\hat{a}_n|^p\|\hat{f}_n\|^p_{L^p(\widetilde{V}^c)} \lesssim_{p,R} \|f\|_p^p, \tag{4.7}$$

with $\widetilde{V}$ being the interval with the same center as V but with three times its diameter.

Proof. We can assume that $|V| \le 1/3$, since otherwise $|\widetilde{V}^c| = 0$ and the left hand side of (4.7) is zero.

We start by estimating $|\hat{a}_n|$. Depending on n, we partition V into intervals $(A_{n,j})_{j=1}^{N_n}$, where except at most two intervals at the boundary of V, we choose $A_{n,j}$ to be a grid point interval in the grid $\hat{\mathcal{T}}_n$. Let $I_{n,\ell} := [\sigma_{n,\ell}, \sigma_{n,\ell+1}]$ be the ℓth grid point interval in $\hat{\mathcal{T}}_n$. Moreover, for a grid point interval I in $\hat{\mathcal{T}}_n$ and all subsets $E \subset I$, we set $C_n(E)$ to be the interval given by Proposition 3.11 that satisfies

$$C_n(E) \supset I \cup \hat{J}_n$$

and we denote by $K_n(C_n(I))$ the number of grid points from $\hat{\mathcal{T}}_n$ that are contained in $C_n(I)$. Next, we define $r_n = \min_{\ell:\, I_{n,\ell}\cap\widetilde{V}^c\neq\emptyset} K_n(C_n(I_{n,\ell}))$, $a_{n,j} = K_n(C_n(A_{n,j}))$ and we choose a number $S > 1$ which we will specify later and estimate by Hölder's inequality (with $p' = p/(p-1)$)

$$\begin{aligned} |\hat{a}_n| = |\langle f, \hat{f}_n\rangle| &= \Big|\sum_{j=1}^{N_n} \int_{A_{n,j}} f(t)\hat{f}_n(t)\,\mathrm{d}t\Big| \\ &\le \sum_{j=1}^{N_n} \Big(\int_{A_{n,j}} |f(t)|^p\,\mathrm{d}t\Big)^{1/p}\Big(\int_{A_{n,j}} |\hat{f}_n(t)|^{p'}\,\mathrm{d}t\Big)^{1/p'} \end{aligned}$$

$$= \sum_{j=1}^{N_n} S^{-a_{n,j}} S^{a_{n,j}} \Big(\int_{A_{n,j}} |f(t)|^p \, \mathrm{d}t \Big)^{1/p} \Big(\int_{A_{n,j}} |\hat{f}_n(t)|^{p'} \, \mathrm{d}t \Big)^{1/p'}$$

$$\leq \Big(\sum_{j=1}^{N_n} S^{-p' a_{n,j}} \Big)^{1/p'} \Big(\sum_{j=1}^{N_n} S^{p a_{n,j}} \int_{A_{n,j}} |f(t)|^p \, \mathrm{d}t \cdot \Big(\int_{A_{n,j}} |\hat{f}_n(t)|^{p'} \, \mathrm{d}t \Big)^{p-1} \Big)^{1/p}.$$

Since the first sum above is a geometric series, and by using Corollary 3.12 for the integral of $\hat{f}_n$, we obtain

$$|\hat{a}_n| \lesssim \Big(\sum_{j=1}^{N_n} S^{p a_{n,j}} \int_{A_{n,j}} |f(t)|^p \, \mathrm{d}t \cdot |\hat{J}_n|^{p/2} \frac{\hat{q}^{p a_{n,j}} |A_{n,j}|^{p-1}}{|C_n(A_{n,j})|^p} \Big)^{1/p}. \tag{4.8}$$

We also estimate $\|\hat{f}_n\|^p_{L^p(\widetilde{V}^c)}$ by applying Corollary 3.12:

$$\|\hat{f}_n\|^p_{L^p(\widetilde{V}^c)} \lesssim |\hat{J}_n|^{p/2} \hat{q}^{p r_n} \sum_{\ell\colon \widetilde{V}^c \cap I_{n,\ell} \neq \emptyset} \frac{|\widetilde{V}^c \cap I_{n,\ell}|}{|C_n(I_{n,\ell})|^p}$$

$$= |\hat{J}_n|^{p/2} \hat{q}^{p r_n} \int_{\widetilde{V}^c} \sum_{\ell\colon \widetilde{V}^c \cap I_{n,\ell} \neq \emptyset} \frac{\mathbb{1}_{I_{n,\ell}}(t)}{|C_n(I_{n,\ell})|^p} \, \mathrm{d}t.$$

By integration of the function $t \mapsto t^{-p}$, this is dominated by

$$\frac{|\hat{J}_n|^{p/2} \hat{q}^{p r_n}}{\min_{\ell\colon \widetilde{V}^c \cap I_{n,\ell} \neq \emptyset} |C_n(I_{n,\ell})|^{p-1}}. \tag{4.9}$$

For every $E \subset \mathbb{T}$, let $\ell_0(E)$ be an index such that $I_{n,\ell_0(E)} \cap E \neq \emptyset$ and

$$|C_n(I_{n,\ell_0(E)})| = \min_{\ell\colon I_{n,\ell} \cap E \neq \emptyset} |C_n(I_{n,\ell})|.$$

Now we introduce one more notation: let $B_n(E) \subset C_n(I_{n,\ell_0(E)})$ be the largest interval B such that $B \cap E = \hat{J}_n \cap E$. Obviously $B_n(E) \supset \hat{J}_n$ for every E. Using this notation, we estimate (4.9) and conclude

$$\|\hat{f}_n\|^p_{L^p(\widetilde{V}^c)} \lesssim \frac{|\hat{J}_n|^{p/2} \hat{q}^{p r_n}}{|B_n(\widetilde{V}^c)|^{p-1}}. \tag{4.10}$$

Combining (4.8) and (4.10) yields

$$\sum_n R^{p \hat{d}_n(V)} |\hat{a}_n|^p \|\hat{f}_n\|^p_{L^p(\widetilde{V}^c)} \lesssim \sum_n |\hat{J}_n|^p \hat{q}^{p r_n} R^{p \hat{d}_n(V)} |B_n(\widetilde{V}^c)|^{1-p}$$

$$\cdot \Big(\sum_{j=1}^{N_n} (\hat{q} S)^{p a_{n,j}} \int_{A_{n,j}} |f(t)|^p \, \mathrm{d}t \cdot \frac{|A_{n,j}|^{p-1}}{|C_n(A_{n,j})|^p} \Big).$$

Since $(A_{n,j})_{j=1}^{N_n}$ is a partition of V for any n, we further write

$$\sum_n R^{p\hat{d}_n(V)}|\hat{a}_n|^p\|\hat{f}_n\|^p_{L^p(\widetilde{V}^c)} \lesssim \int_V \sum_n \left(\frac{|\hat{J}_n|}{|B_n(\widetilde{V}^c)|}\right)^{p-1} \hat{q}^{pr_n} R^{p\hat{d}_n(V)} \times \sum_{j=1}^{N_n} (\hat{q}S)^{pa_{n,j}} \frac{|\hat{J}_n|\,|A_{n,j}|^{p-1}}{|C_n(A_{n,j})|^p} \mathbb{1}_{A_{n,j}}(t)|f(t)|^p\,\mathrm{d}t.$$

To estimate this by $\int_V |f(t)|^p\,\mathrm{d}t$, we will estimate pointwise for fixed $t \in V$. To do this, we first observe that we have to estimate the expression

$$\sum_n \left(\frac{|\hat{J}_n|}{|B_n(\widetilde{V}^c)|}\right)^{p-1} \hat{q}^{pr_n} R^{p\hat{d}_n(V)} (\hat{q}S)^{pa_{n,j(n)}} \frac{|\hat{J}_n|\,|A_{n,j(n)}|^{p-1}}{|C_n(A_{n,j(n)})|^p},$$

where $A_{n,j(n)}$ is just the interval $A_{n,j}$ such that $t \in A_{n,j}$. Next, we split the summation index set into $\bigcup T_s$, where

$$T_s = \{n : r_n + a_{n,j(n)} = s\}.$$

Since $\hat{d}_n(V) \le a_{n,j(n)}$, we see that if $R, S > 1$ with $RS\hat{q} < 1$, then there exists $\alpha < 1$, depending only on k, such that the above expression is $\lesssim$

$$\sum_{s=0}^{\infty} \alpha^s \sum_{n \in T_s} \left(\frac{|\hat{J}_n|}{|B_n(\widetilde{V}^c)|}\right)^{p-1} \frac{|\hat{J}_n|\,|A_{n,j(n)}|^{p-1}}{|C_n(A_{n,j(n)})|^p}. \tag{4.11}$$

Now, we split the analysis of this expression into two cases:

CASE 1: Summing over $n \in T_{s,1} := \{n \in T_s : |B_n(\widetilde{V}^c)| \le |B_n(V)|$ or $|V| \le |\hat{J}_n|\}$. The inner sum in (4.11), taken over $n \in T_{s,1}$, is immediately estimated by

$$\sum_{n \in T_{s,1}} \frac{|\hat{J}_n|\,|A_{n,j(n)}|^{p-1}}{|C_n(A_{n,j(n)})|^p}.$$

To estimate this sum, we further split $T_{s,1}$ into

$$S_1 = \{n \in T_{s,1} : \hat{J}_n \text{ contains at least one of the two endpoints of } V\},$$
$$S_2 = T_{s,1} \setminus S_1.$$

By the conditions of Case 1 and the definition of $\widetilde{V}$, if $n \in S_1$ we have $|\hat{J}_n| \ge |V|$ and a geometric decay in the length of $\hat{J}_n$ by Corollary 3.15, and therefore

$$\sum_{n \in S_1} \frac{|\hat{J}_n|\,|A_{n,j(n)}|^{p-1}}{|C_n(A_{n,j(n)})|^p} \le \sum_{n \in S_1} \frac{|\hat{J}_n|\,|V|^{p-1}}{|C_n(A_{n,j(n)})|^p} \le \sum_{n \in S_1} \left(\frac{|V|}{|\hat{J}_n|}\right)^{p-1} \lesssim 1.$$

Next, observe that under the conditions in Case 1 and the definition of S_2, we have $|\hat{J}_n \cap V| = 0$ for $n \in S_2$. Since additionally $(A_{n,j(n)})$ is a decreasing family of subsets of V and since $r_n + a_{n,j(n)} = s$ for $n \in T_s$, we can split S_2 into $S_{2,1}$ and $S_{2,2}$ such that for any distinct $n_1, n_2 \in S_{2,i}$ for $i \in \{1,2\}$, the corresponding intervals $\hat{J}_{n_1}$ and $\hat{J}_{n_2}$ are either disjoint or share an endpoint.

If $n \in S_{2,i}$ then an endpoint a of $B_n(V)$ coincides with an endpoint of V (since $\hat{J}_n \subset V^c$). In this case, for $t \in B_n(V)$ we let $B_n(t) \subset B_n(V)$ be the interval with endpoints t and a. Let $\hat{J}_n^\beta$ for $\beta = 1/4$ be the interval characterized by the properties

$$\hat{J}_n^\beta \subset \hat{J}_n, \qquad \operatorname{center}(\hat{J}_n^\beta) = \operatorname{center}(\hat{J}_n), \qquad |\hat{J}_n^\beta| = |\hat{J}_n|/4.$$

By Lemma 3.14, for each point $u \in \mathbb{T}$, there exist at most F_k indices in $S_{2,i}$ such that $u \in \hat{J}_n^\beta$. We now enumerate the intervals $\hat{J}_n$ with $n \in S_{2,i}$ in the following way: Since the intervals are nested, we write $\hat{J}_{n_{\ell,1}}$ for the maximal ones and we assume that $\hat{J}_{n_{\ell,j+1}} \subset \hat{J}_{n_{\ell,j}}$ for all j. Since the two intervals $\hat{J}_{n_2} \subset \hat{J}_{n_1}$ for $n_1, n_2 \in S_{2,i}$ have one endpoint in common, for each maximal interval $\hat{J}_{n_{\ell,1}}$ we have at most two sequences of this form.

Using this enumeration, we write

$$\sum_{n \in S_{2,i}} \frac{|A_{n,j(n)}|^{p-1} |\hat{J}_n|}{|C_n(A_{n,j(n)})|^p} \le 2\beta^{-1} |V|^{p-1} \sum_{\ell,j} \int_{\hat{J}^\beta_{n_{\ell,j}}} \frac{\mathrm{d}t}{|B_{n_{\ell,j}}(t)|^p}.$$

Observe that the function

$$x \mapsto |\{\ell, j, t : t \in \hat{J}^\beta_{n_{\ell,j}},\, x = |B_{n_{\ell,j}}(t)|\}|$$

is uniformly bounded by $4F_k$ for all $x \ge 0$. Since we also have the estimate

$$|V|/2 \le |B_n(t)|, \qquad n \in S_{2,i},\, t \in \hat{J}_n^\beta,$$

we conclude that

$$\beta^{-1}|V|^{p-1} \sum_{\ell,j} \int_{\hat{J}^\beta_{n_{\ell,j}}} \frac{\mathrm{d}t}{|B_{n_{\ell,j}}(t)|^p} \le 4F_k \beta^{-1} |V|^{p-1} \int_{|V|/2}^{\infty} \frac{\mathrm{d}x}{x^p} \le C_k$$

where C_k is some constant only depending on k. This finishes the proof in the case $n \in T_{s,1}$.

Case 2: Summing over $T_{s,2} := \{n \in T_s : |B_n(V)| \le |B_n(\widetilde{V}^c)|$ and $|\hat{J}_n| \le |V|\}$. Observe that for $n \in T_{s,2}$ we have $\hat{J}_n \subset \widetilde{V}$. Next, we subdivide $T_{s,2}$ into generations $\mathcal{G}_{s,\ell}$ such that for two indices n_1, n_2 in the same generation, the corresponding characteristic intervals $\hat{J}_{n_1}$ and $\hat{J}_{n_2}$ are disjoint. We observe that the geometric decay of characteristic intervals yields $|\hat{J}_n|/|V| \lesssim \kappa^\ell$ for some $\kappa < 1$ and $n \in \mathcal{G}_{s,\ell}$. Therefore, by introducing $\beta < 1$ such that $\beta(p-1) < 1$ we continue estimating (4.11) by using the inequality

$|V| \lesssim |B_n(\widetilde{V}^c)|$ for $n \in T_{s,2}$,

$$\sum_{n\in T_{s,2}} \Big(\frac{|\hat{J}_n|}{|B_n(\widetilde{V}^c)|}\Big)^{p-1} \frac{|\hat{J}_n|\,|A_{n,j(n)}|^{p-1}}{|C_n(A_{n,j(n)})|^p} \lesssim \sum_{\ell=0}^{\infty} \kappa^{\ell(1-\beta)(p-1)} \sum_{n\in\mathcal{G}_{s,\ell}} \frac{|\hat{J}_n|^{1+\beta(p-1)}|A_{n,j(n)}|^{p-1}}{|V|^{\beta(p-1)}|C_n(A_{n,j(n)})|^p}.$$

We further split $\mathcal{G}_{s,\ell}$ into

$$\mathcal{G}_{s,\ell}^{(1)} = \{n \in \mathcal{G}_{s,\ell} : |C_n(A_{n,j(n)})| \ge 1 - 2|V|\}, \qquad \mathcal{G}_{s,\ell}^{(2)} = \mathcal{G}_{s,\ell} \setminus \mathcal{G}_{s,\ell}^{(1)}.$$

Since $|V| \le 1/3$ and the intervals $\hat{J}_n$ for $n \in \mathcal{G}_{s,\ell}^{(1)}$ are disjoint, we immediately see that $\sum_{n\in\mathcal{G}_{s,\ell}^{(1)}} |\hat{J}_n|\,|A_{n,j(n)}|^{p-1} \le 1$, so we next consider

$$\sum_{n\in\mathcal{G}_{s,\ell}^{(2)}} \frac{|\hat{J}_n|^{1+\beta(p-1)}|A_{n,j(n)}|^{p-1}}{|V|^{\beta(p-1)}|C_n(A_{n,j(n)})|^p}. \tag{4.12}$$

To analyze this expression, we define $C_n'(A_{n,j(n)})$ as $C_n(A_{n,j(n)})$ if $\partial C_n(A_{n,j(n)}) \cap \overline{A_{n,j(n)}} \ne \emptyset$, and as the smallest interval which is a subset of $C_n(A_{n,j(n)})$ that contains $\hat{J}_n$ and $\partial V \cap \overline{A_{n,j(n)}}$ if $\partial C_n(A_{n,j(n)}) \cap \overline{A_{n,j(n)}} = \emptyset$. The canonical case is the first one; the second case can only occur if $A_{n,j(n)}$ is not a grid point interval in grid n, which happens only if $A_{n,j(n)}$ lies at the boundary of V. With this definition, we consider the set of different endpoints of $C_n'(A_{n,j(n)})$ intersecting $\overline{A_{n,j(n)}}$, i.e.,

$$E_{s,\ell} = \{x \in \partial C_n'(A_{n,j(n)}) \cap \overline{A_{n,j(n)}} : n \in \mathcal{G}_{s,\ell}^{(2)}\},$$

enumerate it as the sequence $(x_r)_{r=1}^{\infty}$ which by definition is entirely contained in $\overline{V}$, and split the collection $\mathcal{G}_{s,\ell}^{(2)}$ according to those different endpoints into

$$\mathcal{G}_{s,\ell,r}^{(2)} = \{n \in \mathcal{G}_{s,\ell}^{(2)} : r \text{ is minimal with } x_r \in \partial C_n'(A_{n,j(n)}) \cap \overline{A_{n,j(n)}}\}.$$

If we set $\Lambda_{s,\ell} = \{r : \mathcal{G}_{s,\ell,r}^{(2)} \ne \emptyset\}$, we can write (4.12) as

$$\sum_{r\in\Lambda_{s,\ell}} \sum_{n\in\mathcal{G}_{s,\ell,r}^{(2)}} \frac{|\hat{J}_n|^{1+\beta(p-1)}|A_{n,j(n)}|^{p-1}}{|V|^{\beta(p-1)}|C_n(A_{n,j(n)})|^p} \lesssim \frac{1}{|V|^{\beta(p-1)}} \sum_{r\in\Lambda_{s,\ell}} \sum_{n\in\mathcal{G}_{s,\ell,r}^{(2)}} \frac{|\hat{J}_n|}{|C_n'(A_{n,j(n)})|^{1-\beta(p-1)}}.$$

Since the $\hat{J}_n$'s in the above sum are disjoint, $\hat{J}_n \subset \widetilde{V}$ and x_r is an endpoint of $C_n'(A_{n,j(n)})$ for all $n \in \mathcal{G}^{(2)}_{s,\ell,r}$, we can estimate

$$\frac{1}{|V|^{\beta(p-1)}} \sum_{r\in\Lambda_{s,\ell}} \sum_{n\in\mathcal{G}^{(2)}_{s,\ell,r}} \frac{|\hat{J}_n|}{|C_n'(A_{n,j(n)})|^{1-\beta(p-1)}} \lesssim \frac{1}{|V|^{\beta(p-1)}} \sum_{r\in\Lambda_{s,\ell}} \int_0^{2|V|} \frac{1}{t^{1-\beta(p-1)}}\,\mathrm{d}t \lesssim |\Lambda_{s,\ell}|.$$

In order to finish our estimate, we show that $|\Lambda_{s-1,\ell}| < 8s^2+1 =: N$. If we assume the contrary, let $(n_i)_{i=1}^N$ be an increasing sequence such that

$$n_i \in \mathcal{G}^{(2)}_{s,\ell,r_{n_i}}$$

for some different values r_{n_i}. Consider $F := A_{n_N,j(n_N)}$. Since the $\hat{J}_n$'s corresponding to n_i are disjoint, one of the two connected components of $\widetilde{V}\setminus F$ contains $(N-1)/2 = 4s^2$ intervals $\hat{J}_{n_i}$, $i=1,\dots,N$. Enumerate them as $\hat{J}_{m_1},\dots,\hat{J}_{m_{(N-1)/2}}$.

Since any real sequence of length s^2+1 has a monotone subsequence of length s, we only have the following two possibilities:

(1) There is a subsequence $(\ell_i)_{i=1}^s$ of (m_i) such that, for each i,

$$\operatorname{conv}(\hat{J}_{\ell_i}\cup F) \subset \operatorname{conv}(\hat{J}_{\ell_{i+1}}\cup F).$$

(2) There is a subsequence $(\ell_i)_{i=1}^s$ of (m_i) such that, for each i,

$$\operatorname{conv}(\hat{J}_{\ell_{i+1}}\cup F) \subset \operatorname{conv}(\hat{J}_{\ell_i}\cup F).$$

Here by $\operatorname{conv}(U)$ for $U\subset\widetilde{V}$ we mean the smallest interval contained in $\widetilde{V}$ that contains U.

We observe that $\operatorname{conv}(\hat{J}_{n_i}\cup F)\subset C(A_{n_i,j(n_i)})$ for all i since the sequence $(A_{n_i,j(n_i)})_i$ is decreasing and therefore, in case (1), we have $a_{\ell_i,j(\ell_i)}\ge i$ and hence $a_{\ell_s,j(\ell_s)}\ge s$, which is in conflict with the definition of $T_{s-1,2}$.

We now recall that $r_n = \min_{r\in\mathcal{I}_n(\widetilde{V}^c)} K_n(C_n(I_{n,\ell}))$. We let $i(n)$ be an index such that

$$r_n = K_n(C_n(I_{n,i(n)})).$$

In case (2), we distinguish two cases:

(a) $C_{\ell_s}(I_{\ell_s,i(\ell_s)}) \supset \bigcup_{j=1}^s \hat{J}_{\ell_j}$,
(b) $C_{\ell_s}(I_{\ell_s,i(\ell_s)})$ contains $\{x_{r_{\ell_1}},\dots,x_{r_{\ell_s}}\}$.

If we are in case (a), we have of course $r_n\ge s$, in contradiction to the definition of $T_{m,2}$. If we are in case (b), since the points $x_{r_{\ell_i}}$ are all different by definition of $\mathcal{G}^{(2)}_{s,\ell,r}$ and they are all (except possibly the two endpoints of V) part of the grid points in the grid corresponding to the index ℓ_s, we have $r_n \geq s$ as well, which shows that $|\Lambda_{s-1,\ell}| \leq 8s^2 + 1$; therefore, by collecting all estimates and summing geometric series over ℓ and s,

$$\sum_n R^{p\hat{d}_n(V)} |\hat{a}_n|^p \|\hat{f}_n\|^p_{L^p(\widetilde{V}^c)} \lesssim \|f\|_p^p,$$

which finishes the proof of the lemma. ■

5. Proof of the main theorem. In this section, we prove our main result Theorem 1.1, that is, unconditionality of periodic orthonormal spline systems corresponding to an arbitrary admissible point sequence $(s_n)_{n\geq 1}$ in $L^p(\mathbb{T})$ for $p \in (1, \infty)$.

Proof of Theorem 1.1. We recall the notation

$$Sf(t) = \Big(\sum_{n\geq N(k)} |a_n \hat{f}_n(t)|^2\Big)^{1/2}, \quad Mf(t) = \sup_{m\geq N(k)} \Big|\sum_{n=N(k)}^{m} a_n \hat{f}_n(t)\Big|$$

when

$$f = \sum_{n\geq N(k)} a_n \hat{f}_n.$$

Since $(\hat{f}_n)_{n=1}^\infty$ is a basis in $L^p(\mathbb{T})$, $1 \leq p < \infty$, by Theorem 3.1, to show its unconditionality, it suffices to show that $(\hat{f}_n)_{n\geq N(k)}$ is an unconditional basic sequence in $L^p(\mathbb{T})$. Khinchin's inequality implies that a necessary and sufficient condition for this is

$$\|Sf\|_p \sim_p \|f\|_p, \quad f \in L^p(\mathbb{T}). \tag{5.1}$$

We will prove (5.1) for $1 < p < 2$ since the cases $p > 2$ then follow by a duality argument.

We first prove the inequality

$$\|f\|_p \lesssim_p \|Sf\|_p. \tag{5.2}$$

To begin, let $f \in L^p(\mathbb{T})$ with $f = \sum_{n=N(k)}^\infty a_n f_n$. Without loss of generality, we may assume that the sequence $(a_n)_{n\geq N(k)}$ has only finitely many non-zero entries. We will prove (5.2) by showing that $\|Mf\|_p \lesssim_p \|Sf\|_p$. We first observe that

$$\|Mf\|_p^p = p \int_0^\infty \lambda^{p-1} \psi(\lambda)\, \mathrm{d}\lambda \tag{5.3}$$

with $\psi(\lambda) := [Mf > \lambda] := \{t \in \mathbb{T} : Mf(t) > \lambda\}$. Next we decompose f into two parts φ_1, φ_2 and estimate the corresponding distribution functions. We first set, for $\lambda > 0$,

$$E_\lambda := [Sf > \lambda], \qquad B_\lambda := [\hat{\mathcal{M}} \mathbb{1}_{E_\lambda} > 1/2],$$
$$\Gamma := \{n : \hat{J}_n \subset B_\lambda,\ N(k) \leq n < \infty\}, \qquad \Lambda := \Gamma^c,$$

where we recall that $\hat{J}_n$ is the characteristic interval corresponding to the grid point s_n and the function $\hat{f}_n$. Then let

$$\varphi_1 := \sum_{n\in\Gamma} a_n \hat{f}_n \quad \text{and} \quad \varphi_2 := \sum_{n\in\Lambda} a_n \hat{f}_n.$$

Now we estimate $\psi_1 = [M\varphi_1 > \lambda/2]$:

$$\begin{aligned}\psi_1(\lambda) &= |\{t \in B_\lambda : M\varphi_1(t) > \lambda/2\}| + |\{t \notin B_\lambda : M\varphi_1(t) > \lambda/2\}| \\ &\le |B_\lambda| + \frac{2}{\lambda} \int_{B_\lambda^c} M\varphi_1(t)\,\mathrm{d}t \le |B_\lambda| + \frac{2}{\lambda}\int_{B_\lambda^c} \sum_{n\in\Gamma} |a_n \hat{f}_n(t)|\,\mathrm{d}t.\end{aligned}$$

We decompose the open set B_λ into a disjoint collection of open subintervals of $\mathbb{T}$ and apply Lemma 4.1 to each of those intervals to conclude from the latter expression:

$$\begin{aligned}\psi_1(\lambda) &\lesssim |B_\lambda| + \frac{1}{\lambda}\int_{B_\lambda} Sf(t)\,\mathrm{d}t \\ &= |B_\lambda| + \frac{1}{\lambda}\int_{B_\lambda\setminus E_\lambda} Sf(t)\,\mathrm{d}t + \frac{1}{\lambda}\int_{E_\lambda\cap B_\lambda} Sf(t)\,\mathrm{d}t \\ &\le |B_\lambda| + |B_\lambda\setminus E_\lambda| + \frac{1}{\lambda}\int_{E_\lambda} Sf(t)\,\mathrm{d}t,\end{aligned}$$

where in the last inequality, we have simply used the definition of E_λ. Since the Hardy–Littlewood maximal function operator $\hat{\mathcal{M}}$ is of weak type $(1,1)$, $|B_\lambda| \lesssim |E_\lambda|$ and thus finally

$$\psi_1(\lambda) \lesssim |E_\lambda| + \frac{1}{\lambda}\int_{E_\lambda} Sf(t)\,\mathrm{d}t. \tag{5.4}$$

From Theorem 3.13 and the fact that $\hat{\mathcal{M}}$ is a bounded operator on $L^2[0,1]$, we obtain

$$\begin{aligned}\psi_2(\lambda) &\lesssim \frac{1}{\lambda^2}\|\hat{\mathcal{M}}\varphi_2\|_2^2 \lesssim \frac{1}{\lambda^2}\|\varphi_2\|_2^2 = \frac{1}{\lambda^2}\|S\varphi_2\|_2^2 \\ &= \frac{1}{\lambda^2}\Big(\int_{E_\lambda} S\varphi_2(t)^2\,\mathrm{d}t + \int_{E_\lambda^c} S\varphi_2(t)^2\,\mathrm{d}t\Big).\end{aligned}$$

We apply Lemma 4.3 to get

$$\psi_2(\lambda) \lesssim \frac{1}{\lambda^2}\int_{E_\lambda^c} S\varphi_2(t)^2\,\mathrm{d}t. \tag{5.5}$$

Thus, by combining (5.4) and (5.5),

$$\begin{aligned}\psi(\lambda) &\le \psi_1(\lambda) + \psi_2(\lambda)\\ &\lesssim |E_\lambda| + \frac{1}{\lambda}\int\limits_{E_\lambda} Sf(t)\,\mathrm{d}t + \frac{1}{\lambda^2}\int\limits_{E_\lambda^c} Sf(t)^2\,\mathrm{d}t.\end{aligned}$$

Inserting this inequality into (5.3) gives

$$\begin{aligned}\|Mf\|_p^p &\lesssim p\int\limits_0^\infty \lambda^{p-1}|E_\lambda|\,\mathrm{d}\lambda + p\int\limits_0^\infty \lambda^{p-2}\int\limits_{E_\lambda} Sf(t)\,\mathrm{d}t\,\mathrm{d}\lambda\\ &\quad + p\int\limits_0^\infty \lambda^{p-3}\int\limits_{E_\lambda^c} Sf(t)^2\,\mathrm{d}t\,\mathrm{d}\lambda\\ &= \|Sf\|_p^p + p\int\limits_0^1 Sf(t)\int\limits_0^{Sf(t)} \lambda^{p-2}\,\mathrm{d}\lambda\,\mathrm{d}t\\ &\quad + p\int\limits_0^1 Sf(t)^2\int\limits_{Sf(t)}^\infty \lambda^{p-3}\,\mathrm{d}\lambda\,\mathrm{d}t,\end{aligned}$$

and thus, since $1 < p < 2$,

$$\|Mf\|_p \lesssim_p \|Sf\|_p.$$

So, the inequality $\|f\|_p \lesssim_p \|Sf\|_p$ is proved.

We now turn to the proof of

$$\|Sf\|_p \lesssim_p \|f\|_p, \qquad 1 < p < 2. \tag{5.6}$$

It is enough to show that S is of weak type (p,p) for $1 < p < 2$. Indeed, S is (clearly) also of strong type 2 and we can use the Marcinkiewicz interpolation theorem to obtain (5.6).

Thus we have to show

$$|[Sf > \lambda]| \lesssim_p \frac{\|f\|_p^p}{\lambda^p}, \qquad f \in L^p(\mathbb{T}),\ \lambda > 0. \tag{5.7}$$

We fix f and $\lambda > 0$, define $G_\lambda := [\hat{\mathcal{M}}f > \lambda]$ for $\lambda > 0$ and observe that

$$|G_\lambda| \lesssim_p \frac{\|f\|_p^p}{\lambda^p}, \tag{5.8}$$

since $\hat{\mathcal{M}}$ is of weak type (p,p), and, by the Lebesgue differentiation theorem,

$$|f| \le \lambda \quad \text{a.e. on } G_\lambda^c. \tag{5.9}$$

We decompose the open set $G_\lambda \subset [0,1]$ into a collection $(V_j)_{j=1}^\infty$ of disjoint open subintervals of $[0,1]$ and split the function f into

$$h := f\cdot \mathbb{1}_{G_\lambda^c} + \sum_{j=1}^\infty T_{V_j} f, \qquad g := f - h,$$

where for fixed j, $T_{V_j} f$ is the projection of $f \cdot \mathbb{1}_{V_j}$ onto the space of polynomials of order k on the interval V_j.

We treat the functions h, g separately. The definition of h implies

$$\|h\|_2^2 = \int_{G_\lambda^c} |f(t)|^2 \,\mathrm{d}t + \sum_{j=1}^{\infty} \int_{V_j} (T_{V_j} f)(t)^2 \,\mathrm{d}t, \tag{5.10}$$

since the intervals V_j are disjoint. For the second summand, by Corollary 2.2,

$$\int_{V_j} (T_{V_j} f)(t)^2 \,\mathrm{d}t \sim |V_j|^{-1} \Big(\int_{V_j} |T_{V_j} f(t)| \,\mathrm{d}t \Big)^2.$$

Since T_{V_j} is bounded on L^1 (a very special instance of Shadrin's theorem, Theorem 2.3), we have

$$\int_{V_j} (T_{V_j} f)(t)^2 \,\mathrm{d}t \lesssim |V_j|^{-1} \Big(\int_{V_j} |f(t)| \,\mathrm{d}t \Big)^2 \lesssim (\hat{\mathcal{M}} f(x))^2 |V_j| \le \lambda^2 |V_j|,$$

where x is a boundary point of V_j and the last inequality follows from the defining property of V_j. By using this estimate, from (5.10) we obtain

$$\|h\|_2^2 \lesssim \lambda^{2-p} \int_{G_\lambda^c} |f(t)|^p \,\mathrm{d}t + \lambda^2 |G_\lambda|,$$

and thus, in view of (5.8),

$$\|h\|_2^2 \lesssim_p \lambda^{2-p} \|f\|_p^p.$$

This inequality allows us to estimate

$$|[Sh > \lambda/2]| \le \frac{4}{\lambda^2} \|Sh\|_2^2 = \frac{4}{\lambda^2} \|h\|_2^2 \lesssim_p \frac{\|f\|_p^p}{\lambda^p},$$

which concludes the proof of (5.7) for the part h.

We turn to the proof of (5.7) for g. Since $p < 2$, we have

$$Sg(t)^p = \Big(\sum_{n \ge N(k)} |\langle g, f_n \rangle|^2 f_n(t)^2 \Big)^{p/2} \le \sum_{n \ge N(k)} |\langle g, f_n \rangle|^p |f_n(t)|^p. \tag{5.11}$$

For each j, we define $\widetilde{V}_j$ to be the open interval with the same center as V_j but with five times its length. Then set $\widetilde{G}_\lambda := \bigcup_{j=1}^\infty \widetilde{V}_j$ and observe that $|\widetilde{G}_\lambda| \le 5|G_\lambda|$. We get

$$|[Sg > \lambda/2]| \le |\widetilde{G}_\lambda| + \frac{2^p}{\lambda^p} \int_{\widetilde{G}_\lambda^c} Sg(t)^p \,\mathrm{d}t.$$

By (5.8) and (5.11), this becomes

$$|[Sg > \lambda/2]| \lesssim_p \lambda^{-p} \Big(\|f\|_p^p + \sum_{n \ge N(k)}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, \hat{f}_n \rangle|^p |\hat{f}_n(t)|^p \,\mathrm{d}t \Big).$$

But by definition of g and the fact that T_{V_j} is bounded on L^p,

$$\|g\|_p^p = \sum_j \int_{V_j} |f(t) - T_{V_j} f(t)|^p \,\mathrm{d}t \lesssim_p \sum_j \int_{V_j} |f(t)|^p \lesssim \|f\|_p^p,$$

so to prove $|[Sg > \lambda/2]| \le \lambda^{-p}\|f\|_p^p$ it is enough to show that

$$\sum_{n \ge N(k)} \int_{\widetilde{G}_\lambda^c} |\langle g, \hat{f}_n\rangle|^p |\hat{f}_n(t)|^p \,\mathrm{d}t \lesssim \|g\|_p^p. \tag{5.12}$$

We now let $g_j := g \cdot \mathbb{1}_{V_j}$. The supports of g_j are disjoint, and so $\|g\|_p^p = \sum_{j=1}^\infty \|g_j\|_p^p$. Furthermore $g = \sum_{j=1}^\infty g_j$ with convergence in L^p. Thus for each n,

$$\langle g, \hat{f}_n\rangle = \sum_{j=1}^\infty \langle g_j, \hat{f}_n\rangle,$$

and it follows from the definition of g_j that

$$\int_{V_j} g_j(t) p(t) \,\mathrm{d}t = 0$$

for each polynomial p on V_j of order k. This implies that $\langle g_j, \hat{f}_n\rangle = 0$ for $n < \mathrm{n}(V_j)$, where

$$\mathrm{n}(V) := \min\{n : \hat{\mathcal{T}}_n \cap V \ne \emptyset\}.$$

Hence for all $R > 1$ and every n,

(5.13)

$$\begin{aligned}
|\langle g, \hat{f}_n\rangle|^p &= \Big| \sum_{j:\, n \ge \mathrm{n}(V_j)} \langle g_j, \hat{f}_n\rangle \Big|^p \le \Big(\sum_{j:\, n \ge \mathrm{n}(V_j)} R^{\hat{d}_n(V_j)} |\langle g_j, \hat{f}_n\rangle| R^{-\hat{d}_n(V_j)} \Big)^p \\
&\le \Big(\sum_{j:\, n \ge \mathrm{n}(V_j)} R^{p\hat{d}_n(V_j)} |\langle g_j, \hat{f}_n\rangle|^p \Big) \Big(\sum_{j:\, n \ge \mathrm{n}(V_j)} R^{-p'\hat{d}_n(V_j)} \Big)^{p/p'},
\end{aligned}$$

where $p' = p/(p-1)$. If we fix $n \ge \mathrm{n}(V_j)$, there is at least one point of the partition $\hat{\mathcal{T}}_n$ contained in V_j. This implies that for each fixed $s \ge 0$, there are at most two indices j such that $n \ge \mathrm{n}(V_j)$ and $\hat{d}_n(V_j) = s$. Therefore,

$$\Big(\sum_{j:\, n \ge \mathrm{n}(V_j)} R^{-p'\hat{d}_n(V_j)} \Big)^{p/p'} \lesssim_p 1,$$

and from (5.13) we obtain

$$|\langle g, \hat{f}_n\rangle|^p \lesssim_p \sum_{j:\, n \ge \mathrm{n}(V_j)} R^{p\hat{d}_n(V_j)} |\langle g_j, \hat{f}_n\rangle|^p.$$

Now we insert this in (5.12) to get

$$\begin{aligned}\sum_{n=N(k)}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, \hat{f}_n\rangle|^p |\hat{f}_n(t)|^p \,\mathrm{d}t \\ &\lesssim_p \sum_{n=N(k)}^{\infty} \sum_{j:\, n\geq \mathrm{n}(V_j)} R^{p\hat{d}_n(V_j)} |\langle g_j, \hat{f}_n\rangle|^p \int_{\widetilde{G}_\lambda^c} |\hat{f}_n(t)|^p \,\mathrm{d}t \\ &\leq \sum_{n=N(k)}^{\infty} \sum_{j:\, n\geq \mathrm{n}(V_j)} R^{p\hat{d}_n(V_j)} |\langle g_j, \hat{f}_n\rangle|^p \int_{\widetilde{V}_j^c} |\hat{f}_n(t)|^p \,\mathrm{d}t \\ &\leq \sum_{j=1}^{\infty} \sum_{n\geq \mathrm{n}(V_j)} R^{p\hat{d}_n(V_j)} |\langle g_j, \hat{f}_n\rangle|^p \int_{\widetilde{V}_j^c} |\hat{f}_n(t)|^p \,\mathrm{d}t.\end{aligned}$$

We choose $R > 1$ such that we can apply Lemma 4.4 to obtain

$$\sum_{n=N(k)}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, \hat{f}_n\rangle|^p |\hat{f}_n(t)|^p \,\mathrm{d}t \lesssim_p \sum_{j=1}^{\infty} \|g_j\|_p^p = \|g\|_p^p,$$

proving (5.12) and hence $\|Sf\|_p^p \lesssim_p \|f\|_p^p$. Thus the proof of Theorem 1.1 is complete. ∎

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CHAPTER 7

On almost everywhere convergence of tensor product spline projections

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On Almost Everywhere Convergence of Tensor Product Spline Projections

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Abstract. Let $d \in \mathbb{N}$, and let f be a function in the Orlicz class $L(\log^+ L)^{d-1}$ defined on the unit cube $[0,1]^d$ in $\mathbb{R}^d$. Given knot sequences $\Delta_1, \dots, \Delta_d$ on $[0,1]$, we first prove that the orthogonal projection $P_{(\Delta_1,\dots,\Delta_d)}(f)$ onto the space of tensor product splines with arbitrary orders $(k_1, \dots, k_d)$ and knots $\Delta_1, \dots, \Delta_d$ converges to f almost everywhere as the mesh diameters $|\Delta_1|, \dots, |\Delta_d|$ tend to zero. This extends the one-dimensional result in [9] to arbitrary dimensions.

In the second step, we show that this result is optimal, that is, given any "bigger" Orlicz class $X = \sigma(L)L(\log^+ L)^{d-1}$ with an arbitrary function σ tending to zero at infinity, there exist a function $\varphi \in X$ and partitions of the unit cube such that the orthogonal projections of φ do not converge almost everywhere.

1. Introduction and Main Results

The notion of splines is originally motivated by concepts used in shipbuilding design and was first introduced by Schoenberg in his 1946 paper [13] to approach problems of approximation. The particular interest in tensor product splines, besides a purely mathematical one, is due to their various applications in high-dimensional problems. For instance, in statistics, they are used in nonparametric and semiparametric multiple regression, where high-dimensional vectors of covariates are considered for each observation (see, e.g., [16]) and in the approximation of finite window roughness penalty smoothers [6]. In data mining, they appear in predictive modeling with multivariate regression splines in the form of popular MARS or MARS-like algorithms [17]. Further applications appear in problems related to high-dimensional numerical integration. With this paper, we contribute to a better understanding of theoretical aspects of tensor product splines.

One of the major mathematical achievements in the last years is Shadrin's proof of de Boor's conjecture [15], where he showed that the max-norm of the orthogonal projection P_Δ onto spline spaces of arbitrary order k with knots Δ is bounded independently of the knot-sequence Δ. In particular, this result implies the L_p-convergence ($1 \le p < \infty$) of orthogonal spline projections, that is, for all

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$f \in L_p[a,b]$,

$$P_\Delta(f) \xrightarrow{L_p} f,$$

provided that the mesh diameter $|\Delta|$ tends to zero. A similar result holds for the L_∞-norm if one replaces the space L_∞ with the space of continuous functions. Recently, in [9], Shadrin and the first named author extended this result. They proved that the max-norm boundedness of P_Δ implies the almost everywhere the (a.e.) convergence of orthogonal projections $P_\Delta(f)$ with arbitrary knot-sequences Δ and $f \in L_1[a,b]$, provided that the mesh diameter $|\Delta|$ tends to zero. Their proof is based on a classical approach, where a.e. convergence is proved on a dense subset of L_1, and where it is shown that the maximal projection operator is of weak $(1,1)$-type. The main tool in the proof of this theorem is a sharp decay inequality for inverses of B-spline Gram matrices.

This leaves open the natural question of a corresponding result in higher dimensions. In the first step in this work, we extend the one-dimensional result obtained in [9] to arbitrary dimensions $d \in \mathbb{N}$, where the function f, defined on the unit cube in $\mathbb{R}^d$, belongs to the Orlicz class $L(\log^+ L)^{d-1}$ (details are given further). In the second step, which is the main result of this paper, we prove that this is in fact optimal.

Let us present our results in more detail. We write $P_{\mathbf{\Delta}}$ for the orthogonal projection operator from $L_2[0,1]^d$ onto the linear span of the sequence of tensor product B-splines and denote by $|\mathbf{\Delta}|$ the maximal directional mesh width. The first result of this paper is the a.e. convergence of $P_{\mathbf{\Delta}} f$ to f for the Orlicz class $L(\log^+ L)^{d-1}$:

THEOREM 1.1. *Let $f \in L(\log^+ L)^{d-1}$. Then, as $|\mathbf{\Delta}| \to 0$,*

$$P_{\mathbf{\Delta}} f \to f \quad a.e.$$

The second and main result of this work shows that this result is optimal.

THEOREM 1.2. *For any positive function σ on $[0,\infty)$ with $\liminf_{t\to\infty} \sigma(t) = 0$, there exists a nonnegative function φ on $[0,1]^d$ such that*

(i) *the function $\sigma(\varphi)\cdot\varphi\cdot(\log^+\varphi)^{d-1}$ is integrable, and*
(ii) *there exist a subset $B \subset [0,1]^d$ of positive Lebesgue measure and a sequence of partitions $(\mathbf{\Delta}_n)$ of $[0,1]^d$ with $|\mathbf{\Delta}_n| \to 0$ such that, for all $x \in B$,*

$$\limsup_{n\to\infty} |P_{\mathbf{\Delta}_n}\varphi(x)| = \infty.$$

The paper is organized as follows. In Section 2, we present the notation and notions used throughout this work and present some preliminary results. In Section 3, we give the proof of Theorem 1.1. The proof of Theorem 1.2, showing the optimality of Theorem 1.1, is presented in Section 4. We conclude the paper in Section 5 with some final remarks and an open problem that we consider to be of further interest.

2. Notation and Preliminaries

In this section, we introduce the notation used throughout the text and present some background material such as a multidimensional version of the Remez inequality, which we will use later, and recall the definition of tensor product B-splines.

2.1. General Notation

We denote by card$[A]$ the cardinality of a set A. The symbol $|\cdot|$ will be used for the modulus, the mesh width, and the Lebesgue measure; the meaning and the dimension of the Lebesgue measure will be always clear from the context. Given a compact metric space M, we denote by $C(M)$ the space of continuous functions on M. As usual, for $1 \le p \le \infty$ and a measure space (E, Σ, μ), we denote by $L_p(E)$ the space of (equivalence classes of) measurable functions $f : E \to \mathbb{R}$ for which

$$\|f\|_{L_p(E)} := \left(\int_E |f|^p \,\mathrm{d}\mu \right)^{1/p} < \infty$$

for $1 \le p < \infty$ and

$$\|f\|_{L_\infty(E)} := \inf\{\rho \ge 0 : \mu(|f| > \rho) = 0\} < \infty$$

for $p = \infty$. We will also write $\|f\|_p$ instead of $\|f\|_{L_p(E)}$ when the choice of E is clear from the context. More generally, given a convex function $M : [0, \infty) \to [0, \infty)$ with $M(0) = 0$, the set of all (equivalence classes of) measurable functions $f : E \to \mathbb{R}$ such that, for some (and thus for all) $\lambda > 0$,

$$\int_E M\left(\frac{|f|}{\lambda}\right) \mathrm{d}\mu < \infty$$

is called the Orlicz space associated with M and is denoted by $L_M(E)$. This space becomes a Banach space when it is supplied with the Luxemburg norm

$$\|f\|_M = \inf\left\{ \lambda > 0 : \int_E M\left(\frac{|f|}{\lambda}\right) \mathrm{d}\mu \le 1 \right\}.$$

In this work, we consider functions f defined on the unit cube $[0, 1]^d$ that belong to the Orlicz space $L(\log^+ L)^j$, that is, $|f|(\log^+ |f|)^j$ is integrable over $[0, 1]^d$ with respect to the Lebesgue measure, where $\log^+(\cdot) := \max\{0, \log(\cdot)\}$. More information and a detailed exposition of the theory of Orlicz spaces can be found, for instance, in [8; 7; 10; 11].

2.2. Remez Inequality for Polynomials

We will need the following multidimensional version of the Remez theorem (see [4; 1]). If $p(x) = \sum_{\alpha \in I} a_\alpha x^\alpha$ is a d-variate polynomial where I is a finite set containing d-dimensional multiindices, then the degree of p is defined as $\max\{\sum_{i=1}^d \alpha_i : \alpha \in I\}$. Recall that a convex body in $\mathbb{R}^d$ is a compact convex set with nonempty interior.

THEOREM 2.1 (Remez, Brudnyi, Ganzburg). *Let $d \in \mathbb{N}$, let $V \subset \mathbb{R}^d$ be a convex body, and let $E \subset V$ be a measurable subset. Then, for all polynomials p of degree k on V,*

$$\|p\|_{L_\infty(V)} \leq \left(4d\frac{|V|}{|E|}\right)^k \|p\|_{L_\infty(E)}.$$

We have the following corollary.

COROLLARY 2.2. *Let p be a polynomial of degree k on a convex body $V \subset \mathbb{R}^d$. Then*

$$|\{x \in V : |p(x)| \geq (8d)^{-k}\|p\|_{L_\infty(V)}\}| \geq |V|/2.$$

Proof. This follows from an application of Theorem 2.1 to the set $E = \{x \in V : |p(x)| \leq (8d)^{-k}\|p\|_{L_\infty(V)}\}$. □

2.3. Tensor Product B-Splines

We will now provide some background information on tensor product splines. For more information, we refer the reader to [14, Section 12.2]. Let $d \in \mathbb{N}$, and for $\mu \in \{1, \ldots, d\}$, let k_μ be the order of polynomials in the direction of the μth standard unit vector in $\mathbb{R}^d$, where the order of a univariate polynomial refers to the degree plus 1. For each such μ, we define the partition of the interval [0, 1] by

$$\Delta_\mu = (t_i^{(\mu)})_{i=1}^{n_\mu+k_\mu}, \quad n_\mu \in \mathbb{N},$$

where, for all $i < n_\mu + k_\mu$ and $j \leq n_\mu$,

$$t_i^{(\mu)} \leq t_{i+1}^{(\mu)} \quad \text{and} \quad t_j^{(\mu)} < t_{j+k_\mu}^{(\mu)},$$

and

$$t_1^{(\mu)} = \cdots = t_{k_\mu}^{(\mu)} = 0 \quad \text{and} \quad 1 = t_{n_\mu+1}^{(\mu)} = \cdots = t_{n_\mu+k_\mu}^{(\mu)}.$$

A boldface letter always denotes a vector of d entries, and its coordinates are denoted by the same letter, for instance, $\mathbf{n} = (n_1, \ldots, n_d)$, $\mathbf{k} = (k_1, \ldots, k_d)$, or $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_d)$. We let $(N_i^{(\mu)})_{i=1}^{n_\mu}$ be the sequence of B-splines of order k_μ on the partition Δ_μ with the properties

$$\operatorname{supp} N_i^{(\mu)} = [t_i^{(\mu)}, t_{i+k_\mu}^{(\mu)}], \qquad N_i^{(\mu)} \geq 0, \quad \text{and} \quad \sum_{i=1}^{n_\mu} N_i^{(\mu)} \equiv 1.$$

The space spanned by those B-spline functions consists of piecewise polynomials p of order k_μ with grid points Δ_μ, which satisfy the following smoothness conditions at those grid points: if the point t occurs m times in Δ_μ, then the function p is $k_\mu - 1 - m$ times continuously differentiable at t. In particular, if $m = k_\mu$, then there is no smoothness condition at the point t.

The tensor product B-splines are defined as

$$N_{\mathbf{i}}(x_1, \ldots, x_d) := N_{i_1}^{(1)}(x_1) \cdots N_{i_d}^{(d)}(x_d), \quad \mathbf{1} \leq \mathbf{i} \leq \mathbf{n},$$

where $\mathbf{1}$ is the d-dimensional vector consisting of d entries equal to one, and where we say that $\mathbf{i} \leq \mathbf{n}$ if $i_\mu \leq n_\mu$ for all $\mu \in \{1, \ldots, d\}$. Furthermore, $P_{\mathbf{\Delta}}$ is defined to be the orthogonal projection operator from $L_2[0,1]^d$ onto the linear span of the functions $(N_{\mathbf{i}})_{\mathbf{1} \leq \mathbf{i} \leq \mathbf{n}}$ with respect to the standard inner product $\langle \cdot, \cdot \rangle$. This operator can be naturally extended to L_1-functions since B-splines are contained in L_∞ (see Lemma 3.4). For $\mu \in \{1, \ldots, d\}$, we define the mesh width in the direction of μ by $|\Delta_\mu| := \max_i |t_{i+1}^{(\mu)} - t_i^{(\mu)}|$ and the mesh width by

$$|\mathbf{\Delta}| := \max_{1 \leq \mu \leq d} |\Delta_\mu|.$$

3. Almost Everywhere Convergence

In this section, we prove Theorem 1.1 on a.e. convergence. Its proof follows along the lines of the one-dimensional case proved in [9] and is based on the standard approach of verifying the following two conditions that imply the a.e. convergence of $P_{\mathbf{\Delta}} f$ for $f \in L(\log^+ L)^{d-1}$ (see [5, pp. 3–4]):

(a) there is a dense subset $\mathcal{F}$ of $L(\log^+ L)^{d-1}$ on which we have a.e. convergence,

(b) the maximal operator $P^* f := \sup_{\mathbf{\Delta}} |P_{\mathbf{\Delta}} f|$ satisfies some weak-type inequality.

Let us now discuss conditions (a) and (b) in more detail. Concerning (a), we first note that, for $d = 1$, Shadrin [15] proved that the one-dimensional projection operator P_Δ is uniformly bounded on L_∞ for any spline order k, that is,

$$\|P_\Delta\|_\infty \leq c_k,$$

where the constant $c_k \in (0, \infty)$ depends only on k and not on the partition Δ. A direct consequence of this result and of the tensor structure of the underlying operator $P_{\mathbf{\Delta}}$ is that this assertion also holds in higher dimensions:

Corollary 3.1. *For any $d \in \mathbb{N}$, there exists a constant $c_{d,\mathbf{k}} \in (0, \infty)$ that only depends on d and $\mathbf{k}$ such that*

$$\|P_{\mathbf{\Delta}}\|_\infty \leq c_{d,\mathbf{k}}.$$

In particular, $c_{d,\mathbf{k}}$ is independent of the partitions $\mathbf{\Delta}$.

This can be used to prove the uniform convergence of $P_{\mathbf{\Delta}} g$ to g for continuous functions g as $|\mathbf{\Delta}|$ tends to zero:

Proposition 3.2. *Let $g \in C([0,1]^d)$. Then, as $|\mathbf{\Delta}| \to 0$,*

$$\|P_{\mathbf{\Delta}} g - g\|_\infty \to 0.$$

Therefore we may choose $\mathcal{F}$ to be the space of continuous functions on $[0,1]^d$, which is dense in $L(\log^+ L)^{d-1}$ (see, e.g., [7, Chapter 7]).

We now turn to the discussion of condition (b) and define the strong maximal function $\mathrm{M_S} f$ of $f \in L_1[0,1]^d$ by

$$\mathrm{M_S} f(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \, \mathrm{d}y, \quad x \in [0,1]^d,$$

where the supremum is taken over all d-dimensional rectangles $I \subset [0,1]^d$, which are parallel to the coordinate axes and contain the point x. The strong maximal function satisfies the weak-type inequality

$$|\{x : \mathrm{M_S} f(x) > \lambda\}| \leq c_M \int_{[0,1]^d} \frac{|f(x)|}{\lambda} \Big(1 + \log^+ \frac{|f(x)|}{\lambda}\Big)^{d-1} \mathrm{d}x, \tag{3.1}$$

where $|A|$ denotes the d-dimensional Lebesgue measure of a set A, and $c_M \in (0,\infty)$ is a constant independent of f and λ (see, e.g., [3] and [18, Chapter 17]). To get this kind of weak-type inequality for the maximal operator P^*, we prove the following pointwise estimate for $P_{\mathbf{\Delta}}$ by the strong maximal function.

PROPOSITION 3.3. *There exists a constant $c \in (0,\infty)$ that only depends on the dimension d and the spline orders $\mathbf{k}$ such that, for all $f \in L_1[0,1]^d$, $x \in [0,1]^d$, and all partitions $\mathbf{\Delta}$,*

$$|P_{\mathbf{\Delta}} f(x)| \leq c \cdot \mathrm{M_S} f(x).$$

We now present the proof Theorem 1.1 and defer the proofs of Propositions 3.2 and 3.3.

Proof of Theorem 1.1. Let $f \in L(\log^+ L)^{d-1}$ and define

$$R(f,x) := \limsup_{|\mathbf{\Delta}| \to 0} P_{\mathbf{\Delta}} f(x) - \liminf_{|\mathbf{\Delta}| \to 0} P_{\mathbf{\Delta}} f(x).$$

Let $g \in C([0,1]^d)$. Since, by Proposition 3.2, $R(g,x) \equiv 0$ for continuous functions g, and because $P_{\mathbf{\Delta}}$ is a linear operator,

$$R(f,x) \leq R(f-g,x) + R(g,x) = R(f-g,x).$$

Let $\delta > 0$. Then by Proposition 3.3 we have

$$\begin{aligned} |\{x : R(f,x) > \delta\}| &\leq |\{x : R(f-g,x) > \delta\}| \\ &\leq |\{x : 2c \cdot \mathrm{M_S}(f-g)(x) > \delta\}|. \end{aligned}$$

Now we employ the weak-type inequality (3.1) for $\mathrm{M_S}$ to find

$$\begin{aligned} &|\{x : R(f,x) > \delta\}| \\ &\quad \leq c_M \int_{[0,1]^d} \frac{2c \cdot |(f-g)(x)|}{\delta} \Big(1 + \log^+ \frac{2c \cdot |(f-g)(x)|}{\delta}\Big)^{d-1} \mathrm{d}x. \end{aligned}$$

By assumption, the expression on the right-hand side of the latter display is finite. Choosing a suitable sequence of continuous functions (g_n) (first approximate f by a bounded function and then apply Lusin's theorem), this expression tends to zero, and we obtain

$$|\{x : R(f,x) > \delta\}| = 0.$$

Since $\delta > 0$ is arbitrary, $R(f, x) = 0$ for a.e. $x \in [0, 1]^d$. This means that $P_{\mathbf{\Delta}} f$ converges almost everywhere as $|\mathbf{\Delta}| \to 0$. It remains to show that this limit equals f a.e. This is obtained by a similar argument as before by replacing $R(f, x)$ by $|\lim_{|\mathbf{\Delta}|\to 0} P_{\mathbf{\Delta}} f(x) - f(x)|$. □

The rest of this section is devoted to the proofs of Propositions 3.2 and 3.3.

Proof of Proposition 3.2. By Corollary 3.1, $P_{\mathbf{\Delta}}$ is a bounded projection operator, and so, for all functions h in the range of $P_{\mathbf{\Delta}}$, we have

$$\|P_{\mathbf{\Delta}} g - g\|_\infty \le \|P_{\mathbf{\Delta}}(g - h)\|_\infty + \|h - g\|_\infty \le (1 + c_{d,\mathbf{k}})\|g - h\|_\infty.$$

Taking the infimum over all such h, we have

$$\|P_{\mathbf{\Delta}} g - g\|_\infty \le (1 + c_{d,\mathbf{k}}) \cdot E_{\mathbf{\Delta}}(g), \tag{3.2}$$

where $E_{\mathbf{\Delta}}(g)$ is the error of best approximation of g by splines in the span of tensor product B-splines $(N_{\mathbf{i}})_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}}$. It is known that

$$E_{\mathbf{\Delta}}(g) \le c \cdot \sum_{\mu=1}^{d} \sup_{h_\mu \le |\Delta_\mu|} \sup_x |(D_{h_\mu}^{k_\mu} g_{\mu,x})(x_\mu)|,$$

where $g_{\mu,x}(s) := g(x_1, \dots, x_{\mu-1}, s, x_{\mu+1}, \dots, x_d)$, and D_{h_μ} is the forward difference operator with step size h_μ (see, e.g., [14, Theorem 12.8 and Example 13.27]). This is the sum of moduli of smoothness in each direction μ of the function g with respect to the mesh diameters $|\Delta_1|, \dots, |\Delta_d|$, respectively. As these diameters tend to zero, the right-hand side of the inequality also tends to zero since g is continuous. Together with (3.2), this proves the proposition. □

Next, we present the proof of Proposition 3.3. It is essentially a consequence of a pointwise estimate involving the Dirichlet kernel of the projection operator $P_{\mathbf{\Delta}}$. With the notation

$$I_i^{(\mu)} := [t_i^{(\mu)}, t_{i+1}^{(\mu)}], \qquad I_{ij}^{(\mu)} := \mathrm{convexhull}(I_i^{(\mu)}, I_j^{(\mu)}), \quad \mu \in \{1, \dots, d\},$$

its one-dimensional version, where we suppress the superindex (μ), reads as follows.

Lemma 3.4 ([9, Lemma 2.1]). *Let K_Δ be the Dirichlet kernel of the projection operator P_Δ, that is, K_Δ is defined by the equation*

$$P_\Delta f(x) = \int_0^1 K_\Delta(x, y) f(y)\,\mathrm{d}y, \quad f \in L_1[0, 1], x \in [0, 1].$$

Then K_Δ satisfies the inequality

$$|K_\Delta(x, y)| \le C\gamma^{|i-j|}|I_{ij}|^{-1}, \quad x \in I_i, y \in I_j,$$

where $C \in (0, \infty)$ and $\gamma \in (0, 1)$ are constants that depend only on the spline order k.

Proof of Proposition 3.3. We first note that the estimate given in Lemma 3.4 carries over to the Dirichlet kernel $K_{\mathbf{\Delta}}$ of $P_{\mathbf{\Delta}}$ for dimension d, which is defined by the relation

$$P_{\mathbf{\Delta}} f(x) = \int_{[0,1]^d} K_{\mathbf{\Delta}}(x,y) f(y)\,\mathrm{d}y, \qquad f \in L_1[0,1]^d, x \in [0,1]^d. \tag{3.3}$$

Indeed, since $P_{\mathbf{\Delta}}$ is the tensor product of the one-dimensional projections $P_{\Delta_1}, \dots, P_{\Delta_d}$, the Dirichlet kernel $K_{\mathbf{\Delta}}$ is the product of the one-dimensional Dirichlet kernels $K_{\Delta_1}, \dots, K_{\Delta_d}$. Thus, Lemma 3.4 implies the inequality

$$|K_{\mathbf{\Delta}}(x,y)| \le C\gamma^{|\mathbf{i}-\mathbf{j}|_1} |I_{\mathbf{ij}}|^{-1}, \qquad x \in I_{\mathbf{i}}, y \in I_{\mathbf{j}}, \tag{3.4}$$

where we set

$$|\mathbf{i}-\mathbf{j}|_1 := \sum_{\mu=1}^{d} |i_\mu - j_\mu|, \qquad I_{\mathbf{i}} := \prod_{\mu=1}^{d} I_i^{(\mu)}, \qquad I_{\mathbf{ij}} := \prod_{\mu=1}^{d} I_{ij}^{(\mu)},$$

and $C \in (0,\infty)$ and $\gamma \in (0,1)$ are constants only depending on d and $\mathbf{k}$.

Let $x \in [0,1]^d$ and $\mathbf{i}$ be such that $x \in I_{\mathbf{i}}$ and $|I_{\mathbf{i}}| > 0$. By equation (3.3),

$$|P_{\mathbf{\Delta}} f(x)| = \Big| \int_{[0,1]^d} K_{\mathbf{\Delta}}(x,y) f(y)\,\mathrm{d}y \Big| = \Big| \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} \int_{I_{\mathbf{j}}} K_{\mathbf{\Delta}}(x,y) f(y)\,\mathrm{d}y \Big|.$$

Using estimate (3.4) on the Dirichlet kernel, we obtain

$$|P_{\mathbf{\Delta}} f(x)| \le C \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} \frac{\gamma^{|\mathbf{i}-\mathbf{j}|_1}}{|I_{\mathbf{ij}}|} \int_{I_{\mathbf{j}}} |f(y)|\,\mathrm{d}y,$$

where $C \in (0,\infty)$ is the constant in (3.4). Since $I_{\mathbf{j}} \subset I_{\mathbf{ij}}$ and $x \in I_{\mathbf{i}} \subset I_{\mathbf{ij}}$, we conclude

$$|P_{\mathbf{\Delta}} f(x)| \le C \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} \gamma^{|\mathbf{i}-\mathbf{j}|_1} \mathrm{M_S} f(x),$$

which, after summing a geometric series, concludes the proof. □

4. Optimality of the Result

In this section, we prove the optimality result, Theorem 1.2. The choice of the function φ is based on the following result of Saks [12].

THEOREM 4.1. *For any function $\sigma : [0,\infty) \to [0,\infty)$ with $\liminf_{t\to\infty} \sigma(t) = 0$, there exists a nonnegative function $\varphi := \varphi_\sigma$ on $[0,1]^d$ such that*

(i) *the function $\sigma(\varphi)\cdot\varphi\cdot(\log^+\varphi)^{d-1}$ is integrable,*
(ii) *for all $x \in [0,1]^d$,*

$$\limsup_{\operatorname{diam} I\to 0, I\ni x} \frac{1}{|I|} \int_I \varphi(y)\,\mathrm{d}y = \infty,$$

where lim sup *is taken over all d-dimensional rectangles I that are parallel to the coordinate axes and contain the point x.*

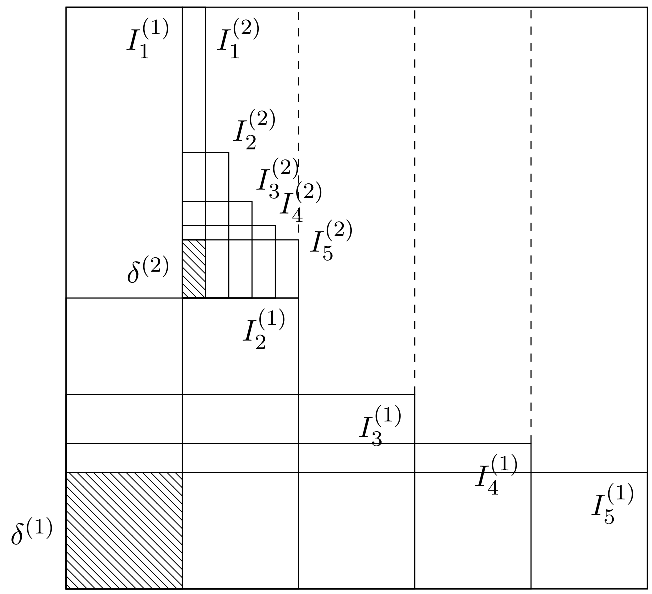

Figure 1 First sets in the enumeration (4.1) for $N = 5$

We will show that the same function φ, constructed in the proof of the previous theorem, also has the properties stated in Theorem 1.2. The definition of φ rests on a construction due to H. Bohr, which appears in the first edition of [2, pp. 689–691] from 1918 for dimension $d = 2$. Let us begin by recalling Bohr's construction and Saks' definition of the function φ.

Bohr's Construction

Let $N \in \mathbb{N}$, and let $S := [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ be a rectangle. Using the splitting parameter N, we define subsets of this rectangle as follows:

$$I_j^{(1)} := \left[a_1, a_1 + \frac{j(b_1 - a_1)}{N}\right] \times \left[a_2, a_2 + \frac{b_2 - a_2}{j}\right], \quad 1 \le j \le N.$$

The part $S \setminus \bigcup_{j=1}^N I_j^{(1)}$ consists of $N - 1$ disjoint rectangles, to which we apply the same splitting as we did with S (see Figure 1). This procedure is carried out until the area of the remainder is less than $|S|/N^2$. The remainder is again a disjoint union of rectangles $J^{(1)}, \dots, J^{(r)}$. Thus, we obtain a sequence of rectangles whose union is S,

$$I_1^{(1)}, \dots, I_N^{(1)}; I_1^{(2)}, \dots, I_N^{(2)}; \dots; I_1^{(s)}, \dots, I_N^{(s)}; J^{(1)}, \dots, J^{(r)}. \tag{4.1}$$

We can generalize this construction to arbitrary dimensions d as follows: first, notice that the corners of the rectangles $I_j^{(1)}$, $1 \le j \le N$, lie on the curve $(x - a_1)(y - a_2) = (b_1 - a_1)(b_2 - a_2)/N = |S|/N$. Given a rectangle $S := [a_1, b_1] \times \cdots \times [a_d, b_d]$, $d > 2$, we consider rectangles similar to $I_j^{(1)}$ whose corners lie on the variety $(x_1 - a_1)(x_2 - a_2)\cdots(x_d - a_d) = |S|/N^{d-1}$. For $a_1 = \cdots = a_d = 0$

and $b_1 = \cdots = b_d = 1$, we can write those rectangles using $d-1$ parameters as

$$I_{j_1,\dots,j_{d-1}} := \Big[0, \frac{j_1}{N}\Big] \times \cdots \times \Big[0, \frac{j_{d-1}}{N}\Big] \times \Big[0, \frac{1}{j_1 \cdots j_{d-1}}\Big]$$

for $1 \le j_1, \dots, j_{d-1} \le N$. The volume of the union over all those sets is approximately

$$\Big(\frac{\log N}{N}\Big)^{d-1},$$

which can be seen by integration of the function $x_d = (N^{d-1} x_1 \cdots x_{d-1})^{-1}$ over the rectangle $[1/N, 1]^{d-1}$. In what follows, it is important that in Bohr's construction, we only choose those rectangles $I_{j_1,\dots,j_{d-1}}$ for which the product $j_1 \cdots j_{d-1}$ is less than or equal to N, so that the volume V_1 of their union is still approximately $N^{1-d} \log^{d-1} N$, whereas the volume V_2 of their intersection equals N^{-d}. Therefore the quotient V_1/V_2 is of the order $N \log^{d-1} N$. This is crucial for the construction of the function φ in Theorem 4.1.

The function φ from Theorem 4.1 is constructed in [12] in such a way that it satisfies the following additional properties.

THEOREM 4.2. *The function φ from Theorem 4.1 can be chosen in such a way that there exist a sequence $(\varepsilon_i)_{i\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ and a sequence $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of rectangular coverings of $[0,1]^d$ such that*

(i) *the function $\sigma(\varphi)\cdot\varphi\cdot(\log^+\varphi)^{d-1}$ is integrable,*
(ii) *the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ converges to 0,*
(iii) *for each $i \in \mathbb{N}$, $\mathcal{C}_i = (R_{ij})_{j=1}^{M_i}$ with $\bigcup_{j=1}^{M_i} R_{ij} = [0,1]^d$ we have $\operatorname{diam} R_{ij} < 1/i$ and*

$$\frac{1}{|R_{ij}|}\int_{R_{ij}} \varphi(x)\,\mathrm{d}x > \varepsilon_i^{-1} \quad \text{for all } j \in \{1,\dots,M_i\},$$

(iv) *for each $i \in \mathbb{N}$, there exist $L_i, N_i \in \mathbb{N}$ and a partition $(S_{ij})_{j=1}^{L_i}$ of the unit cube $[0,1]^d$ consisting of rectangles with diameter $\le 1/i$ such that for all $j \in \{1,\dots,L_i\}$, the subcollection of rectangles in $\mathcal{C}_i$ that intersect S_{ij} is given by the rectangles in (4.1) (or its higher-dimensional analogue) corresponding to S_{ij} and the splitting parameter N_i.*

Let P_I be the orthogonal projection operator onto the space of d-variate polynomials of order $(k_1, k_2, \dots, k_d)$ on the rectangle I. We now use the Remez inequality to prove that $|P_I\varphi|$ is large on a large subset of I as long as $\frac{1}{|I|}\int_I \varphi\,\mathrm{d}y$ is large enough. This is the first important step in proving (ii) of Theorem 1.2.

LEMMA 4.3. *Let $I \subset \mathbb{R}^d$ be a rectangle. Then, there exists a constant $c_{\mathbf{k}} \in (0,\infty)$, only depending on the polynomial orders $\mathbf{k} = (k_1,\dots,k_d)$, such that, for all positive functions f on I, there exists a subset $A \subset I$ with measure $|A| \ge |I|/2$ such that, for all $x \in A$,*

$$|P_I f(x)| \ge \frac{c_{\mathbf{k}}}{|I|}\int_I f(y)\,\mathrm{d}y.$$

Proof. The operator P_I is the orthogonal projection onto the space of d-variate polynomials of order $(k_1, k_2, \dots, k_d)$ on I. Therefore, the characteristic function χ_I is contained in the range of P_I, and we have

$$\langle P_I f, \chi_I \rangle = \langle f, \chi_I \rangle.$$

Hence, in fact, $\|P_I f\|_{L_\infty(I)} \geq |I|^{-1} \int_I f(y)\,\mathrm{d}y$. Consequently, Corollary 2.2 implies the assertion. □

Considering the properties of φ in Theorem 4.2, the previous proposition applied to φ shows that, for any element $I \in \mathcal{C}_i$, there exists a subset $A := A(I) \subset I$ with measure $\geq |I|/2$, on which $|P_I \varphi| \geq c/\varepsilon_i$ for a constant $c \in (0, \infty)$ only depending on the polynomial orders $(k_1, \dots, k_d)$. In Lemma 4.4, we ensure that the union over those sets A still has large enough measure relatively to the measure of the union over all $I \in \mathcal{C}_i$. To this end, we will use the special structure indicated by Bohr's construction and Theorem 4.2(iv).

Lemma 4.4. *For all $j_1, \dots, j_{d-1} \in \{1, \dots, N\}$, let*

$$I_{j_1,\dots,j_{d-1}} = \left[0, \frac{j_1}{N}\right] \times \cdots \times \left[0, \frac{j_{d-1}}{N}\right] \times \left[0, \frac{1}{j_1 \cdots j_{d-1}}\right]$$

and $\Lambda = \{(j_1, \dots, j_{d-1}) : j_1 \cdots j_d \leq N\}$. For $\lambda \in \Lambda$, let $A_\lambda \subset I_\lambda$ be a Borel measurable subset of I_λ such that

$$|A_\lambda| \geq c|I_\lambda| = \frac{c}{N^{d-1}}$$

for some absolute constant $c \in (0, \infty)$. Then there exist constants $c_1, c_2 \in (0, \infty)$, depend only on c and d, such that

$$\Big|\bigcup_{\lambda \in \Lambda} A_\lambda\Big| \geq c_2 \left(\frac{\log N}{N}\right)^{d-1} \geq c_1 \Big|\bigcup_{\lambda \in \Lambda} I_\lambda\Big|.$$

Proof. Let $M \in \mathbb{N}$ to be specified later and define $q := 1/M$. Define the index set

$$\Gamma = \{(M^{k_1}, \dots, M^{k_{d-1}}) \in \Lambda : k_1, \dots, k_{d-1} \in \mathbb{N}_0\}.$$

Then we can estimate

$$\begin{aligned}\Big|\bigcup_{\lambda \in \Lambda} A_\lambda\Big| \geq \Big|\bigcup_{\lambda \in \Gamma} A_\lambda\Big| &\geq \sum_{\lambda \in \Gamma} |A_\lambda| - \frac{1}{2} \sum_{\substack{\lambda,\mu \in \Gamma \\ \lambda \neq \mu}} |A_\lambda \cap A_\mu| \\ &\geq c \sum_{\lambda \in \Gamma} |I_\lambda| - \frac{1}{2} \sum_{\substack{\lambda,\mu \in \Gamma \\ \lambda \neq \mu}} |I_\lambda \cap I_\mu|.\end{aligned}$$

Now we observe that $\operatorname{card}[\Gamma] = \operatorname{card}[\{k \in \mathbb{N}_0^s : \sum_{j=1}^s k_j \leq L\}] = \binom{\lfloor L \rfloor + s}{s}$ where $L = \log_M N$, $s = d-1$, and $\lfloor L \rfloor$ denotes the largest integer smaller than or equal to L. Therefore $\operatorname{card}[\Gamma] \geq C_s \log_M^{d-1} N$ for some positive constant C_s depending only on s. Thus,

$$\Big|\bigcup_{\lambda\in\Lambda} A_\lambda\Big| \geq c \cdot C_{d-1}\Big(\frac{\log_M N}{N}\Big)^{d-1} - \frac{1}{2}\sum_{\substack{\lambda,\mu\in\Gamma\\ \lambda\neq\mu}} |I_\lambda \cap I_\mu|. \tag{4.2}$$

Next, observe that if $\lambda, \mu \in \Gamma$ have the form $\lambda = (M^{\ell_1}, \ldots, M^{\ell_{d-1}})$ and $\mu = (M^{m_1}, \ldots, M^{m_{d-1}})$, then

$$|I_\lambda \cap I_\mu| = N^{1-d} q^{\sum_{i=1}^{d-1}(\max(\ell_i,m_i)-\min(\ell_i,m_i))},$$

which, by summing geometric series and noting that the condition $\lambda \neq \mu$ implies the existence of at least one index $i \in \{1, \ldots, d-1\}$ such that $\lambda_i \neq \mu_i$, yields

$$\sum_{\substack{\lambda,\mu\in\Gamma\\ \lambda\neq\mu}} |I_\lambda \cap I_\mu| \leq \frac{q}{(1-q)^{d-1}} \sum_{\lambda\in\Gamma} N^{1-d} \leq \frac{q}{(1-q)^{d-1}}\Big(\frac{\log_M N}{N}\Big)^{d-1}.$$

Inserting this inequality into (4.2), we obtain

$$\Big|\bigcup_{\lambda\in\Lambda} A_\lambda\Big| \geq \Big(c \cdot C_{d-1} - \frac{q}{2(1-q)^{d-1}}\Big) \cdot \Big(\frac{\log_M N}{N}\Big)^{d-1}.$$

We can choose $M = 1/q$ (depending only on c and d) sufficiently large to guarantee that $c \cdot C_{d-1} - \frac{q}{2(1-q)^{d-1}} \geq c \cdot C_{d-1}/2$. Then the assertion of the lemma follows with the choice $c_2 = c \cdot C_{d-1}/(2\log^{d-1} M)$. □

Bringing together the previous facts, we are now able to prove our optimality result.

Proof of Theorem 1.2. We subdivide the proof into two parts. In the first part, we show that, for all points x in a set of positive measure, there exists a sequence (I_n) of intervals containing x whose measure tends to zero and such that $|P_{I_n}\varphi(x)| \to \infty$. Based on that observation, we construct the desired sequence of partitions in the second step.

Step 1. Since Theorem 4.1 proves the integrability condition (i) of Theorem 1.2, we only need to prove (ii), that is, the existence of a set $B \subset [0,1]^d$ with positive Lebesgue measure and of a sequence $(\boldsymbol{\Delta}_n)$ of partitions such that, for all $x \in B$, $\limsup_{n\to\infty} |P_{\boldsymbol{\Delta}_n}\varphi(x)| = \infty$. We fix $i \in \mathbb{N}$ and consider the corresponding covering $\mathcal{C}_i$ of $[0,1]^d$ from Theorem 4.2. Then we define

$$\begin{aligned} B_i := \{x \in [0,1]^d : &\text{ there exists a rectangle } I \in \mathcal{C}_i \text{ with } x \in I \\ &\text{ and } |P_I\varphi(x)| \geq c_{\mathbf{k}}/\varepsilon_i\}, \end{aligned}$$

where $c_{\mathbf{k}} \in (0,\infty)$ is the constant that appears in Lemma 4.3, and (ε_i) is the sequence from Theorem 4.2. Recall that $\varepsilon_i \to 0$ as $i \to \infty$. We will show that $|B_i| \geq c > 0$ for all $i \in \mathbb{N}$ and some suitable constant $c \in (0,\infty)$.

Let $I \in \mathcal{C}_i$. Due to Theorem 4.2, we have $\operatorname{diam} I \le 1/i$ and

$$\frac{1}{|I|}\int_I \varphi \, \mathrm{d}x \ge \varepsilon_i^{-1}.$$

Thus, Lemma 4.3 provides a set $A(I) \subset I$ with $|A(I)| \ge |I|/2$ such that, for all $x \in A(I)$,

$$|P_I \varphi(x)| \ge \frac{c_{\mathbf{k}}}{\varepsilon_i}.$$

This means that $A(I) \subset B_i$. For fixed j, let $(I_m^{(\ell)})$ and $(J^{(\ell)})$ be the collections of rectangles (4.1) contained in $\mathcal{C}_i$ forming a covering of S_{ij} (see Theorem 4.2, part (iv)). As a consequence of the latter bound, Lemma 4.4 and the fact that the rectangles $J^{(\ell)}$ are disjoint, we find

$$\begin{aligned} |S_{ij} \cap B_i| &\ge \sum_\ell \Big| \bigcup_{m=1}^{N_i} A(I_m^{(\ell)}) \Big| + \sum_\ell |A(J^{(\ell)})| \\ &\ge c_1 \sum_\ell \Big| \bigcup_{m=1}^{N_i} I_m^{(\ell)} \Big| + \frac{1}{2}\sum_\ell |J^{(\ell)}| \ge c_2 |S_{ij}|, \end{aligned}$$

where $c_2 := \min\{c_1, \frac{1}{2}\}$. Consequently,

$$|B_i| = \sum_{j=1}^{L_i} |S_{ij} \cap B_i| \ge c_2 \sum_{j=1}^{L_i} |S_{ij}| = c_2 |[0,1]^d| = c_2.$$

Since all sets B_i satisfy this uniform lower bound, the set $B := \limsup_n B_n$ has a positive measure as well, because

$$|B| = \lim_n \Big| \bigcup_{m \ge n} B_m \Big| \ge \limsup_n |B_n| \ge c > 0.$$

Step 2: We now proceed with the construction of the desired sequence of partitions $(\mathbf{\Delta}_n)$. Let $(R_{ij})_{j=1}^{M_i}$ be the rectangles contained in the collection $\mathcal{C}_i$. For $1 \le j \le M_i$, we define the partition $\mathbf{\Delta}^{(i,j)} = (\Delta_1^{(i,j)}, \ldots, \Delta_d^{(i,j)})$ such that each R_{ij} is a grid point interval of $\mathbf{\Delta}^{(i,j)}$ and, for $\mu \in \{1, \ldots, d\}$, the μth coordinate projection of the vertices of R_{ij} has multiplicity k_μ in the partition $\Delta_\mu^{(i,j)}$. We give this multiplicity condition in order to have, for all $x \in R_{ij}$,

$$P_{\mathbf{\Delta}^{(i,j)}} f(x) = P_{R_{ij}} f(x), \quad f \in L_1[0,1]^d.$$

Other knots of the partition $\mathbf{\Delta}^{(i,j)}$ are chosen arbitrarily, with the only condition $|\mathbf{\Delta}^{(i,j)}| \le 1/i$. Observe that this is possible since $\operatorname{diam} R_{ij} \le 1/i$. Now we define the sequence $(\mathbf{\Delta}_n)$ as

$$(\mathbf{\Delta}_n) := (\mathbf{\Delta}^{(1,1)}, \ldots, \mathbf{\Delta}^{(1,M_1)}, \mathbf{\Delta}^{(2,1)}, \ldots, \mathbf{\Delta}^{(2,M_2)}, \ldots).$$

Observe that this sequence of partitions is not nested. To prove the assertion of the theorem, we fix some $x \in B$. By the definition of B, for infinitely many indices $i \in \mathbb{N}$, there exists a rectangle $R_{i\ell_i}$ in the collection $\mathcal{C}_i$ such that $x \in R_{i\ell_i}$, $\operatorname{diam} R_{i\ell_i} \le$

$1/i$, and $|P_{\mathbf{\Delta}^{(i,\ell_i)}}\varphi(x)| = |P_{R_{i\ell_i}}\varphi(x)| \geq c_{\mathbf{k}}/\varepsilon_i$. Therefore, since $\varepsilon_i \to 0$, we have, for all $x \in B$,

$$\limsup_{n\to\infty} |P_{\mathbf{\Delta}_n}\varphi(x)| = \infty.$$

This completes the proof of the theorem. □

5. Final Remarks and Open Problems

It is natural to ask whether the rather general structure of the partitions $\mathbf{\Delta}$, whose mesh diameter tends to zero in Theorem 1.1, can be relaxed to obtain a.e. convergence for a class larger than $L(\log^+ L)^{d-1}$. A result in this direction is supported by the fact that in the case of piecewise constant functions, we get a.e. convergence for all L_1-functions, provided that the underlying sequence of partitions is nested. This holds as the sequence of projection operators applied to an L_1-function then forms a martingale. Although it first seems that approaching this problem for general spline orders under the same framework should lead to a positive or negative answer, we must say that it is far from clear if such a result holds. On the other hand, it is unclear how to generalize Saks' construction from [12] to this setting, since the sequence of partitions constructed in the proof of Theorem 1.2 is not nested.

We close this work with the following open problem.

PROBLEM 1. Is it true that the a.e. convergence in Theorem 1.1 holds for all $f \in L_1$ under the assumption that the sequence of partitions is nested?

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CHAPTER 8

Almost everywhere convergence of spline sequences

ALMOST EVERYWHERE CONVERGENCE OF SPLINE SEQUENCES

PAUL F. X. MÜLLER AND MARKUS PASSENBRUNNER

Abstract. We prove the analogue of the Martingale Convergence Theorem for polynomial spline sequences. Given a natural number k and a sequence (t_i) of knots in $[0, 1]$ with multiplicity $\leq k-1$, we let P_n be the orthogonal projection onto the space of spline polynomials in $[0, 1]$ of degree $k-1$ corresponding to the grid $(t_i)_{i=1}^n$. Let X be a Banach space with the Radon-Nikodým property. Let (g_n) be a bounded sequence in the Bochner-Lebesgue space $L^1_X[0, 1]$ satisfying

$$g_n = P_n(g_{n+1}), \qquad n \in \mathbb{N}.$$

We prove the existence of $\lim_{n\to\infty} g_n(t)$ in X for almost every $t \in [0, 1]$. Already in the scalar valued case $X = \mathbb{R}$ the result is new.

1. Introduction

In this paper we prove a convergence theorem for splines in vector valued L^1-spaces. By way of introduction we consider the analogous convergence theorems for martingales with respect to a filtered probability space $(\Omega, (\mathcal{A}_n), \mu)$. We first review two classical theorems for scalar valued martingales in $L^1 = L^1(\Omega, \mu)$. See Neveu [6].

(M1) Let $g \in L^1$. If $g_n = \mathbb{E}(g|\mathcal{A}_n)$ then $\|g_n\|_1 \leq \|g\|_1$ and (g_n) converges almost everywhere and in L^1.

(M2) Let (g_n) be a bounded sequence in L^1 such that $g_n = \mathbb{E}(g_{n+1}|\mathcal{A}_n)$. Then (g_n) converges almost everywhere and $g = \lim g_n$ satisfies $\|g\|_1 \leq \sup \|g_n\|_1$.

Next we turn to vector valued martingales. We fix a Banach space X and let $L^1_X = L^1_X(\Omega, \mu)$ denote the Bochner-Lebesgue space. The Radon-Nikodým property (RNP) of the Banach space X is intimately tied to martingales in Banach spaces. We refer to the book by Diestel and Uhl [3] for the following basic and well known results.

(M3) Let $g \in L^1_X$. If $g_n = \mathbb{E}(g|\mathcal{A}_n)$ then $\|g_n\|_{L^1_X} \leq \|g\|_{L^1_X}$. The sequence (g_n) converges almost everywhere in X and in L^1_X. (This holds for any Banach space X.)

(M4) Let (g_n) be a bounded sequence in L^1_X such that $g_n = \mathbb{E}(g_{n+1}|\mathcal{A}_n)$. If the Banach space X satisfies the Radon-Nikodým property, then (g_n) converges almost everywhere in X and $g = \lim g_n$ satisfies $\|g\|_{L^1_X} \leq \sup \|g_n\|_{L^1_X}$. Moreover the L^1_X-density of the μ-absolutely continuous part of the vector

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measure

$$\nu(E) = \lim_{n\to\infty} \int_E g_n d\mu, \qquad E \in \cup \mathcal{A}_n$$

determines $g = \lim g_n$.

(M5) Conversely if X fails to satisfy the Radon Nikodým property, then there exists a filtered probability space $(\Omega, (\mathcal{A}_n), \mu)$ and bounded sequence in $L^1_X(\Omega, \mu)$ satisfying $g_n = \mathbb{E}(g_{n+1}|\mathcal{A}_n)$ such that (g_n) fails to converge almost everywhere in X.

In the present paper we establish a new link between probability (almost sure convergence of martingales, the RNP) and approximation theory (projections onto splines in $[0, 1]$).

We review the basic setting pertaining to spline projections. (See for instance [12], [9], [11].) So, fix an integer $k \geq 2$, let (t_i) a sequence of grid points in $(0, 1)$ where each t_i occurs at most $k - 1$ times. We emphasize that in contrast to [9], in the present paper we don't assume that the sequence of grid points is dense in $(0, 1)$.

Let S_n denote the space of splines on the interval $[0, 1]$ of order k (degree $k - 1$) corresponding to the grid $(t_i)_{i=1}^n$. Let λ denote Lebesgue measure on the unit interval $[0, 1]$. Let P_n be the orthogonal projection with respect to $L^2([0, 1], \lambda)$ onto the space of splines S_n. By Shadrin's theorem [12], P_n admits an extension to $L^1([0, 1], \lambda)$ such that

$$\sup_{n\in\mathbb{N}} \|P_n : L^1([0, 1], \lambda) \to L^1([0, 1], \lambda)\| < \infty.$$

Assuming that the sequence (t_i) is dense in the unit interval $[0, 1]$, the second named author and A. Shadrin [9] proved – in effect – that for any $g \in L^1_X([0, 1], \lambda)$ the sequence $g_n = P_n g$ converges almost everywhere in X. The vector valued version of [9] holds true without any condition on the underlying Banach space X. Thus the paper [9] established the spline analogue of the martingale properties (M1) and (M3) – under the restriction that (t_i) is dense in the unit interval $[0, 1]$.

Our main theorem – extending [9] – shows that the vector valued martingale convergence theorem has a direct counterpart in the context of spline projections. Theorem 1.1 gives the spline analogue of the martingale properties (M2) and (M4). The first step in the proof of Theorem 1.1 consists in showing that the restrictive density condition on (t_i) may be lifted from the assumptions in [9].

Theorem 1.1 (Spline Convergence Theorem). *Let X be a Banach space with RNP and (g_n) be a sequence in L^1_X with the properties*

(1) $\sup_n \|g_n\|_{L^1_X} < \infty$,

(2) $P_m g_n = g_m$ *for all* $m \leq n$.

Then, g_n converges λ-a.e. to some L^1_X function.

Already in the scalar case $X = \mathbb{R}$ Theorem 1.1 is a new result. In the course of its proof we *intrinsically* describe the pointwise limit of the sequence (g_n). At the end of Section 6 we formulate a refined version of Theorem 1.1 employing the tools we developed for its proof. This includes an explicit expression of $\lim g_n$ in terms of B-splines.

We point out that only under significant restrictions on the geometry of the grid points (t_i), is it true that the spline projections P_n are Calderon-Zygmund operators (with constants independent of n). See [4].

Our present paper should be seen in context with the second named author's work [7], where Burkholder's martingale inequality

$$\big\| \sum \pm(\mathbb{E}(f|\mathcal{A}_n) - \mathbb{E}(f|\mathcal{A}_{n-1}))\big\|_{L^p(\Omega,\mu)} \leq C_p \|f\|_{L^p(\Omega,\mu)},$$

was given a counterpiece for spline projections as follows

$$\big\| \sum \pm(P_n(g) - P_{n-1}(g))\big\|_{L^p([0,1])} \leq C_p \|g\|_{L^p([0,1])},$$

where $1 < p < \infty$, and $C_p \sim p^2/(p-1)$. The corresponding analogue for vector valued spline projections is still outstanding. (See however [5] for a special case.)

Organization. The presentation is organized as follows. In Section 2, we collect some important facts and tools used in this article. Section 3 treats the convergence of $P_n g$ for L^1_X-functions g. Section 4 contains special spline constructions associated to the point sequence (t_i). In Section 5, we give a measure theoretic lemma that is subsequently employed and may be of independent interest in the theory of splines. Finally, in Section 6, we give the proof of the Spline Convergence Theorem.

2. Preliminaries

2.1. **Basics about vector measures.** We refer to the book [3] by J. Diestel and J.J. Uhl for basic facts on martingales and vector measures. Let $(\Omega, \mathcal{A})$ be a measure space and X a Banach space. Every σ-additive map $\nu : \mathcal{A} \to X$ is called a *vector measure*. The *variation* $|\nu|$ of ν is the set function

$$|\nu|(E) = \sup_{\pi} \sum_{A \in \pi} \|\nu(A)\|_X,$$

where the supremum is taken over all partitions π of E into a finite numer of pairwise disjoint members of $\mathcal{A}$. If ν is of bounded variation, i.e., $|\nu|(\Omega) < \infty$, the variation $|\nu|$ is σ-additive. If $\mu : \mathcal{A} \to [0, \infty)$ is a measure and $\nu : \mathcal{A} \to X$ is a vector measure, ν is called *μ-continuous*, if $\lim_{\mu(E)\to 0} \nu(E) = 0$ for all $E \in \mathcal{A}$.

Definition 2.1. A Banach space X has the *Radon-Nikodým property (RNP)* if for every measure space $(\Omega, \mathcal{A})$, for every positive measure μ on $(\Omega, \mathcal{A})$ and for every μ-continuous vector measure ν of bounded variation, there exists a function $f \in L^1_X(\Omega, \mathcal{A}, \mu)$ such that

$$\nu(A) = \int_A f \, \mathrm{d}\mu, \qquad A \in \mathcal{A}.$$

Theorem 2.2 (Lebesgue decomposition of vector measures)**.** *Let $(\Omega, \mathcal{A})$ be a measure space, X a Banach space, $\nu : \mathcal{A} \to X$ a vector measure and $\mu : \mathcal{A} \to [0, \infty)$ a measure. Then, there exist unique vector measures $\nu_c, \nu_s : \mathcal{A} \to X$ such that*

(1) $\nu = \nu_c + \nu_s$,
(2) ν_c is μ-continuous,
(3) $x^*\nu_s$ and μ are mutually singular for each $x^* \in X^*$.

If ν is of bounded variation, ν_c and ν_s are of bounded variation as well, $|\nu|(E) = |\nu_c|(E) + |\nu_s|(E)$ for each $E \in \mathcal{A}$ and $|\nu_s|$ and μ are mutually singular.

The following theorem provides the fundamental link between convergence of vector valued martingales and the RNP of the underlying Banach space X. See Diestel-Uhl [3, Theorem V.2.9]. It is the point of reference for our present work on convergence of spline projections.

Theorem 2.3 (Martingale convergence theorem)**.** *Let $(\Omega, \mathcal{A})$ be a measure space and $\mu : \mathcal{A} \to [0, \infty)$ a measure. Let $(\mathcal{A}_n)$ be a sequence of increasing sub-σ-algebras of $\mathcal{A}$. Let X be a Banach space, let (g_n) be a bounded sequence in $L^1_X(\Omega, \mathcal{A}_n, \mu)$, such that $g_n = \mathbb{E}(g_{n+1}|\mathcal{A})$ and let*

$$\nu(E) = \lim_{n\to\infty} \int_E g_n d\mu, \qquad E \in \cup\mathcal{A}_n.$$

Let $\nu = \nu_c + \nu_s$ denote the Lebesgue decomposition of ν with respect to μ. Then $\lim_{n\to\infty} g_n$ exists almost everywhere with respect to μ if and only if ν_c has a Radon-Nikodým derivative $f \in L^1_X(\Omega, \mu)$. In this case

$$\lim_{n\to\infty} g_n = \mathbb{E}(f|\mathcal{A}_\infty),$$

where $\mathcal{A}_\infty$ is the σ-algebra generated by $\cup\mathcal{A}_n$.

Let X be a Banach space, let $v \in L^1(\Omega, \mathcal{A}, m)$ and $x \in X$. We recall that $v \otimes x : \Omega \to X$ is defined by $v \otimes x(\omega) = v(\omega)x$ and that

$$L^1(\Omega, \mathcal{A}, m) \otimes X = \operatorname{span}\{v_i \otimes x_i : v_i \in L^1(\Omega, \mathcal{A}, m), x_i \in X\}.$$

The following lemmata are taken from [10].

Lemma 2.4. *For any Banach space X, the algebraic tensor product $L^1(\Omega, \mathcal{A}, m) \otimes X$ is a dense subspace of the Bochner-Lebesgue space $L^1_X(\Omega, \mathcal{A}, m)$.*

Lemma 2.5. *Given a bounded operator $T : L^1(\Omega, \mathcal{A}, m) \to L^1(\Omega', \mathcal{A}', m')$ there exists a unique bounded linear map $\widetilde{T} : L^1_X(\Omega, \mathcal{A}, m) \to L^1_X(\Omega', \mathcal{A}', m')$ such that*

$$\widetilde{T}(\varphi \otimes x) = T(\varphi)x, \qquad \varphi \in L^1(\Omega, \mathcal{A}, m), x \in X.$$

Moreover, $\|\widetilde{T}\| = \|T\|$.

Lemma 2.6. *Let X_0 be a separable closed subspace of a Banach space X. Then, there exists a sequence (x_n^*) in the unit ball of the dual X^* of X such that*

$$\|x\| = \sup_n |x_n^*(x)|, \qquad x \in X_0.$$

2.2. **Tools from Real Analysis.** We use the book by E. Stein [13] as our basic reference to Vitali's covering Lemma and weak-type estimates for the Hardy-Littlewood maximal function.

Lemma 2.7 (Vitali covering lemma)**.** *Let $\{C_x : x \in \Lambda\}$ be an arbitrary collection of balls in $\mathbb{R}^d$ such that $\sup\{\operatorname{diam}(C_x) : x \in \Lambda\} < \infty$. Then, there exists a countable subcollection $\{C_x : x \in J\}$, $J \subset \Lambda$ of balls from the original collection that are disjoint and satisfy*

$$\bigcup_{x\in\Lambda} C_x \subset \bigcup_{x\in J} 5C_x.$$

Vitali's covering Lemma implies weak type estimates for the Hardy-Littlewood maximal function.

Theorem 2.8. *Let $f \in L^1_X$ and $\mathcal{M}f(t) := \sup_{I\ni t} \frac{1}{\lambda(I)} \int_I \|f(s)\|_X \,\mathrm{d}s$ the Hardy-Littlewood maximal function. Then $\mathcal{M}$ satisfies the weak type estimate*

$$\lambda(\{\mathcal{M}f > u\}) \le \frac{C\|f\|_{L^1_X}}{u}, \qquad u > 0,$$

where $C > 0$ is an absolute constant.

2.3. **Spline spaces.** Denote by $|\Delta_n|$ the maximal mesh width of the grid $\Delta_n = (t_i)_{i=1}^n$ augmented with k times the boundary points $\{0,1\}$. Recall that P_n is the orthogonal projection operator onto the space S_n of splines corresponding to the grid Δ_n, which is a conditional expectation operator for $k=1$.

For the following, we introduce the notation $A(t) \lesssim B(t)$ to indicate the existence of a constant $c > 0$ that only depends on k such that $A(t) \leq cB(t)$, where t denotes all explicit or implicit dependences that the expressions A and B might have. As is shown by A. Shadrin, the sequence (P_n) satisfies L^1 estimates as follows:

Theorem 2.9 ([12]). *The orthogonal projection P_n admits a bounded extension to L^1 such that*

$$\sup_n \|P_n : L^1 \to L^1\| \lesssim 1.$$

By Lemma 2.5, the operator P_n can be extended to the vector valued L^1 space L^1_X with the same norm so that for all $\varphi \in L^1$ and $x \in X$, we have $P_n(\varphi \otimes x) = (P_n\varphi)x$. We also have the identity

$$\int_0^1 P_n g(t) \cdot f(t) \,\mathrm{d}\lambda(t) = \int_0^1 g(t) \cdot P_n f(t) \,\mathrm{d}\lambda(t), \qquad g \in L^1_X, f \in L^\infty, \tag{2.1}$$

which is just the extension of the fact that P_n is selfadjoint on L^2.

Fix $f \in C[0,1]$. Consider the kth forward differences of f given by

$$D_h^k f(t) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(t+jh).$$

The kth modulus of smoothness of f in L^∞ is defined as

$$\omega_k(f,\delta) = \sup_{0 \leq h \leq \delta} \sup_{0 \leq t \leq 1-kh} |D_h^k f(t)|,$$

where $0 \leq \delta \leq 1/k$. We have $\lim_{\delta \to 0} \omega_k(f,\delta) = 0$ for any $f \in C[0,1]$. Any continuous function can be approximated by spline functions satisfying the following quantitative error estimate.

Theorem 2.10 ([11, Theorem 6.27]). *Let $f \in C[0,1]$. Then,*

$$d(f, S_n)_\infty \lesssim \omega_k(f, |\Delta_n|),$$

where $d(f,S_n)_\infty$ is the distance between f and S_n in the sup*-norm. Therefore, if $|\Delta_n| \to 0$, we have $d(f,S_n)_\infty \to 0$.*

Denote by $(N_i^{(n)})_i$ the B-spline basis of S_n normalized such that it forms a partition of unity and by $(N_i^{(n)*})_i$ its corresponding dual basis in S_n. Observe that

$$P_n f(t) = \sum_i \langle f, N_i^{(n)} \rangle N_i^{(n)*}(t), \qquad f \in L^2.$$

Since the B-spline functions $N_i^{(n)}$ are contained in $C[0,1]$, we can also insert L^1-functions as well as measures in the above formula.

If we set $a_{ij}^{(n)} = \langle N_i^{(n)*}, N_j^{(n)*} \rangle$, we can expand the dual B-spline functions as a linear combination of B-spline functions with those coefficients:

$$N_i^{(n)*} = \sum_j a_{ij}^{(n)} N_j^{(n)}. \tag{2.2}$$

Moreover, for $t \in [0,1]$ denote by $I_n(t)$ a smallest grid point interval of positive length in the grid Δ_n that contains the point t. We denote by $i_n(t)$ the largest index i such that $I_n(t) \subset \operatorname{supp} N_i^{(n)}$. Additionally, denote by $h_{ij}^{(n)}$ the length of the convex hull of the union of the supports of $N_i^{(n)}$ and $N_j^{(n)}$.

With this notation, we can give the following estimate for the numbers $a_{ij}^{(n)}$ and, a fortiori, for $N_i^{(n)*}$:

Theorem 2.11 ([9])**.** *There exists $q \in (0,1)$ depending only on the spline order k, such that the numbers $a_{ij}^{(n)} = \langle N_i^{(n)*}, N_j^{(n)*}\rangle$ satisfy the inequality*

$$|a_{ij}^{(n)}| \lesssim \frac{q^{|i-j|}}{h_{ij}^{(n)}},$$

and therefore, in particular, for all i,

$$|N_i^{(n)*}(t)| \lesssim \frac{q^{|i-i_n(t)|}}{\max\big(\lambda(I_n(t)), \lambda(\operatorname{supp} N_i^{(n)})\big)}, \qquad t \in [0,1].$$

Proof. The first inequality is proved in [9] and the second one is an easy consequence of the first one inserted in formula (2.2) for $N_i^{(n)*}$. □

An almost immediate consequence of this estimate is the following pointwise maximal inequality for $P_n g$:

Theorem 2.12 ([9])**.** *For all $g \in L^1_X$,*

$$\sup_n \|P_n g(t)\|_X \lesssim \mathcal{M}g(t), \qquad t \in [0,1],$$

where $\mathcal{M}g(t) = \sup_{I \ni t} \frac{1}{\lambda(I)} \int_I \|g(s)\|_X \,\mathrm{d}s$ denotes the Hardy-Littlewood maximal function.

This result and Theorem 2.10, combined with Theorem 2.8, imply the a.e. convergence of $P_n g$ to g for any L^1-function g provided that the point sequence (t_i) is dense in the unit interval $[0,1]$, cf. [9].

As the spline spaces S_n form an increasing sequence of subspaces of L^2, we can write the B-spline function $N_i^{(n)}$ as a linear combination of the finer B-spline functions $(N_j^{(n+1)})$. The exact form of this expansion is given by Böhm's algorithm [1] and it states in particular that the following result is valid:

Proposition 2.13. *Let $f = \sum_i \alpha_i N_i^{(m)} \in S_m$ for some m. Then, there exists a sequence (β_i) of coefficients so that*

$$f \equiv \sum_i \beta_i N_i^{(m+1)}$$

and, for all i, β_i is a convex combination of α_{i-1} and α_i.

By induction, an immediate consequence of this result is

Corollary 2.14. *For any positive integers $n \geq m$ and any index i, the B-spline function $N_i^{(m)}$ can be represented as*

$$N_i^{(m)} \equiv \sum_j \lambda_j N_j^{(n)},$$

with coefficients $\lambda_j \in [0,1]$ for all j.

In the following theorem it is convenient to display explicitly the order k of the B-splines $N_i^{(n)} = N_{i,k}^{(n)}$. The relation between the sequences $(N_{i,k}^{(n)})_i$ and $(N_{i,k-1}^{(n)})_i$ is given by well known recursion formulae, for which we refer [2]. See also [11].

Theorem 2.15. *Let $[a,b] = \operatorname{supp} N_{i,k}^{(n)}$. Then, the B-spline function $N_{i,k}^{(n)}$ of order k can be expressed in terms of two B-spline functions of order $k-1$ as follows:*

$$N_{i,k}^{(n)}(t) = \frac{t-a}{\lambda(\operatorname{supp} N_{i,k-1}^{(n)})} N_{i,k-1}^{(n)}(t) + \frac{b-t}{\lambda(\operatorname{supp} N_{i+1,k-1}^{(n)})} N_{i+1,k-1}^{(n)}(t).$$

3. Convergence of $P_n g$

As we are considering arbitrary sequences of grid points (t_i) which are not necessarily dense in $[0,1]$, as a first stage in the proof of the Spline Convergence Theorem, we examine the convergence of $P_n g$ for $g \in L_X^1$.

We first notice that $P_n g$ converges in L^1. Indeed, this is a consequence of the uniform boundedness of P_n on L^1 as we will now show. Observe that for $g \in L^2$, we get that if we define S_∞ as the L^2 closure of $\cup S_n$ and P_∞ as the orthogonal projection onto S_∞,

$$\|P_n g - P_\infty g\|_{L^2} \to 0.$$

Next, we show that this definition of P_∞ can be extended to L^1 functions g. So, let $g \in L^1$ and $\varepsilon > 0$. Since L^2 is dense in L^1, we can choose $f \in L^2$ with the property $\|g - f\|_1 < \varepsilon$. Now, choose N_0 sufficiently large that for all $m, n > N_0$, we have $\|(P_n - P_m)f\|_2 < \varepsilon$. Then, we obtain

$$\begin{aligned} \|(P_n - P_m)g\|_{L^1} &\le \|(P_n - P_m)(g-f)\|_{L^1} + \|(P_n - P_m)f\|_{L^1} \\ &\le 2C\varepsilon + \|(P_n - P_m)f\|_{L^2} \\ &\le (2C+1)\varepsilon \end{aligned}$$

for a constant C depending only on k by Theorem 2.9. This means that $P_n g$ converges in L^1 to some limit that we will again call $P_\infty g$. It actually coincides with the operator P_∞ on L^2 and satisfies the same L^1 bound as the sequence (P_n). Summing up we have

$$\|P_n g - P_\infty g\|_{L^1} \to 0,$$

for any $g \in L^1$. Applying Lemma 2.5 to $(P_n - P_\infty)$ we obtain the following vector valued extension. For any Banach space X

$$\|P_n g - P_\infty g\|_{L_X^1} \to 0,$$

for $g \in L_X^1$.

The next step is to show pointwise convergence of $P_n g$ for continuous functions g. We define U to be the complement of the set of all accumulation points of the given knot sequence (t_i). This set U is open, so it can be written as a disjoint union of open intervals

$$U = \cup_{j=1}^\infty U_j.$$

Lemma 3.1. *Let $g \in C[0,1]$. Then, $P_n g$ converges pointwise a.e. to $P_\infty g$ with respect to Lebesgue measure.*

Proof. We first show that on each interval U_j, $P_n g$ converges locally uniformly. Let $A \subset U_j$ be a compact subset. Then the definition of U_j implies that $s := \inf\{\lambda(I_n(t)) : t \in A, n \in \mathbb{N}\}$ is positive. Observe that of course, since in particular

$g \in L^1[0,1]$, the sequence $P_n g$ converges in L^1. Therefore, for $\varepsilon > 0$, we can choose M so large that for all $n, m \geq M$, $\|P_n g - P_m g\|_{L^1} \leq \varepsilon s$. We then estimate by Theorem 2.11 for $n \geq m \geq M$ and $t \in A$:

$$\begin{aligned}
|(P_n - P_m)g(t)| &= |P_n(P_n - P_m)g(t)| \\
&= \Big| \sum_i \langle (P_n - P_m)g, N_i^{(n)} \rangle N_i^{(n)*}(t) \Big| \\
&\lesssim \sum_i \frac{q^{|i - i_n(t)|}}{\lambda(I_n(t))} \|(P_n - P_m)g\|_{L^1(\operatorname{supp} N_i^{(n)})} \\
&\leq \|(P_n - P_m)g\|_{L^1([0,1])} \sum_i \frac{q^{|i - i_n(t)|}}{s} \\
&\lesssim \frac{\|(P_n - P_m)g\|_{L^1[0,1]}}{s} \leq \varepsilon,
\end{aligned}$$

so $P_n g$ converges uniformly on A.

If $t \in U^c$, we can assume that on both sides of t, there is a subsequence of grid points converging to t, since if there is a side that does not have a sequence of grid points converging to t, the point t would be an endpoint of an interval U_j and the union over all endpoints of U_j is countable and therefore a Lebesgue zero set. Let $\varepsilon > 0$ and let ℓ be such that

$$q^\ell \|g\|_{L^\infty} \leq \varepsilon. \tag{3.1}$$

We choose M so large that for any $m \geq M$ on each side of t there are ℓ grid points of Δ_m and each of those grid point intervals has the property that the length is $< \delta$ with $\delta > 0$ being such that $\omega_k(g, \delta) < \varepsilon$, where ω_k is the kth modulus of smoothness. With this choice, by Theorem 2.10, there exists a function $f \in S_M$ with $\|f\|_{L^\infty} \lesssim \|g\|_{L^\infty}$ that approximates g well on the smallest interval B that contains $\ell - k$ grid points to the left of t and $\ell - k$ grid points to the right of t in Δ_M in the sense that

$$\|f - g\|_{L^\infty(B)} \lesssim \omega_k(g, \delta) \leq \varepsilon. \tag{3.2}$$

Therefore, we can write for $n, m \geq M$

$$(P_n - P_m)g(t) = P_n(g - f)(t) + P_m(f - g)(t).$$

Next, estimate $P_n(g - f)(t)$ for $n \geq M$ by Theorem 2.11:

$$\begin{aligned}
|P_n(g - f)(t)| &= \Big| \sum_i \langle g - f, N_i^{(n)} \rangle N_i^{(n)*}(t) \Big| \\
&\lesssim \sum_i \|g - f\|_{L^\infty(\operatorname{supp}(N_i^{(n)}))} \lambda(\operatorname{supp} N_i^{(n)}) \frac{q^{|i - i_n(t)|}}{\lambda(\operatorname{supp} N_i^{(n)})} \\
&= \sum_i q^{|i - i_n(t)|} \|g - f\|_{L^\infty(\operatorname{supp} N_i^{(n)})}.
\end{aligned}$$

In estimating the above series we distinguish two cases for the value of i:

$$|i - i_n(t)| \leq \ell - 2k, \qquad \text{and} \qquad |i - i_n(t)| > \ell - 2k.$$

Using $\|g-f\|_{L^\infty(\operatorname{supp} N_i^{(n)})} \leq \|g-f\|_{L^\infty(B)}$ and (3.2) we get

$$\sum_{i:|i-i_n(t)|\leq \ell-2k} q^{|i-i_n(t)|}\|g-f\|_{L^\infty(\operatorname{supp} N_i^{(n)})} \lesssim \varepsilon.$$

Using $\|g-f\|_{L^\infty(\operatorname{supp} N_i^{(n)})} \lesssim \|g\|_{L^\infty}$ and (3.1) gives

$$\sum_{i:|i-i_n(t)|> \ell-2k} q^{|i-i_n(t)|}\|g-f\|_{L^\infty(\operatorname{supp} N_i^{(n)})} \lesssim \varepsilon.$$

This yields $|P_n(g-f)(t)| \lesssim \varepsilon$ for $n \geq M$ and therefore $P_n g(t)$ converges as $n \to \infty$. $\square$

The following theorem establishes the spline analogue of the martingale results (M1) and (M3). The role of Lemma 3.1 in the proof given below is to free the main theorem in [9] from the restriction that the sequence of knots (t_i) is dense in $[0,1]$.

Theorem 3.2. *Let X be any Banach space. For $f \in L^1_X$, there exists $E \subset [0,1]$ with $\lambda(E) = 0$ such that*

$$\lim_{n\to\infty} P_n f(t) = P_\infty f(t),$$

for any $t \in [0,1] \setminus E$.

Proof. The proof uses standard arguments involving Lemma 3.1, Theorems 2.12 and 2.8. (See [9].)

Step 1: (The scalar case.) Fix $v \in L^1$ and $\ell \in \mathbb{N}$. Put

$$A^{(\ell)}(v) = \bigcap_N \bigcup_{m,n\geq N} \{t : |P_n v(t) - P_m v(t)| > 1/\ell\}.$$

By Lemma 3.1, for any $u \in C[0,1]$,

$$\lambda(A^{(\ell)}(v)) = \lambda(A^{(\ell)}(v-u))$$

Let $P^*(v-u)(t) = \sup_n |P_n(v-u)(t)|$. Clearly we have

$$\lambda(A^{(\ell)}(v-u)) \leq \lambda(\{t : 2P^*(v-u)(t) \geq 1/\ell\}).$$

By Theorem 2.12, P^* is dominated pointwise by the Hardy-Littlewood maximal function and the latter is of weak type 1-1. Hence

$$\lambda(\{t : P^*(v-u)(t) \geq 1/\ell\}) \lesssim \ell\|v-u\|_{L^1}$$

Now fix $\varepsilon > 0$. Since $C[0,1]$ is dense in L^1, there exists $u \in C[0,1]$ such that $\|v-u\|_{L^1} \leq \varepsilon/\ell$. Thus, we obtained $\lambda(A^{(\ell)}(v)) < \varepsilon$ for any $\varepsilon > 0$, or $\lambda(A^{(\ell)}(v)) = 0$. It remains to observe that

$$\lambda(\{t : P_n v(t) \text{ does not converge}\}) = \lambda\Big(\bigcup_\ell A^{(\ell)}(v)\Big) = 0.$$

Step 2:(Vector valued extension.) Let $g_m = v_m \otimes x_m$ where $v_m \in L^1$ and $x_m \in X$ and let $g \in L^1 \otimes X$ be given as

$$g = \sum_{m=1}^M g_m.$$

Applying Step 1 to v_m shows that $P_n g(t)$ converges in X for λ-almost every $t \in [0,1]$. Taking into account that $L^1 \otimes X$ is dense in L^1_X, we may now repeat the argument above to finish the proof. Details are as follows: Fix $f \in L^1_X$ and $\ell \in \mathbb{N}$. Put

$$A^{(\ell)}(f) = \bigcap_N \bigcup_{m,n \geq N} \{t : \|P_n f(t) - P_m f(t)\|_X > 1/\ell\}.$$

Then

$$\lambda(\{t : P_n f(t) \text{ does not converge in } X\}) = \lambda\Big(\bigcup_\ell A^{(\ell)}(f)\Big).$$

It remains to prove that $\lambda(A^{(\ell)}(f)) = 0$. To this end observe that for $g \in L^1 \otimes X$ we have $\lambda(A^{(\ell)}(f)) = \lambda(A^{(\ell)}(f-g))$. Define the maximal function $P^*(f-g)(t) = \sup_n \|P_n(f-g)(t)\|_X$. Clearly we have

$$\lambda(A^{(\ell)}(f-g)) \leq \lambda(\{t : 2P^*(f-g)(t) \geq 1/\ell\}).$$

By Theorem 2.12, and the weak type 1-1 estimate for the Hardy-Littlewood maximal function,

$$\lambda(\{t : P^*(f-g)(t) \geq 1/\ell\}) \lesssim \ell \|f-g\|_{L^1_X}.$$

Fix $\varepsilon > 0$, choose $g \in L^1 \otimes X$ such that $\|f-g\|_{L^1_X} \leq \varepsilon/\ell$. This gives $\lambda(A^{(\ell)}(f)) \lesssim \varepsilon$ for any $\varepsilon > 0$, proving that $\lambda(A^{(\ell)}(f)) = 0$. $\square$

4. B-spline constructions

Recall that we defined U to be the complement of the set of all accumulation points of the sequence (t_i). This set U is open, so it can be written as a disjoint union of open intervals

$$U = \cup_{j=1}^\infty U_j.$$

Observe that, since a boundary point a of U_j is an accumulation point of the sequence (t_j), there exists a subsequence of grid points converging to a. Let

$$B_j := \{a \in \partial U_j : \text{ there is } \textit{no} \text{ sequence of grid points contained in } U_j \text{ that converges to } a\}$$

Now we set $V_j := U_j \cup B_j$ and $V := \cup_j V_j$.

Consider an arbitrary interval V_{j_0} and set $a = \inf V_{j_0}$, $b = \sup V_{j_0}$. We define the sequences (s_j) and $(s_j^{(n)})$ – rewritten in increasing order with multiplicities included – to be the points in (t_j) and $(t_j)_{j=1}^n$, respectively, that are contained in V_{j_0}. If $a \in V_{j_0}$, the sequence (s_j) is finite to the left and we extend the sequences (s_j) and $(s_j^{(n)})$ so that they contain the point a k times and they are still increasing. Similarly, if $b \in V_{j_0}$, the sequence (s_j) is finite to the right and we extend the sequences (s_j) and $(s_j^{(n)})$ so that they contain the point b k times and they are still increasing. Observe that if $a \notin V_{j_0}$ or $b \notin V_{j_0}$, the sequence (s_j) is infinite to the left or infinite to the right, respectively. We choose the indices of the sequences (s_j) and $(s_j^{(n)})$ so that for fixed j and n sufficiently large, we have $s_j = s_j^{(n)}$. Let $(\bar{N}_j)$ and $(\bar{N}_j^{(n)})$ be the sequences of B-spline functions corresponding to the sequences (s_j) and $(s_j^{(n)})$, respectively. Observe that the choice of the sequences (s_j) and $(s_j^{(n)})$ implies for all j that $\bar{N}_j \equiv \bar{N}_j^{(n)}$ if n is sufficiently large. Let $(N_j^{(n)})$ be the sequence of those B-spline functions from Section 2 whose supports intersect the set V_{j_0} on a set of positive Lebesgue measure, but do not contain any of the points $\partial U_{j_0} \setminus B_{j_0}$

and without loss of generality, we assume that this sequence is enumerated in such a way that starting index and ending index coincide with the ones of the sequence $(\bar N_j^{(n)})_j$. Then, the relation between $(N_j^{(n)})_j$ and $(\bar N_j)_j$ is given by the following lemma:

Lemma 4.1. *For all j, the sequence of functions $(N_j^{(n)}\mathbb{1}_{V_{j_0}})$ converges uniformly to some function that coincides with $\bar N_j$ on U_{j_0}.*

Proof. If the support of $N_i^{(n)}$ is a subset of V_{j_0} for sufficiently large n, the sequence $n \mapsto N_i^{(n)}$ is eventually constant and coincides by definition with $\bar N_i$. In the other case, this follows by the recursion formula (Theorem 2.15) for B-splines and observing that for piecewise linear B-splines, this is clear. □

In view of the above lemma, we may assume that $\bar N_i$ coincides with the uniform limit of the sequence $(N_i^{(n)}\mathbb{1}_{V_{j_0}})$. Define $(\bar N_j^{(n)*})$ to be the dual B-splines to $(\bar N_j^{(n)})$. For $t \in [0,1]$ denote by $\bar I_n(t)$ a smallest grid point interval of positive length in the grid $(s_j^{(n)})$ that contains the point t. We denote by $\bar\imath_n(t)$ the largest index i such that $\bar I_n(t) \subset \operatorname{supp} \bar N_i^{(n)}$. Additionally, denote by $\bar h_{ij}^{(n)}$ the length of the convex hull of the union of the supports of $\bar N_i^{(n)}$ and $\bar N_j^{(n)}$. Similarly we let $\bar I(t)$ denote a smallest grid point interval of positive length in the grid (s_j) containing $t \in [0,1]$. We denote by $\bar\imath(t)$ the largest index i such that $\bar I(t) \subset \operatorname{supp} \bar N_i$. Next, we identify dual functions to the sequence $(\bar N_j)$:

Lemma 4.2. *For each j, the sequence $\bar N_j^{(n)*}$ converges uniformly on each interval $[s_i, s_{i+1}]$ to some function $\bar N_j^*$ that satisfies*

(1) $\langle \bar N_j^*, \bar N_i\rangle = \delta_{ij}$ for all i,
(2) for all $t \in U_{j_0}$,

$$|\bar N_j^*(t)| \lesssim \frac{q^{|j-\bar\imath(t)|}}{\lambda(\bar I(t))}, \tag{4.1}$$

where $q \in (0,1)$ is given by Theorem 2.11.

Proof. We fix the index j, the point $t \in U_{j_0}$ and $\varepsilon > 0$. Next, we choose M sufficiently large so that for all $m \geq M$ and all ℓ with the property $|\ell - \bar\imath(t)| \leq L$ we have $s_\ell^{(m)} = s_\ell$, where L is chosen so that $q^L/\lambda(\bar I(t)) \leq \varepsilon$ and $|j - \bar\imath(t)| \leq L - k$. For $n \geq m \geq M$, we can expand the function $\bar N_j^{(m)*}$ in the basis $(\bar N_i^{(n)*})$ and write

$$\bar N_j^{(m)*} = \sum_i \alpha_{ji} \bar N_i^{(n)*}. \tag{4.2}$$

We now turn to estimating the coefficients α_{ji} defined by equation (4.2). Observe that for ℓ with $|\ell - \bar\imath(t)| \leq L - k$, we have $\bar N_\ell^{(m)} \equiv \bar N_\ell^{(n)}$, and therefore, for such ℓ,

$$\delta_{j\ell} = \langle \bar N_j^{(m)*}, \bar N_\ell^{(m)}\rangle = \langle \bar N_j^{(m)*}, \bar N_\ell^{(n)}\rangle = \sum_i \alpha_{ji}\langle \bar N_i^{(n)*}, \bar N_\ell^{(n)}\rangle = \alpha_{j\ell},$$

which means that the expansion (4.2) takes the form

$$\bar N_j^{(m)*} = \bar N_j^{(n)*} + \sum_{\ell : |\ell - \bar\imath(t)| > L-k} \alpha_{j\ell} \bar N_\ell^{(n)*}. \tag{4.3}$$

Next we show that $|\alpha_{j\ell}|$ is bounded by a constant independently of j, ℓ and m, n. Recall $\bar{h}_{ij}^{(m)}$ denotes the length of the smallest interval containing $\operatorname{supp} \bar{N}_i^{(m)} \cup \operatorname{supp} \bar{N}_j^{(m)}$. By Theorem 2.11, applied to the matrix $(\bar{a}_{ij}^{(m)}) = (\langle \bar{N}_i^{(m)*}, \bar{N}_j^{(m)*} \rangle)$, we get

$$
\begin{aligned}
|\alpha_{j\ell}| = |\langle \bar{N}_j^{(m)*}, \bar{N}_\ell^{(n)} \rangle| &= \Big| \Big\langle \sum_i \bar{a}_{ij}^{(m)} \bar{N}_i^{(m)}, \bar{N}_\ell^{(n)} \Big\rangle \Big| \\
&\lesssim \sum_i \frac{q^{|i-j|}}{\bar{h}_{ij}^{(m)}} \langle \bar{N}_i^{(m)}, \bar{N}_\ell^{(n)} \rangle \le \sum_i \frac{q^{|i-j|}}{\bar{h}_{ij}^{(m)}} \lambda(\operatorname{supp} \bar{N}_i^{(m)}) \\
&\le \sum_i q^{|i-j|} \lesssim 1.
\end{aligned}
$$

This can be used to obtain an estimate for the difference between $\bar{N}_j^{(m)*}(t)$ and $\bar{N}_j^{(n)*}(t)$ by inserting it into (4.3) and applying again Theorem 2.11:

$$
\begin{aligned}
|(\bar{N}_j^{(m)*} - \bar{N}_j^{(n)*})(t)| &\le \sum_{\ell : |\ell - \bar{i}(t)| > L-k} |\alpha_{j\ell}| |\bar{N}_\ell^{(n)*}(t)| \\
&\lesssim \sum_{\ell : |\ell - \bar{i}(t)| > L-k} \frac{q^{|\ell - \bar{i}_n(t)|}}{\lambda(\bar{I}_n(t))} \lesssim \frac{q^L}{\lambda(\bar{I}(t))} \le \varepsilon.
\end{aligned}
$$

This finishes the proof of the convergence part. Estimate (4.1) now follows from the corresponding estimate for $\bar{N}_j^{(n)*}$ in Theorem 2.11.

Now, we turn to the proof of property (1). Let j, i be arbitrary. Choose M sufficiently large so that for all $n \ge M$, we have $\bar{N}_i \equiv \bar{N}_i^{(n)}$ on U_{j_0}, therefore,

$$
\begin{aligned}
|\langle \bar{N}_j^*, \bar{N}_i \rangle - \delta_{ij}| &= |\langle \bar{N}_j^*, \bar{N}_i \rangle - \langle \bar{N}_j^{(n)*}, \bar{N}_i^{(n)} \rangle| = |\langle \bar{N}_j^* - \bar{N}_j^{(n)*}, \bar{N}_i^{(n)} \rangle| \\
&\le \|\bar{N}_j^* - \bar{N}_j^{(n)*}\|_{L^\infty(\operatorname{supp} \bar{N}_i^{(n)})} \cdot \lambda(\operatorname{supp} \bar{N}_i^{(n)}),
\end{aligned}
$$

which, by the local uniform convergence of $\bar{N}_j^{(n)*}$ to $\bar{N}_j^*$, tends to zero. □

5. A measure estimate

Let σ be a measure defined on the unit interval. Recall that $P_n(\sigma)$ is defined by duality. In view of Theorem 2.11, localized and pointwise estimates for $P_n(\sigma)$ are controlled by terms of the form

$$
\sum_{i,j} \frac{q^{|i-j|}}{h_{ij}^{(n)}} |\sigma|(\operatorname{supp} N_i^{(n)}) N_j^{(n)}.
$$

Subsequently the following Lemma will be used to show that $P_n(\sigma)$ converges a.e. to zero, for any measure σ singular to the Lebesgue measure.

Lemma 5.1. *Let F_r be a Borel subset of V^c and θ a positive measure on $[0,1]$ with $\theta(F_r) = 0$ so that for all $x \in F_r$, we have*

$$
\limsup_n b_n(x) > 1/r,
$$

where $b_n(x)$ is a positive function satisfying

$$
b_n(x) \lesssim \sum_{i,j} \frac{q^{|i-j|}}{h_{ij}^{(n)}} \theta(\operatorname{supp} N_i^{(n)}) N_j^{(n)}(x), \qquad x \in F_r.
$$

Then, $\lambda(F_r) = 0$.

Proof. First observe that we can assume that each point in F_r can be approximated from both sides with points of the sequence (t_i), since the set of points in V^c for which this is not possible is a subset of $\cup_j \partial V_j$ and therefore of Lebesgue measure zero.

Step 1: For an arbitrary positive number ε, by the regularity of θ, we can take an open set $U_\varepsilon \subset [0,1]$ with $U_\varepsilon \supset F_r$ and $\theta(U_\varepsilon) \leq \varepsilon$. Then, for $x \in F_r$, we choose a ball $B_x \subset U_\varepsilon$ with center x, define $s_m(x) = \{j : N_j^{(m)}(x) \neq 0\}$ and calculate

$$\begin{aligned} b_m(x) &\lesssim \sum_{i,j} \frac{q^{|i-j|}}{h_{ij}^{(m)}} \theta(\operatorname{supp} N_i^{(m)}) N_j^{(m)}(x) \\ &\lesssim \sum_{j \in s_m(x)} \sum_i \frac{q^{|i-j|}}{h_{ij}^{(m)}} \theta(\operatorname{supp} N_i^{(m)}) \\ &\lesssim \max_{j \in s_m(x)} \sum_i \frac{q^{|i-j|}}{h_{ij}^{(m)}} \theta(\operatorname{supp} N_i^{(m)}) \\ &= C \max_{j \in s_m(x)} (\Sigma_{1,j}^{(m)} + \Sigma_{2,j}^{(m)}), \end{aligned}$$

for some constant C and where

$$\Sigma_{1,j}^{(m)} := \sum_{i \in \Lambda_1^{(m)}} \frac{q^{|i-j|}}{h_{ij}^{(m)}} \theta(\operatorname{supp} N_i^{(m)}), \qquad \Sigma_{2,j}^{(m)} := \sum_{i \in \Lambda_2^{(m)}} \frac{q^{|i-j|}}{h_{ij}^{(m)}} \theta(\operatorname{supp} N_i^{(m)})$$

and

$$\Lambda_1^{(m)} = \{i : \operatorname{supp} N_i^{(m)} \subset B_x\}, \qquad \Lambda_2^{(m)} = (\Lambda_1^{(m)})^c$$

Step 2: Next, we show that it is possible to choose m sufficiently large to have $\Sigma_{2,j}^{(m)} \leq 1/(2Cr)$ for all $j \in s_m(x)$.

To do that, let $j_m \in s_m(x)$ and observe that

$$\Sigma_{2,j_m}^{(m)} = \sum_{i \in \Lambda_2^{(m)}} \frac{q^{|i-j_m|} \theta(\operatorname{supp} N_i^{(m)})}{h_{ij_m}^{(m)}} \leq \sum_{i \in \Lambda_2^{(m)}} \frac{q^{|i-j_m|} \theta(\operatorname{supp} N_i^{(m)})}{d(x, \operatorname{supp} N_i^{(m)})} =: A_{2,j_m}^{(m)},$$

where $d(x, \operatorname{supp} N_i^{(m)})$ denotes the Euclidean distance between x and $\operatorname{supp} N_i^{(m)}$. Now, for $n > m$ sufficiently large, we get

$$\begin{aligned} A_{2,j_n}^{(n)} &= \sum_{\ell \in \Lambda_2^{(n)}} \frac{q^{|\ell - j_n|} \theta(\operatorname{supp} N_\ell^{(n)})}{d(x, \operatorname{supp} N_\ell^{(n)})} \\ &\leq \sum_{i \in \Lambda_2^{(m)}} \sum_{\substack{\ell \in \Lambda_2^{(n)}, \\ \operatorname{supp} N_\ell^{(n)} \subset \operatorname{supp} N_i^{(m)}}} \frac{q^{|\ell - j_n|} \theta(\operatorname{supp} N_\ell^{(n)})}{d(x, \operatorname{supp} N_\ell^{(n)})}. \end{aligned} \tag{5.1}$$

Define $L_{n,m}$ to be the cardinality of the set $\{t_i : m < i \leq n\} \cap B_x \cap [0,x]$ and $R_{n,m}$ the cardinality of $\{t_i : m < i \leq n\} \cap B_x \cap [x,1]$. Put

$$K_{n,m} = \min\{L_{n,m}, R_{n,m}\}.$$

The term (5.1) admits the following upper bound

$$
\begin{aligned}
&q^{K_{n,m}} \sum_{i\in\Lambda_2^{(m)}} \frac{q^{|i-j_m|}}{d(x,\operatorname{supp} N_i^{(m)})} \sum_{\substack{\ell\in\Lambda_2^{(n)},\\ \operatorname{supp} N_\ell^{(n)}\subset \operatorname{supp} N_i^{(m)}}} \theta(\operatorname{supp} N_\ell^{(n)}) \\
&\lesssim q^{K_{n,m}} \sum_{i\in\Lambda_2^{(m)}} \frac{q^{|i-j_m|}}{d(x,\operatorname{supp} N_i^{(m)})}\theta(\operatorname{supp} N_i^{(m)}) = q^{K_{n,m}} A_{2,j_m}^{(m)},
\end{aligned}
$$

Since x can be approximated by grid points from both sides, $\lim_{n\to\infty} K_{n,m} = \infty$, and we can choose m sufficiently large to guarantee

$$\Sigma_{2,j}^{(m)} \leq A_{2,j}^{(m)} \leq \frac{1}{2Cr}.$$

Step 3: Next, we show that for any $x \in F_r$, there exists an open interval $C_x \subset B_x$ such that $\theta(C_x)/\lambda(C_x) \gtrsim 1/(2Cr)$.

By Step 2 and the fact that $\limsup b_n(x) > 1/r$ for $x \in F_r$, there exists an integer m and an index $j_0 \in s_m(x)$ with

$$\Sigma_{1,j_0}^{(m)} \geq \frac{1}{2Cr},$$

which means that

$$
\begin{aligned}
\frac{1}{2Cr} &\leq \sum_{i\in\Lambda_1^{(m)}} \frac{q^{|i-j_0|}}{h_{ij_0}^{(m)}}\theta(\operatorname{supp} N_i^{(m)}) \\
&\leq \sum_{i\in\Lambda_1^{(m)}} \frac{q^{|i-j_0|}}{h_{ij_0}^{(m)}}\theta\big(\operatorname{conv}(\operatorname{supp} N_i^{(m)} \cup \operatorname{supp} N_{j_0}^{(m)})\big),
\end{aligned}
$$

where $\operatorname{conv}(A)$ denotes the convex hull of the set A. Since $\sum_{i\in\Lambda_1^{(m)}} q^{|i-j_0|} \lesssim 1$, there exists a constant c depending only on q and an index i with $\operatorname{supp} N_i^{(m)} \subset B_x$ and

$$\frac{\theta\big(\operatorname{conv}(\operatorname{supp} N_i^{(m)} \cup \operatorname{supp} N_{j_0}^{(m)})\big)}{h_{ij_0}^{(m)}} \geq \frac{c}{2Cr},$$

which means that there exists an open interval C_x with $x \in C_x \subset B_x$ with the property $\theta(C_x)/\lambda(C_x) \geq c/(2Cr)$.

Step 4: Now we finish with a standard argument using the Vitali covering lemma (Lemma 2.7): there exists a countable collection J of points $x \in F_r$ such that $\{C_x : x \in J\}$ are disjoint sets and

$$F_r \subset \bigcup_{x\in F_r} C_x \subset \bigcup_{x\in J} 5C_x.$$

Combining this with Steps 1-3, we conclude

$$\lambda(F_r) \leq \lambda\Big(\bigcup_{x\in J} 5C_x\Big) \leq 5\sum_{x\in J}\lambda(C_x) \leq \frac{10Cr}{c}\sum_{x\in J}\theta(C_x) \leq \frac{10Cr}{c}\theta(U_\varepsilon) \leq \frac{10Cr}{c}\varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, we get that $\lambda(F_r) = 0$. □

6. Proof of the Spline Convergence Theorem

In this section, we prove the Spline Convergence Theorem 1.1. For $f \in S_m$, a consequence of (2.1) is

$$\begin{aligned}\int_0^1 g_n(t) \cdot f(t)\, \mathrm{d}\lambda(t) &= \int_0^1 g_n(t) \cdot P_m f(t)\, \mathrm{d}\lambda(t) = \int P_m g_n(t) \cdot f(t)\, \mathrm{d}\lambda(t) \\ &= \int_0^1 g_m(t) \cdot f(t)\, \mathrm{d}\lambda(t), \qquad n \geq m.\end{aligned}$$

This means in particular that for all $f \in \cup_n S_n$, the limit of $\int_0^1 g_n(t) \cdot f(t)\, \mathrm{d}\lambda(t)$ exists, so we can define the linear operator

$$T : \cup S_n \to X, \qquad f \mapsto \lim_n \int_0^1 g_n(t) \cdot f(t)\, \mathrm{d}\lambda(t).$$

By Alaoglu's theorem, we may choose a subsequence k_n such that the bounded sequence of measures $\|g_{k_n}\|_X \, \mathrm{d}\lambda$ converges in the weak*-topology to some scalar measure μ. Then, as each $f \in \cup_n S_n$ is continuous,

$$\|Tf\|_X \leq \int_0^1 |f(t)|\, \mathrm{d}\mu(t), \qquad f \in \cup S_n. \tag{6.1}$$

We let W denote the $L^1([0,1],\mu)$-closure of $\cup_n S_n$. By (6.1), the operator T extends to W with norm bounded by 1.

We set

$$(P_n T)(t) := \sum_i (T N_i^{(n)}) N_i^{(n)*}(t)$$

which is well defined. Moreover,

$$\begin{aligned}(P_n T)(t) &= \sum_i (T N_i^{(n)}) N_i^{(n)*}(t) \\ &= \sum_i \lim_m \int g_m N_i^{(n)}\, \mathrm{d}\lambda \cdot N_i^{(n)*}(t) \\ &= \sum_i \langle g_n, N_i^{(n)} \rangle N_i^{(n)*}(t) = (P_n g_n)(t) = g_n(t).\end{aligned}$$

Thus we verify a.e. convergence of g_n, by showing a.e. convergence of $P_n T$ below.

Lemma 6.1. *For all $f \in \cup S_n$, the function $f \mathbb{1}_{V_j}$ is contained in W and also $f \mathbb{1}_V$ is contained in W. Additionally, on the complement of $V = \cup V_j$, the σ-algebra $\mathcal{F} = \{A \in \mathcal{B} : \mathbb{1}_A \in W\}$ coincides with the Borel σ-algebra $\mathcal{B}$, i.e., $V^c \cap \mathcal{F} = V^c \cap \mathcal{B}$.*

Proof. Since W is a linear space, it suffices to show the assertion for each B-spline function $N_i^{(m)}$ contained in some S_m. By Corollary 2.14, it can be written as a linear combination of finer B-spline functions ($n \geq m$)

$$N_i^{(m)} = \sum_\ell \lambda_\ell^{(n)} N_\ell^{(n)},$$

where each coefficient $\lambda_\ell^{(n)}$ satisfies the inequality $|\lambda_\ell^{(n)}| \leq 1$. We set

$$h_n := \sum_{\ell \in \Lambda_n} \lambda_\ell^{(n)} N_\ell^{(n)},$$

where the index set Λ_n is defined to contain precisely those indices ℓ so that $\operatorname{supp} N_\ell^{(n)}$ intersects V_j but does not contain any of the points $\partial U_j \setminus B_j$. The function h_n is contained in S_n and satisfies $|h_n| \leq 1$. Observe that $\operatorname{supp} h_n \subset O_n$ for some open set O_n and $h_n \equiv N_i^{(m)}$ on some compact set $A_n \subset V_j$ that satisfy $O_n \setminus A_n \downarrow \emptyset$ as $n \to \infty$ and thus,

$$\|N_i^{(m)} \mathbb{1}_{V_j} - h_n\|_{L^1(\mu)} \lesssim \mu(O_n \setminus A_n) \to 0.$$

This shows that $N_i^{(m)} \mathbb{1}_{V_j} \in W$.

Since μ is a finite measure, $\lim_n \mu(\cup_{j \geq n} V_j) = 0$, and therefore, $f \mathbb{1}_V = f \mathbb{1}_{\cup_j V_j}$ is also contained in W.

Similarly, we see that the collection $\mathcal{F} = \{A \in \mathcal{B} : \mathbb{1}_A \in W\}$ is a σ-algebra. So, in order to show $V^c \cap \mathcal{F} = V^c \cap \mathcal{B}$ we will show that for each interval (c, d) contained in $[0, 1]$, we can find an interval $I \in \mathcal{F}$ with the property $V^c \cap (c, d) = V^c \cap I$. By the same reasoning as in the approximation of $N_i^{(m)} \mathbb{1}_{V_j}$ by finer spline functions, we can give the following sufficient condition for an interval I to be contained in $\mathcal{F}$: if for all $a \in \{\inf I, \sup I\}$ we have either

$$a \in I \text{ and there exists a seq. of grid points conv. from outside of } I \text{ to } a$$

or

$$a \notin I \text{ and there exists a seq. of grid points conv. from inside of } I \text{ to } a,$$

then $I \in \mathcal{F}$. Let now (c, d) be an arbitrary interval and assume first that $c, d \notin \cup_j \partial U_j$. For arbitrary points $x \in [0, 1]$, define

$$I(x) := \begin{cases} V_j, & \text{if } x \in U_j, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, by the above sufficient criterion, the set $I = (c, d) \setminus (I(c) \cup I(d))$ is contained in $\mathcal{F}$. Moreoever, $V^c \cap (c, d) = V^c \cap I$ and this shows that $(c, d) \cap V^c \in \mathcal{F} \cap V^c$. In general, since the set $\cup_j \partial U_j$ is countable, we can find sequences $c_n \geq c$ and $d_n \leq d$ with $c_n, d_n \notin \bigcup_j \partial U_j$, $c_n \to c$, $d_n \to d$, and

$$(c, d) \cap V^c = (\cup_n (c_n, d_n)) \cap V^c \in \mathcal{F} \cap V^c,$$

since $\mathcal{F} \cap V^c$ is a σ-algebra. This shows the fact that $\mathcal{F} \cap V^c = \mathcal{B} \cap V^c$. $\square$

Proof of Theorem 1.1. PART 1: $t \in V^c$: By Lemma 6.1, we can decompose

$$\begin{aligned} g_n(t) = (P_n T)(t) &= \sum_i T(N_i^{(n)}) N_i^{(n)*}(t) \\ &= \sum_i T(N_i^{(n)} \mathbb{1}_V) N_i^{(n)*}(t) + \sum_i T(N_i^{(n)} \mathbb{1}_{V^c}) N_i^{(n)*}(t) \\ &=: \Sigma_1^{(n)}(t) + \Sigma_2^{(n)}(t). \end{aligned}$$

PART 1.A: $\Sigma_1^{(n)}(t)$ FOR $t \in V^c$: We will show that $\Sigma_1^{(n)}(t)$ converges to zero a.e. on V^c. This is done by defining the measure

$$\theta(E) := \mu\big(E \cap V\big), \qquad E \in \mathcal{B},$$

and

$$F_r = \{t \in V^c : \limsup_n \|\Sigma_1^{(n)}(t)\|_X > 1/r\} \subset V^c.$$

Observe that $\theta(F_r) = 0$ and, by (6.1) and Theorem 2.11,

$$\|\Sigma_1^{(n)}(t)\|_X \lesssim \sum_{i,j} \frac{q^{|i-j|}}{h_{ij}^{(n)}} \theta(\operatorname{supp} N_i^{(n)}) N_j^{(n)}(t), \qquad t \in F_r,$$

which allows us to apply Lemma 5.1 on F_r and θ to get $\lambda(F_r) = 0$ for all $r > 0$, i.e., $\Sigma_1^{(n)}(t)$ converges to zero a.e. on V^c.

PART 1.B: $\Sigma_2^{(n)}(t)$ FOR $t \in V^c$: Let $\mathcal{B}_{V^c} = V^c \cap \mathcal{B}$. Thus $\mathcal{B}_{V^c}$ is the restriction of the Borel σ-algebra $\mathcal{B}$ to V^c. In this case, we define the vector measure ν of bounded variation on $(V^c, \mathcal{B}_{V^c})$ by

$$\nu(A) := T(\mathbb{1}_A), \qquad A \in \mathcal{B}_{V^c}.$$

Here we use the second part of Lemma 6.1 to guarantee that the right hand side is defined and (6.1) ensures $|\nu| \le \mu$. Apply Lebesgue decomposition Theorem 2.2 to get

$$\mathrm{d}\nu = g\,\mathrm{d}\lambda + \mathrm{d}\nu_s \tag{6.2}$$

where $g \in L^1_X$ and $|\nu_s|$ is singular to λ. Observe that for all $f \in \cup S_n$, we have

$$\int f\,\mathrm{d}\nu = T(f\mathbb{1}_{V^c}). \tag{6.3}$$

Indeed, this holds for indicator functions by definition and each $f \in \cup S_n$ can be approximated in $L^1(\mu)$ by linear combinations of indicator functions. Therefore, (6.3) is established, since both sides of (6.3) are continuous in $L^1(\mu)$. So,

$$\begin{aligned}\Sigma_2^{(n)}(t) &= \sum_i \int N_i^{(n)}\,\mathrm{d}\nu \cdot N_i^{(n)*}(t) \\ &= \sum_i \int N_i^{(n)} g\,\mathrm{d}\lambda \cdot N_i^{(n)*}(t) + \sum_i \int N_i^{(n)}\,\mathrm{d}\nu_s \cdot N_i^{(n)*}(t).\end{aligned}$$

The first part is $P_n g$ for an L^1_X function g and this converges by Theorem 3.2 a.e. to g.

To treat the second part $P_n\nu_s$, let $A \in \mathcal{B}_{V^c}$ be a subset of V^c with the property $\lambda(V^c \setminus A) = |\nu_s|(A) = 0$, which is possible since $|\nu_s|$ is singular to λ. For $x^* \in X^*$, we define the set

$$F_{r,x^*} := \{t \in A : \limsup_n |(x^* P_n \nu_s)(t)| > 1/r\}.$$

Since by Theorem 2.11

$$\begin{aligned}|x^* P_n \nu_s(t)| &= \Big|\sum_{i,j} a_{ij}^{(n)} \int N_i^{(n)}\,\mathrm{d}(x^* \circ \nu_s) \cdot N_j^{(n)}(t)\Big| \\ &\lesssim \sum_{i,j} \frac{q^{|i-j|}}{h_{ij}^{(n)}} |x^* \circ \nu_s|(\operatorname{supp} N_i^{(n)}) \cdot N_j^{(n)}(t),\end{aligned}$$

we can apply Lemma 5.1 to F_{r,x^*} and the measure $\theta(B) = |x^* \circ \nu_s|(B \cap V^c)$ to obtain $\lambda(F_{r,x^*}) = 0$. Since the closure X_0 in X of the set $\{P_n\nu_s(t) : t \in [0,1], n \in \mathbb{N}\}$ is a separable subspace of X, by Lemma 2.6, there exists a sequence (x_n^*) of elements in X^* such that for all $x \in X_0$ we have $\|x\| = \sup_n |x_n^*(x)|$. This means that we can write

$$F := \{t \in A : \limsup_n \|P_n \nu_s(t)\| > 0\} = \bigcup_{n,\ell=1}^{\infty} F_{\ell, x_n^*},$$

and thus, $\lambda(F) = 0$, which shows that $P_n\nu_s$ tends to zero almost everywhere on V^c with respect to Lebesgue measure.

PART 2: $t \in V$:

Now, we consider $t \in V$ or more precisely $t \in U$. This makes no difference for considering a.e. convergence since the difference between V and U is a Lebesgue zero set. We choose the index j_0 such that $t \in U_{j_0}$ and based on the location of t, we decompose (using Lemma 6.1)

$$\begin{aligned} g_n(t) = P_n T(t) &= \sum_i T(N_i^{(n)}) N_i^{(n)*}(t) \\ &= \sum_i T(N_i^{(n)} \mathbb{1}_{V_{j_0}}) \cdot N_i^{(n)*}(t) + \sum_i T(N_i^{(n)} \mathbb{1}_{V_{j_0}^c}) \cdot N_i^{(n)*}(t) \\ &=: \Sigma_1^{(n)}(t) + \Sigma_2^{(n)}(t). \end{aligned}$$

PART 2.A: $\Sigma_1^{(n)}(t)$ FOR $t \in U_{j_0}$:

We now consider

$$\Sigma_1^{(n)} = \sum_i T(N_i^{(n)} \mathbb{1}_{V_{j_0}}) N_i^{(n)*}(t), \qquad t \in U_{j_0},$$

and perform the construction of the B-splines $(\bar{N}_j)$ and their dual functions $(\bar{N}_j^*)$ corresponding to V_{j_0} described in Section 4. Define the function

$$u(t) := \sum_j T(\bar{N}_j) \bar{N}_j^*(t), \qquad t \in U_{j_0}, \tag{6.4}$$

and first note that $\bar{N}_j \in W$ since by Lemma 4.1 it is the uniform limit of the functions $(N_j^{(n)} \mathbb{1}_{V_{j_0}})$, which, in turn, are contained in W by Lemma 6.1. Therefore, $T(\bar{N}_j)$ is defined. Moreover, the series in (6.4) converges pointwise for $t \in U_{j_0}$, since $\lambda(\bar{I}(t)) > 0$, the sequence $j \mapsto \bar{N}_j^*(t)$ admits a geometric decay estimate by (4.1) and the inequality $\|T(\bar{N}_i)\|_X \le \mu(\operatorname{supp} \bar{N}_i)$. If one additionally notices that (4.1) implies the estimate $\|\bar{N}_j^*\|_{L^1} \lesssim 1$ we see that the convergence in (6.4) takes place in L^1_X as well. This implies $\langle u, \bar{N}_i \rangle = T(\bar{N}_i)$ for all i by Lemma 4.2.

Next, we show that if for all n, (a_i) and $(a_i^{(n)})$ are sequences in X so that for all i we have $\lim_n a_i^{(n)} = a_i$, and $\sup_i \|a_i\|_X + \sup_{i,n} \|a_i^{(n)}\|_X \lesssim 1$ it follows that

$$\lim_n \sum_i (a_i^{(n)} - a_i) N_i^{(n)*}(t) = 0, \qquad t \in U_{j_0}. \tag{6.5}$$

Indeed, let $\varepsilon > 0$, the integer L such that $q^L \le \varepsilon \cdot \inf_n \lambda(I_n(t))$ and M sufficiently large that for all $n \ge M$ and all i with $|i - i_n(t)| \le L$, we have $\|a_i^{(n)} - a_i\|_X \le \varepsilon \cdot \inf_n \lambda(I_n(t))$. Then, by Theorem 2.11,

$$\Big\| \sum_i (a_i^{(n)} - a_i) N_i^{(n)*}(t) \Big\|_X \le \sum_i \|a_i^{(n)} - a_i\|_X \frac{q^{|i - i_n(t)|}}{\lambda(I_n(t))}$$

$$= \Big(\sum_{i:|i-i_n(t)|\le L} + \sum_{i:|i-i_n(t)|>L} \Big) \|a_i^{(n)} - a_i\|_X \frac{q^{|i-i_n(t)|}}{\lambda(I_n(t))}$$
$$\lesssim \sum_{i:|i-i_n(t)|\le L} \varepsilon q^{|i-i_n(t)|} + \sum_{i:|i-i_n(t)|>L} \frac{q^{|i-i_n(t)|}}{\lambda(I_n(t))} \lesssim \varepsilon$$

We now use these remarks to show that

$$\lim_n \|\Sigma_1^{(n)}(t) - P_n u(t)\|_X = 0, \qquad t \in U_{j_0}.$$

Indeed, since $\langle u, \bar N_i\rangle = T(\bar N_i)$ for all i,

$$\begin{aligned}\Sigma_1^{(n)}(t) - P_n u(t) &= \sum_i \big(T(N_i^{(n)} \mathbb{1}_{V_{j_0}}) - \langle u, N_i^{(n)}\rangle\big) N_i^{(n)*}(t) \\ &= \sum_i \big(T(N_i^{(n)} \mathbb{1}_{V_{j_0}}) - T(\bar N_i)\big) N_i^{(n)*}(t) \\ &\qquad + \sum_i \big(\langle u, \bar N_i\rangle - \langle u, N_i^{(n)}\rangle\big) N_i^{(n)*}(t).\end{aligned}$$

Now, observe that for all i, we have $T(N_i^{(n)} \mathbb{1}_{V_{j_0}}) \to T(\bar N_i)$ and $\langle u, N_i^{(n)}\rangle \to \langle u, \bar N_i\rangle$ since by Lemma 4.1, $N_i^{(n)}$ converges uniformly to $\bar N_i$ on V_{j_0} and $u \in L^1$. Moreover all the expressions $T(N_i^{(n)} \mathbb{1}_{V_{j_0}}), T(\bar N_i), \langle u, N_i^{(n)}\rangle$ are bounded in i and n. As a consequence, we can apply (6.5) to both of the sums in the above display to conclude

$$\lim_n \|\Sigma_1^{(n)}(t) - P_n u(t)\|_X = 0, \qquad t \in U_{j_0}.$$

But we know that $P_n u(t)$ converges a.e. to $u(t)$ by Theorem 3.2, this means that also $\Sigma_1^{(n)}(t)$ converges to u a.e.

Part 2.b: $\Sigma_2^{(n)}(t)$ for $t \in U_{j_0}$: We show that $\Sigma_2^{(n)}(t) = \sum_i T(N_i^{(n)} \mathbb{1}_{V_{j_0}^c}) \cdot N_i^{(n)*}(t)$ converges to zero for $t \in U_{j_0}$. Let $\varepsilon > 0$ and set $s = \inf_n \lambda(I_n(t))$, where we recall that $I_n(t)$ is the grid interval in Δ_n that contains the point t. Since $s > 0$ we can choose an open interval O with the property $\mu(O \setminus V_{j_0}) \le \varepsilon s$. Then, due to the fact that $t \in U_{j_0}$, we can choose M sufficiently large that both intervals $(\inf O, t)$ and $(t, \sup O)$ contain L points of the grid Δ_M where L is such that $q^L \le \varepsilon s/\mu([0,1])$. Thus, we estimate for $n \ge M$ by (6.1) and Theorem 2.11

$$\begin{aligned}\|\Sigma_2^{(n)}(t)\|_X &\le \sum_i \mu(\operatorname{supp} N_i^{(n)} \cap V_{j_0}^c) \frac{q^{|i-i_n(t)|}}{\lambda(I_n(t))} \\ &\le \frac{1}{s} \cdot \Big(\sum_{i:\operatorname{supp} N_i^{(n)} \cap O^c \ne \emptyset} + \sum_{i:\operatorname{supp} N_i^{(n)} \subset O} \Big) \big(\mu(\operatorname{supp} N_i^{(n)} \cap V_{j_0}^c) q^{|i-i_n(t)|}\big) \\ &\lesssim \frac{1}{s} \big(\mu([0,1]) q^L + \mu(O \setminus V_{j_0})\big) \lesssim \varepsilon.\end{aligned}$$

This proves that $\Sigma_2^{(n)}(t)$ converges to zero for $t \in U_{j_0}$. □

By looking at the above proof and employing the notation therein, we have actually proved the following, explicit form of the Spline Convergence Theorem:

Theorem 6.2. *Let X be a Banach space with RNP and (g_n) be sequence in L^1_X with the properties*

(1) $\sup_n \|g_n\|_{L^1_X} < \infty$,
(2) $P_m g_n = g_m$ *for all* $m \leq n$.

Then, g_n *converges a.e. to the* L^1_X*-function*

$$g\mathbb{1}_{V^c} + \sum_{j_0} \sum_j T(\bar{N}_{j_0,j}) \bar{N}^*_{j_0,j} \mathbb{1}_{U_{j_0}}.$$

Here, g *is defined by* (6.2)*, and for each* j_0*,* $(\bar{N}_{j_0,j})$ *and* $(\bar{N}^*_{j_0,j})$ *are the B-splines and their dual functions constructed in Section 4 corresponding to* V_{j_0}.

Remark 6.3. In order to emphasize the pivotal role of the set V and its complement we note that the proof of Theorem 6.2 implies the following: If (g_n) be sequence in L^1_X such that

(1) $\sup_n \|g_n\|_{L^1_X} < \infty$,
(2) $P_m g_n = g_m$ for all $m \leq n$

and if $\lambda(V^c) = 0$ then, *without any* condition on the Banach space X, g_n converges a.e. to

$$\sum_{j_0} \sum_j T(\bar{N}_{j_0,j}) \bar{N}^*_{j_0,j} \mathbb{1}_{U_{j_0}}.$$

Remark 6.4. Based on the results of the present paper, an intrinsic spline characterization of the Radon-Nikodým property in terms of splines was obtained by the second named author in [8]. The result in [8] establishes the full analogy between spline and martingale convergence.

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CHAPTER 9

Spline characterizations of the Radon-Nikodým property

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SPLINE CHARACTERIZATIONS OF THE RADON-NIKODÝM PROPERTY

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ABSTRACT. We give necessary and sufficient conditions for a Banach space X having the Radon-Nikodým property in terms of polynomial spline sequences.

1. Introduction and preliminaries

The aim of this paper is to prove new characterizations of the Radon-Nikodým property for Banach spaces in terms of polynomial spline sequences in the spirit of the corresponding martingale results (see Theorem 1.2). We thereby continue the line of research about extending martingale results to also cover (general) spline sequences that is carried out in [4–8, 11]. We refer to the book [1] by J. Diestel and J. J. Uhl for basic facts on martingales and vector measures; here, we only give the necessary notions to define the Radon-Nikodým property below. Let $(\Omega, \mathcal{A})$ be a measure space and let X be a Banach space. Every σ-additive map $\nu : \mathcal{A} \to X$ is called a *vector measure*. The *variation* $|\nu|$ of ν is the set function

$$|\nu|(E) = \sup_{\pi} \sum_{A \in \pi} \|\nu(A)\|_X,$$

where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of $\mathcal{A}$. If ν is of bounded variation, i.e., $|\nu|(\Omega) < \infty$, then the variation $|\nu|$ is σ-additive. If $\mu : \mathcal{A} \to [0, \infty)$ is a measure and $\nu : \mathcal{A} \to X$ is a vector measure, ν is called *μ-continuous* if $\lim_{\mu(E)\to 0} \nu(E) = 0$ for all $E \in \mathcal{A}$. In the following, $L^p_X = L^p_X(\Omega, \mathcal{A}, \mu)$ will denote the Bochner-Lebesgue space of p-integrable Bochner measurable functions $f : \Omega \to X$, and if $X = \mathbb{R}$, we simply write L^p instead of $L^p_{\mathbb{R}}$.

Definition 1.1. A Banach space X has the *Radon-Nikodým property* (*RNP*) if for every measure space $(\Omega, \mathcal{A})$, for every positive measure μ on $(\Omega, \mathcal{A})$, and for every μ-continuous vector measure ν of bounded variation, there exists a function $f \in L^1_X(\Omega, \mathcal{A}, \mu)$ such that

$$\nu(A) = \int_A f \,\mathrm{d}\mu, \qquad A \in \mathcal{A}.$$

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Additionally, recall that a sequence (f_n) in L^1_X is *uniformly integrable* if the sequence $(\|f_n\|_X)$ is bounded in L^1 and, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \delta \quad \Longrightarrow \quad \sup_n \int_A \|f_n\|_X \,\mathrm{d}\mu < \varepsilon, \qquad A \in \mathcal{A}.$$

We have the following characterization of the Radon-Nikodým property in terms of martingales; see e.g. [9, p. 50].

Theorem 1.2. *For any $p \in (1,\infty)$, the following statements about a Banach space X are equivalent:*

(i) *X has the Radon-Nikodým property (RNP),*
(ii) *every X-valued martingale bounded in L^1_X converges almost surely,*
(iii) *every uniformly integrable X-valued martingale converges almost surely and in L^1_X,*
(iv) *every X-valued martingale bounded in L^p_X converges almost surely and in L^p_X.*

Remark. For the above equivalences, it is enough to consider X-valued martingales defined on the unit interval with respect to Lebesgue measure and the dyadic filtration (cf. [9, p. 54]).

Now, we describe the general framework that allows us to replace properties (ii)–(iv) with their spline versions.

Definition 1.3. A sequence of σ-algebras $(\mathcal{F}_n)_{n\geq 0}$ in $[0,1]$ is called an *interval filtration* if $(\mathcal{F}_n)$ is increasing and each $\mathcal{F}_n$ is generated by a finite partition of $[0,1]$ into intervals of positive Lebesgue measure.

For an interval filtration $(\mathcal{F}_n)$, we define $\Delta_n := \{\partial A : A \text{ is atom of } \mathcal{F}_n\}$ to be the set of all endpoints of atoms in $\mathcal{F}_n$. For a fixed positive integer k, set

$$S_n^{(k)} = \{f \in C^{k-2}[0,1] : \quad f \text{ is a polynomial of order } k \text{ on each atom of } \mathcal{F}_n\},$$

where $C^n[0,1]$ denotes the space of real-valued functions on $[0,1]$ that, for $n \geq 0$, are additionally n times continuously differentiable and the order k of a polynomial p is related to the degree d of p by the formula $k = d+1$.

The finite dimensional space $S_n^{(k)}$ admits a very special basis (N_i) of non-negative and uniformly bounded functions, called B-spline basis, that forms a partition of unity, i.e., $\sum_i N_i(t) = 1$ for all $t \in [0,1]$, and the support of each N_i consists of the union of k neighboring atoms of $\mathcal{F}_n$. If $n \geq m$ and $(N_i), (\tilde{N}_i)$ are the B-spline bases of $S_n^{(k)}$ and $S_m^{(k)}$, respectively, we can write each $f \in S_m^{(k)}$ as $f = \sum a_i \tilde{N}_i = \sum b_i N_i$ for some coefficients $(a_i), (b_i)$ since $S_m^{(k)} \subset S_n^{(k)}$. Those coefficients are related to each other in the way that each b_i is a convex combination of the coefficients (a_i). For more information on spline functions, see [10].

Additionally, we let $P_n^{(k)}$ be the orthogonal projection operator onto $S_n^{(k)}$ with respect to $L^2[0,1]$ equipped with the Lebesgue measure $|\cdot|$. Each space $S_n^{(k)}$ is finite dimensional and B-splines are uniformly bounded. Therefore, $P_n^{(k)}$ can be extended to L^1 and L^1_X satisfying $P_n^{(k)}(f \otimes x) = (P_n^{(k)} f) \otimes x$ for all $f \in L^1$ and $x \in X$, where $f \otimes x$ denotes the function $t \mapsto f(t)x$. Moreover, by $S_n^{(k)} \otimes X$, we denote the space $\operatorname{span}\{f \otimes x : f \in S_n^{(k)}, x \in X\}$.

Definition 1.4. Let X be a Banach space and let $(f_n)_{n\geq 0}$ be a sequence of functions in L^1_X. Then, (f_n) is an (X-valued) *k-martingale spline sequence adapted to* $(\mathcal{F}_n)$ if $(\mathcal{F}_n)$ is an interval filtration and

$$P_n^{(k)} f_{n+1} = f_n, \qquad n \geq 0.$$

This definition resembles the definition of martingales with the conditional expectation operator replaced by $P_n^{(k)}$. For splines of order $k = 1$, i.e., piecewise constant functions, the operator $P_n^{(k)}$ even is the conditional expectation operator with respect to the σ-algebra $\mathcal{F}_n$.

Many of the results that are true for martingales (such as Doob's inequality, the martingale convergence theorem, or Burkholder's inequality) in fact carry over to k-martingale spline sequences corresponding to an *arbitrary* interval filtration as the following two theorems show:

Theorem 1.5. *For any positive integer k, any interval filtration $(\mathcal{F}_n)$, and any Banach space X, the following assertions are true:*

(i) *there exists a constant C_k depending only on k such that*

$$\sup_n \|P_n^{(k)} : L^1_X \to L^1_X\| \leq C_k;$$

(ii) *there exists a constant C_k depending only on k such that for any X-valued k-martingale spline sequence (f_n) and any $\lambda > 0$,*

$$|\{\sup_n \|f_n\|_X > \lambda\}| \leq C_k \frac{\sup_n \|f_n\|_{L^1_X}}{\lambda};$$

(iii) *for all $p \in (1, \infty]$ there exists a constant $C_{p,k}$ depending only on p and k such that for all X-valued k-martingale spline sequences (f_n),*

$$\big\| \sup_n \|f_n\|_X \big\|_{L^p} \leq C_{p,k} \sup_n \|f_n\|_{L^p_X};$$

(iv) *if X has the RNP and (f_n) is an L^1_X-bounded k-martingale spline sequence, then (f_n) converges a.s. to some L^1_X-function.*

Item (i) is proved in [11], and (ii)–(iv) are proved (effectively) in [5, 8].

Theorem 1.6 ([6])**.** *For all $p \in (1, \infty)$ and all positive integers k, scalar-valued k-spline differences converge unconditionally in L^p; i.e., for all $f \in L^p$,*

$$\big\| \sum_n \pm (P_n^{(k)} - P_{n-1}^{(k)}) f \big\|_{L^p} \leq C_{p,k} \|f\|_{L^p}$$

for some constant $C_{p,k}$ depending only on p and k.

The martingale version of Theorem 1.6 is Burkholder's inequality, which precisely holds in the vector-valued setting for UMD-spaces X (by the definition of UMD-spaces). It is an open problem whether Theorem 1.6 holds for UMD-valued k-martingale spline sequences in this generality, but see [2] for a special case. For more information on UMD-spaces, see e.g. [9].

Definition 1.7. Let X be a Banach space, let $(\mathcal{F}_n)$ be an interval filtration, and let k be a positive integer. Then, X has the *$((\mathcal{F}_n), k)$-martingale spline convergence property* (*MSCP*) if all L^1_X-bounded k-martingale spline sequences adapted to $(\mathcal{F}_n)$ admit a limit almost surely.

In this work, we prove the following characterization of the Radon-Nikodým property in terms of k-martingale spline sequences.

Theorem 1.8. *Let X be a Banach space, let $(\mathcal{F}_n)$ be an interval filtration, let k be a positive integer, and let V be the set of all accumulation points of $\bigcup_n \Delta_n$. Then, $((\mathcal{F}_n), k)$-MSCP characterizes RNP if and only if $|V| > 0$; i.e., $|V| > 0$ precisely when the following are equivalent:*

(1) *X has the RNP,*
(2) *X has the $((\mathcal{F}_n), k)$-MSCP.*

Proof. If $|V| > 0$, it follows from Theorem 1.5(iv) that RNP implies $((\mathcal{F}_n), k)$-MSCP for any positive integer k and any interval filtration $(\mathcal{F}_n)$. The reverse implication for $|V| > 0$ is a consequence of Theorem 1.10. We even have that if X does not have RNP, we can find an $(\mathcal{F}_n)$-adapted k-martingale spline sequence that does not converge at all points $t \in E$ for a subset $E \subset V$ with $|E| = |V|$. We simply have to choose $E := \limsup E_n$ with (E_n) being the sets from Theorem 1.10.

If $|V| = 0$, it is proved in [5] that any Banach space X has $((\mathcal{F}_n), k)$-MSCP. □

We also have the following spline analogue of Theorem 1.2:

Theorem 1.9. *For any positive integer k and any $p \in (1, \infty)$, the following statements about a Banach space X are equivalent:*

(i) *X has the Radon-Nikodým property,*
(ii) *every X-valued k-martingale spline sequence bounded in L^1_X converges almost surely,*
(iii) *every uniformly integrable X-valued k-martingale spline sequence converges almost surely and in L^1_X,*
(iv) *every X-valued k-martingale spline sequence bounded in L^p_X converges almost surely and in L^p_X.*

Proof. (i)⇒(ii): Theorem 1.5(iv).

(ii)⇒(iii): Clear.

(iii)⇒(iv): If (f_n) is a k-martingale spline sequence bounded in L^p_X for $p > 1$, then (f_n) is uniformly integrable; therefore it has a limit f (a.s. and L^1_X), which, by Fatou's lemma, is also contained in L^p_X. By Theorem 1.5(iii), $\sup_n \|f_n\|_X \in L^p$, and we can apply dominated convergence to obtain $\|f_n - f\|_{L^p_X} \to 0$.

(iv)⇒(i): Follows from Theorem 1.10. □

The rest of the article is devoted to the construction of a suitable non-RNP-valued k-martingale spline sequence, adapted to an arbitrary given filtration $(\mathcal{F}_n)$, so that the associated martingale spline differences are separated away from zero on a large set, which, more precisely, takes the following form:

Theorem 1.10. *Let X be a Banach space without RNP, let $(\mathcal{F}_n)$ be an interval filtration, let V be the set of all accumulation points of $\bigcup_n \Delta_n$, and let k be a positive integer.*

Then, there exists a positive number δ such that for all $\eta \in (0,1)$, there exists an increasing sequence of positive integers (m_j), an L^∞_X-bounded k-martingale spline sequence $(f_j)_{j\geq 0}$ adapted to $(\mathcal{F}_{m_j})$ with $f_j \in S^{(k)}_{m_j} \otimes X$, and a sequence (E_n) of measurable sets $E_n \subset V$ with $|E_n| \geq (1 - 2^{-n}\eta)|V|$ so that for all $n \geq 1$,

$$\|f_n(t) - f_{n-1}(t)\|_X \geq \delta, \qquad t \in E_n.$$

We will use the concept of dentable sets to prove Theorem 1.10 and recall its definition:

Definition 1.11. Let X be a Banach space. A subset $D \subset X$ is called *dentable* if for any $\varepsilon > 0$ there is a point $x \in D$ such that

$$x \notin \overline{\operatorname{conv}}\big(D \setminus B(x,\varepsilon)\big),$$

where $\overline{\operatorname{conv}}$ denotes the closure of the convex hull and where $B(x,\varepsilon) = \{y \in X : \|y - x\| < \varepsilon\}$.

Remark (Cf. [1, p. 138, Theorem 10] and [9, p. 49, Lemma 2.7]). If D is a bounded non-dentable set, then the closed convex hull $\overline{\operatorname{conv}}(D)$ is also bounded and non-dentable. Thus, we may assume that D is convex. Moreover, we can as well assume that each $x \in D$ can be expressed as a finite convex combination of elements in $D \setminus B(x,\delta)$ for some $\delta > 0$ since if $D \subset X$ is a convex set such that $x \in \overline{\operatorname{conv}}\big(D \setminus B(x,\delta)\big)$ for all $x \in D$, then, the enlarged set $\widetilde{D} = D + B(0,\eta)$ is also convex and satisfies

$$x \in \operatorname{conv}\big(\widetilde{D} \setminus B(x,\delta-\eta)\big), \qquad x \in \widetilde{D}.$$

The reason why we are able to use the concept of dentability in the proof of Theorem 1.10 is the following geometric characterization of the RNP (see for instance [1, p. 136]).

Theorem 1.12. *For any Banach space X we have that X has the RNP if and only if every bounded subset of X is dentable.*

We record the following (special case of the) basic composition formula for determinants (see for instance [3, p. 17]):

Lemma 1.13. *Let $(f_i)_{i=1}^n$ and $(g_j)_{j=1}^n$ be two sequences of functions in L^2. Then,*

$$\det\Big(\int_0^1 f_i(t)g_j(t)\,\mathrm{d}t\Big)_{i,j=1}^n = \int_{0\le t_1<\cdots<t_n\le 1} \det(f_i(t_\ell))_{i,\ell=1}^n \cdot \det(g_j(t_\ell))_{j,\ell=1}^n \,\mathrm{d}(t_1,\ldots,t_n).$$

We also note the following simple lemma:

Lemma 1.14. *Let $I \subset [0,1]$ be an interval and let V be an arbitrary measurable subset of $[0,1]$. Then, for all $\varepsilon_1, \varepsilon_2 > 0$, there exists a positive integer n so that for the decomposition of I into intervals $(A_\ell)_{\ell=1}^n$ with $\sup A_\ell \le \inf A_{\ell+1}$ and $n|A_\ell \cap V| = |I \cap V|$ for all ℓ, the index set $\Gamma = \{2 \le \ell \le n-1 : \max(|A_{\ell-1}|, |A_\ell|, |A_{\ell+1}|) \le \varepsilon_1\}$ satisfies*

$$\sum_{\ell\in\Gamma} |A_\ell \cap V| \ge (1-\varepsilon_2)|I \cap V|.$$

2. Construction of non-convergent spline sequences

In this section, we prove Theorem 1.10. In order to do that, we begin by fixing an interval filtration $(\mathcal{F}_n)$, the corresponding endpoints of atoms (Δ_n), and a positive integer k. For the space $S_n^{(k)}$, we will suppress the (fixed) index k and write S_n instead. We will apply the same convention to the corresponding projection operators $P_n = P_n^{(k)}$. We also let $V \subset [0,1]$ be the closed set of all accumulation points of $\bigcup_n \Delta_n$.

The main step in the proof of Theorem 1.10 consists of an inductive application of the construction of a suitable martingale spline difference in the following lemma:

Lemma 2.1. *Let $(x_j)_{j=1}^M$ be in the Banach space X, let $\bar{x} \in S_N \otimes X$ for some non-negative integer N such that $\bar{x} = \sum_{j=1}^M \alpha_j \otimes x_j$ with $\sum_{j=1}^M \alpha_j \equiv 1$, $\|x_j\| \le 1$, $\alpha_j \in S_N$ having non-negative B-spline coefficients for all j, and let $I \subset [0,1]$ be an interval so that $|I \cap V| > 0$.*

Then, for all $\varepsilon \in (0,1)$, there exist a positive integer K and a function $g \in S_K \otimes X$ with the following properties:

(i) $\int_I t^j g(t)\,\mathrm{d}t = 0$ *for all* $j = 0, \dots, k-1$.
(ii) $\operatorname{supp} g \subset \operatorname{int} I$.
(iii) *We have a splitting of the collection $\mathscr{A} = \{A \subset I : A \text{ is atom in } \mathcal{F}_K\}$ into $\mathscr{A}_1 \cup \mathscr{A}_2$ so that*
  (a) *if the functions α_j are all constant, then on each $J \in \mathscr{A}_1$, $\bar{x} + g$ is constant with a value in $\bigcup_i \{x_i\}$; otherwise we still have that on each $J \in \mathscr{A}_1$, $\bar{x} + g$ is constant with a value in $\operatorname{conv}\{x_i : 1 \le i \le M\}$;*
  (b) $|\bigcup_{J \in \mathscr{A}_1} J \cap V| \ge (1-\varepsilon)|I \cap V|$;
  (c) *on each $J \in \mathscr{A}_2$, $\bar{x} + g = \sum_\ell \lambda_\ell \otimes y_\ell$ for some functions $\lambda_\ell \in S_K$ having non-negative B-spline coefficients with $\sum_\ell \lambda_\ell \equiv 1$ and $y_\ell \in \operatorname{conv}\{x_j : 1 \le j \le M\} + B(0,\varepsilon)$.*

Proof. The first step of the construction gives a function g satisfying the desired conditions but having only mean zero instead of vanishing moments in property (i). In the second step, we use this result to construct a function g whose moments also vanish.

Step 1. We start with the (simpler) construction of g when the functions α_j are not constant and condition (iii)(a) has the form that on each $J \in \mathscr{A}_1$, $\bar{x} + g$ is constant with a value in $\operatorname{conv}\{x_i : 1 \le i \le M\}$.

First, decompose I into intervals $(A_\ell)_{\ell=1}^n$ satisfying $n|A_\ell \cap V| = |I \cap V|$ with $\sup A_\ell \le \inf A_{\ell+1}$ and $n \ge 4/\varepsilon$. Then, choose $K \ge N$ so large that A_1, A_2, A_{n-1}, A_n each contains at least $k+1$ atoms of $\mathcal{F}_K$. Denoting by (N_j) the B-spline basis of S_K, we can write

$$\alpha_\ell \equiv \sum_j \alpha_{\ell,j} N_j, \qquad \ell = 1, \dots, M,$$

for some non-negative coefficients $(\alpha_{\ell,j})$. Define

$$h_\ell \equiv \sum_{j : \bigcup_{i=2}^{n-1} A_i \cap \operatorname{supp} N_j \ne \emptyset} \alpha_{\ell,j} N_j.$$

Observe that $\operatorname{supp} h_\ell \subset \operatorname{int} I$ and $h_\ell \equiv \alpha_\ell$ on $\bigcup_{i=2}^{n-1} A_i$. Letting $\widetilde{x} = \sum \beta_\ell x_\ell$ for $\beta_\ell = \int h_\ell / \big(\sum_j \int h_j\big) \in [0,1]$, we define

$$g := -\sum_{\ell=1}^M h_\ell \otimes x_\ell + \Big(\sum_{j=1}^M h_j\Big) \otimes \widetilde{x}.$$

This is a function of the desired form when defining $\mathscr{A}_1 := \{A \subset \bigcup_{i=2}^{n-1} A_i : A \text{ is atom in } \mathcal{F}_K\}$ and $\mathscr{A}_2 = \mathscr{A} \setminus \mathscr{A}_1$, as we will now show by proving $\int g = 0$ and properties (ii), (iii). The fact that $\int g = 0$ follows from a simple calculation. Property (ii) is satisfied by the definition of the functions h_ℓ. Property (iii)(a) follows from the fact that $\bar{x}(t) + g(t) = \tilde{x} \in \operatorname{conv}\{x_j : 1 \le j \le M\}$ for $t \in \bigcup_{i=2}^{n-1} A_i$ since $h_\ell \equiv \alpha_\ell$ on that set for any $\ell = 1, \ldots, M$. Since $|(A_1 \cup A_2 \cup A_{n-1} \cup A_n) \cap V| = 4|I \cap V|/n \le \varepsilon |I \cap V|$, (iii)(b) also follows from the construction of $\mathscr{A}_1$. Since

$$\bar{x}(t) + g(t) = \sum_{\ell=1}^{M} \big(\alpha_\ell(t) - h_\ell(t)\big) x_\ell + \Big(\sum_{j=1}^{M} h_j(t)\Big)\tilde{x},$$

$\tilde{x} \in \operatorname{conv}\{x_j : 1 \le j \le M\}$, $h_\ell \le \alpha_\ell$, and $\sum_\ell \alpha_\ell \equiv 1$, (iii)(c) is also proved.

The next step is to construct the desired function g when α_j are assumed to be constant and (iii)(a) has the form that on each $J \in \mathscr{A}_1$, $\bar{x} + g$ is constant with a value in $\bigcup_i \{x_i\}$. Here, the idea is to construct a function of the form $g(t) = \sum f_j(t)(x_j - \bar{x})$ with $f_j \in S_K$ for some K and $\int f_j \simeq C\alpha_j$ for all j and some constant C independent of j to employ the assumption $\sum \alpha_j (x_j - \bar{x}) = 0$, implying $\int g = 0$.

We begin this construction by successively choosing parameters $\varepsilon_3 \ll \varepsilon_1 \ll \tilde{\varepsilon} < \varepsilon$ obeying certain given conditions depending on ε, $\bar{x}$, (x_j), (α_j), $|I \cap V|$, and $|I|$.

First, set

$\tilde{\varepsilon} = \varepsilon |I \cap V|/(3|I|) > 0$ and

$$\varepsilon_1 = \frac{\varepsilon\tilde{\varepsilon}(1 - \varepsilon/3)|I \cap V|}{72M}. \tag{2.1}$$

Now, we apply Lemma 1.14 with the parameters ε_1 and $\varepsilon_2 = \varepsilon/3$ to get a positive integer n and a partition $(A_\ell)_{\ell=1}^n$ of I consisting of intervals with $n|A_\ell \cap V| = |I \cap V|$ for all $\ell = 1, \ldots, n$ so that

$$\Gamma = \{2 \le \ell \le n-1 : \max(|A_{\ell-1}|, |A_\ell|, |A_{\ell+1}|) \le \varepsilon_1\}$$

satisfies

$$\big(1 - \frac{\varepsilon}{3}\big)|I \cap V| \le \sum_{\ell \in \Gamma} |A_\ell \cap V|. \tag{2.2}$$

Finally, we put $\varepsilon_3 = \varepsilon_1/(2n)$.

Next, for each $\ell = 1, \ldots, n$, we choose a point $p_\ell \in \operatorname{int} A_\ell$ and an integer K_ℓ so that the intersection of $\operatorname{int} A_\ell$ and the ε_3-neighborhood $B(p_\ell, \varepsilon_3)$ of p_ℓ contains at least $k+1$ atoms of $\mathcal{F}_{K_\ell}$ to the left as well as to the right of p_ℓ. This is possible since $|A_\ell \cap V| = |I \cap V|/n$ and V is the set of all accumulation points of $\bigcup_j \Delta_j$. Then set $K = \max_\ell K_\ell$ and let $u_\ell \in A_\ell$ be the leftmost point of Δ_K contained in $B(p_\ell, \varepsilon_3) \cap \operatorname{int} A_\ell$. Similarly, let $v_\ell \in A_\ell$ be the rightmost point of Δ_K contained in $B(p_\ell, \varepsilon_3) \cap \operatorname{int} A_\ell$. Next, for $2 \le \ell \le n-1$, we put $B_\ell := (v_{\ell-1}, u_{\ell+1}) \subset A_{\ell-1} \cup A_\ell \cup A_{\ell+1}$. Observe that the construction of u_ℓ and v_ℓ implies that $B_\ell \cap B_j = \emptyset$ for all $|\ell - j| \ge 2$. Next, let (N_i) be the B-spline basis of the space S_K and let $(\ell(i))_{i=1}^L$ be the increasing sequence of integers so that $\Gamma = \{\ell(i) : 1 \le i \le L\}$ for $L = |\Gamma| \le n$. We then define the set

$$\Lambda(r, s) := \Big\{ j : \operatorname{supp} N_j \cap \Big(\bigcup_{i=r}^{s} B_{\ell(i)} \Big) \neq \emptyset \Big\}$$

to consist of those B-spline indices so that the support of the corresponding B-spline function intersects the set $\bigcup_{i=r}^s B_{\ell(i)}$. Observe that by (2.2),

$$\big(1 - \frac{\varepsilon}{3}\big)|I \cap V| \le \sum_{\ell \in \Gamma} |A_\ell \cap V| = \Big| \bigcup_{\ell \in \Gamma} A_\ell \cap V \Big| \le \Big| \bigcup_i B_{\ell(i)} \cap V \Big| \le \Big| \bigcup_i B_{\ell(i)} \Big|. \tag{2.3}$$

Thus, the definition (2.1) of ε_1 in particular implies that

$$72\varepsilon_1 M \leq \varepsilon \cdot \tilde{\varepsilon} \cdot \Big|\bigcup_i B_{\ell(i)}\Big|. \tag{2.4}$$

We continue defining the functions (f_j) contained in S_K using a stopping time construction and first set $j_0 = -1$ and $C = (1-\tilde{\varepsilon}/3)\big|\bigcup_i B_{\ell(i)}\big| > 0$. For $1 \leq m \leq M$, if j_{m-1} is already chosen, we define j_m to be the smallest integer $\leq L$ so that the function

$$f_m := \sum_{j \in \Lambda(j_{m-1}+2, j_m)} N_j \qquad \text{satisfies} \qquad \int f_m(t)\,\mathrm{d}t > C\alpha_m. \tag{2.5}$$

If no such integer exists, we set $j_m = L$ (however, we will see below that for the current choice of parameters, such an integer always exists). Additionally, we define

$$f_{M+1} := \sum_{j \in \Lambda(j_M+2, L)} N_j.$$

Observe that by the locality of the B-spline basis (N_i), $\operatorname{supp} f_\ell \cap \operatorname{supp} f_m = \emptyset$ for $1 \leq \ell < m \leq M+1$. Based on the collection of functions $(f_m)_{m=1}^{M+1}$, we will define the desired function g. But before we do that, we make a few comments about $(f_m)_{m=1}^{M+1}$.

Note that for $m = 1, \ldots, M$, by the minimality of j_m,

$$\int \sum_{j \in \Lambda(j_{m-1}+2, j_m-1)} N_j(t)\,\mathrm{d}t \leq C\alpha_m,$$

and therefore, again by the locality of the B-splines (N_i),

$$\int f_m(t)\,\mathrm{d}t \leq C\alpha_m + \int \sum_{j \in \Lambda(j_m, j_m)} N_j(t)\,\mathrm{d}t \leq C\alpha_m + 3\varepsilon_1. \tag{2.6}$$

Additionally, employing also the definition of u_ℓ and v_ℓ and the fact that the B-splines (N_i) form a partition of unity, we obtain

$$\begin{aligned} \Big|\bigcup_{i=j_{m-1}+2}^{j_m} B_{\ell(i)}\Big| \leq \int f_m(t)\,\mathrm{d}t &\leq \Big|\bigcup_{i=j_{m-1}+2}^{j_m} (p_{\ell(i)-1}, p_{\ell(i)+1})\Big| \\ &\leq \Big|\bigcup_{i=j_{m-1}+2}^{j_m} B_{\ell(i)}\Big| + 2n\varepsilon_3. \end{aligned} \tag{2.7}$$

Next, we will show that

$$(1-\tilde{\varepsilon})\Big|\bigcup_i B_{\ell(i)}\Big| \leq \Big|\bigcup_{i \leq j_M} B_{\ell(i)}\Big| \leq (1-\tilde{\varepsilon}/6)\Big|\bigcup_i B_{\ell(i)}\Big|. \tag{2.8}$$

Indeed, we calculate on the one hand by (2.7) and (2.6) that

$$\begin{aligned} \Big|\bigcup_{i \leq j_M} B_{\ell(i)}\Big| &\leq \sum_{m=1}^M \Big|\bigcup_{i=j_{m-1}+2}^{j_m} B_{\ell(i)}\Big| + \sum_{m=1}^M |B_{j_m+1}| \leq \sum_{m=1}^M \int f_m(t)\,\mathrm{d}t + 3\varepsilon_1 M \\ &\leq \sum_{m=1}^M (C\alpha_m + 3\varepsilon_1) + 3\varepsilon_1 M = C + 6\varepsilon_1 M. \end{aligned}$$

Recalling that $C = (1 - \tilde{\varepsilon}/3)\big|\bigcup_i B_{\ell(i)}\big|$ and using (2.4) now yield the right hand side of (2.8).

On the other hand, employing (2.7) and (2.5), we obtain

$$\Big|\bigcup_{i\le j_M} B_{\ell(i)}\Big| \ge \sum_{m=1}^{M}\Big|\bigcup_{i=j_{m-1}+2}^{j_m} B_{\ell(i)}\Big| \ge \sum_{m=1}^{M}\Big(\int f_m(t)\,\mathrm{d}t - 2n\varepsilon_3\Big)$$
$$\ge C\sum_{m=1}^{M}\alpha_m - 2nM\varepsilon_3 = C - 2nM\varepsilon_3.$$

The definitions of $C = (1 - \tilde{\varepsilon}/3)\big|\bigcup_i B_{\ell(i)}\big|$ and $\varepsilon_3 = \varepsilon_1/(2n)$, combined with (2.4), give the left hand inequality in (2.8).

The inequality on the right hand side of (2.8), combined with (2.4) again, allows us to give the following lower estimate of $\int f_{M+1}$:

$$\int f_{M+1}(t)\,\mathrm{d}t \ge \Big|\bigcup_{i\ge j_M+2} B_{\ell(i)}\Big| \ge \Big|\bigcup_{i>j_M} B_{\ell(i)}\Big| - 3\varepsilon_1 \ge \frac{\tilde{\varepsilon}}{12}\Big|\bigcup_i B_{\ell(i)}\Big|. \tag{2.9}$$

We are now ready to define the function $g \in S_K \otimes X$ as follows:

$$g \equiv \sum_{j=1}^{M} f_j \otimes (x_j - \bar{x}) + f_{M+1} \otimes \sum_{j=1}^{M}\beta_j(x_j - \bar{x}), \tag{2.10}$$

where

$$\beta_j = \frac{C\alpha_j - \int f_j(t)\,\mathrm{d}t}{\int f_{M+1}(t)\,\mathrm{d}t}, \qquad 1 \le j \le M. \tag{2.11}$$

We proceed by proving $\int g = 0$ and properties (ii)–(iii) for g.

The fact that $\int g = 0$ follows from a straightforward calculation using (2.11) and the assumption $\sum_{j=1}^{M}\alpha_j(x_j - \bar{x}) = 0$. (ii) follows from $\operatorname{supp} g \subset [p_1, p_n] \subset \operatorname{int} I$. Next, observe that by definition of g and $f_1, \ldots, f_{M+1}$, on each $\mathcal{F}_K$-atom contained in the set $B := \bigcup_{m=1}^{M}\bigcup_{i=j_{m-1}+2}^{j_m} B_{\ell(i)}$, the function $\bar{x} + g$ is constant with a value in $\bigcup_i\{x_i\}$. Setting $\mathscr{A}_1 = \{A \subset B : A \text{ is atom in } \mathcal{F}_K\}$ and $\mathscr{A}_2 = \mathscr{A} \setminus \mathscr{A}_1$ now shows (iii)(a). Moreover, by (2.1), (2.3), and (2.8),

$$\Big|\bigcup_{J\in\mathscr{A}_1} J\cap V\Big| = \Big|\bigcup_{m=1}^{M}\bigcup_{i=j_{m-1}+2}^{j_m} B_{\ell(i)}\cap V\Big| \ge \Big|\bigcup_{i\le j_M} B_{\ell(i)}\cap V\Big| - 3M\varepsilon_1$$
$$\ge \Big|\bigcup_i B_{\ell(i)}\cap V\Big| - \Big|\bigcup_{i>j_M} B_{\ell(i)}\Big| - \frac{\varepsilon|I\cap V|}{24}$$
$$\ge \big(1 - \frac{\varepsilon}{3}\big)|I\cap V| - \tilde{\varepsilon}\Big|\bigcup_i B_{\ell(i)}\Big| - \frac{\varepsilon|I\cap V|}{24}.$$

Since $\tilde{\varepsilon}|\bigcup_i B_{\ell(i)}| \le \tilde{\varepsilon}|I| \le \varepsilon|I\cap V|/3$ by definition of $\tilde{\varepsilon}$, we conclude that $|\bigcup_{J\in\mathscr{A}_1} J\cap V| \ge (1-\varepsilon)|I\cap V|$, proving also (iii)(b). Next, we note that for $t \in \operatorname{supp} f_j$ with $j \le M$, we have

$$\bar{x} + g(t) = \bar{x} + f_j(t)(x_j - \bar{x}) = f_j(t)x_j + \big(1 - f_j(t)\big)\bar{x}.$$

Since $f_j(t) \in [0,1]$ and $\bar{x}$ is a convex combination of the elements (x_j), we get (iii)(c) in this case. If $t \in \operatorname{supp} f_{M+1}$, we calculate that

$$
\begin{aligned}
\bar{x} + g(t) &= \bar{x} + f_{M+1}(t) \sum_{j=1}^{M} \beta_j (x_j - \bar{x}) \\
&= \big(1 - f_{M+1}(t)\big)\bar{x} + f_{M+1}(t)\big(\bar{x} + \sum_{j=1}^{M} \beta_j (x_j - \bar{x})\big).
\end{aligned} \tag{2.12}
$$

We have by the lower estimate (2.9) for $\int f_{M+1}$ and by (2.6)

$$
\sum_{j=1}^{M} |\beta_j| \leq \frac{12}{\tilde{\varepsilon}\big|\bigcup_i B_{\ell(i)}\big|} \sum_{j\leq M} \Big(\int f_j - C\alpha_j \Big) \leq \frac{12}{\tilde{\varepsilon}\big|\bigcup_i B_{\ell(i)}\big|} (3\varepsilon_1 M),
$$

which, by (2.4), is smaller than $\varepsilon/2$. Therefore, combining this with (2.12) yields property (iii)(c) for $t \in \operatorname{supp} f_{M+1}$ by setting $\lambda_1 = 1 - f_{M+1}$, $\lambda_2 = f_{M+1}$, $y_1 = \bar{x}$, $y_2 = \bar{x} + \sum_j \beta_j (x_j - \bar{x})$. Thus, we have finished Step 1 of constructing a function g with mean zero and properties (ii), (iii). The next step is to construct a function g so that additionally all of its moments up to order k vanish.

Step 2. Set $\tilde{\varepsilon} = 1 - (1-\varepsilon)^{1/3} > 0$. We write $a = \inf I$, $b = \sup I$, and choose $c \in I$ so that $R := (c, b)$ satisfies $0 < |R \cap V| = \tilde{\varepsilon}|I \cap V|$. Define $L = I \setminus R$. Let (N_i) be the B-spline basis of S_{K_R}, where we choose the integer K_R so that we can select B-spline functions $(N_{m_i})_{i=0}^{k-1}$ that $\operatorname{supp} N_{m_i} \subset \operatorname{int} R$ for any $i = 0, \ldots, k-1$ and $\operatorname{supp} N_{m_i} \cap \operatorname{supp} N_{m_j} = \emptyset$ for $i \neq j$. We then form the $k \times k$-matrix

$$
A = \Big(\int_R t^i N_{m_j}(t)\,\mathrm{d}t \Big)_{i,j=0}^{k-1}.
$$

The matrix $(t_\ell^i)_{i,\ell=0}^{k-1}$ is a Vandermonde matrix having positive determinant for $t_0 < \cdots < t_{k-1}$. Moreover, the matrix $(N_{m_j}(t_\ell))_{j,\ell=0}^{k-1}$ is a diagonal matrix having positive entries if $t_\ell \in \operatorname{int}\operatorname{supp} N_{m_\ell}$ for $\ell = 0, \ldots, k-1$. For other choices of (t_ℓ), the determinant of $(N_{m_j}(t_\ell))_{j,\ell=0}^{k-1}$ vanishes. Therefore, Lemma 1.13 implies that $\det A \neq 0$ and A is invertible.

Next, we choose $\varepsilon_1 = \tilde{\varepsilon}/\big(k(1+\tilde{\varepsilon})\|A^{-1}\|_\infty |L|\big)$ and apply Lemma 1.14 with the parameters ε_1, $\varepsilon_2 = \tilde{\varepsilon}$, and the interval L to obtain a positive integer n so that for the partition $(A_\ell)_{\ell=1}^n$ of L with $n|A_\ell \cap V| = |L \cap V|$ and $\sup A_{\ell-1} = \inf A_\ell$, the set $\Gamma = \{2 \leq \ell \leq n-1 : \max(|A_{\ell-1}|, |A_\ell|, |A_{\ell+1}|) \leq \varepsilon_1\}$ satisfies

$$
\sum_{\ell\in\Gamma} |A_\ell \cap V| \geq (1-\tilde{\varepsilon})|L \cap V|.
$$

We now apply the construction of Step 1 on every set A_ℓ, $\ell \in \Gamma$, with the parameters $\bar{x}$, $(x_j)_{j=1}^M$, $(\alpha_j)_{j=1}^M$, $\tilde{\varepsilon}$ to get functions (g_ℓ) with zero mean having properties (ii), (iii) with I replaced by A_ℓ. On L, we define the function

$$
g(t) := \sum_{\ell\in\Gamma} g_\ell(t), \qquad t \in L.
$$

Let $z_j := \int_L t^j g(t)\,\mathrm{d}t$ for $j = 0, \ldots, k-1$. Observe that since $\int_{A_\ell} g_\ell(t)\,\mathrm{d}t = 0$ and $\|g_\ell\|_{L^\infty_X} \leq 1 + \tilde{\varepsilon}$ by (iii) and $|A_\ell| \leq \varepsilon_1$, we get for all $j = 0, \ldots, k-1$,

$$\begin{aligned}\|z_j\| &= \Big\| \sum_{\ell\in\Gamma} \int_{A_\ell} t^j g_\ell(t)\,\mathrm{d}t \Big\| = \Big\| \sum_{\ell\in\Gamma} \int_{A_\ell} \big(t^j - (\inf A_\ell)^j\big) \cdot g_\ell(t)\,\mathrm{d}t \Big\| \\ &\leq j \sum_{\ell\in\Gamma} |A_\ell| \int_{A_\ell} \|g_\ell(t)\|\,\mathrm{d}t \\ &\leq j\varepsilon_1(1+\tilde{\varepsilon})|L| \leq \tilde{\varepsilon} \cdot \|A^{-1}\|_\infty^{-1}.\end{aligned}$$

In order to have $\int_I t^j g(t)\,\mathrm{d}t = 0$ for all $j = 0, \ldots, k-1$, we want to define g on $R = I \setminus L$ so that

$$\int_R t^j g(t)\,\mathrm{d}t = -z_j, \qquad j = 0, \ldots, k-1. \tag{2.13}$$

Assume that g on R is of the form

$$g(t) = \sum_{i=0}^{k-1} N_{m_i}(t) w_i, \qquad t \in R,$$

for some $(w_i)_{i=0}^{k-1}$ contained in X. Then, (2.13) is equivalent to

$$Aw = -z$$

by writing $w = (w_0, \ldots, w_{k-1})^T$ and $z = (z_0, \ldots, z_{k-1})^T$. Defining $w := -A^{-1}z$ and employing the estimate for $\|z\|_\infty$ above, we obtain

$$\|w\|_\infty \leq \|A^{-1}\|_\infty \|z\|_\infty \leq \tilde{\varepsilon}. \tag{2.14}$$

The definition of g immediately yields properties (i), (ii). From the application of the construction in Step 1 to each A_ℓ, $\ell \in \Gamma$, we obtained collections $\mathscr{A}_1(\ell)$ of disjoint subintervals of A_ℓ that are atoms in $\mathcal{F}_{K_\ell}$ for some positive integer $K_\ell \geq N$ satisfying that $\bar{x} + g_\ell$ is constant on each $J \in \mathscr{A}_1(\ell)$ taking values in $\operatorname{conv}\{x_i : 1 \leq i \leq M\}$ and $|\bigcup_{J\in\mathscr{A}_1(\ell)} J \cap V| \geq (1-\tilde{\varepsilon})|A_\ell \cap V|$. Let $B := \bigcup_\ell \bigcup_{J\in\mathscr{A}_1(\ell)} J$ and define $\mathscr{A}_1$ to be the collection $\{J \subset B : J \text{ is atom in } \mathcal{F}_K\}$, where $K := \max(\max_\ell K_\ell, K_R)$, and define $\mathscr{A} := \{J \subset I : J \text{ is atom in } \mathcal{F}_K\}$, $\mathscr{A}_2 := \mathscr{A} \setminus \mathscr{A}_1$.

Then, (iii)(a) is satisfied by the corresponding property of each g_ℓ. Property (iii)(b) follows from the calculation

$$\begin{aligned}\Big| \bigcup_{J\in\mathscr{A}_1} J \cap V \Big| &\geq (1-\tilde{\varepsilon}) \sum_{\ell\in\Gamma} |A_\ell \cap V| \geq (1-\tilde{\varepsilon})^2 |L \cap V| \\ &\geq (1-\tilde{\varepsilon})^3 |I \cap V| = (1-\varepsilon)|I \cap V|.\end{aligned}$$

Property (iii)(c) on L is a consequence of property (iii)(c) for the functions g_ℓ. We can write $\alpha_j \equiv \sum_\ell \alpha_{j,\ell} N_\ell$ for some non-negative coefficients $(\alpha_{j,\ell})$ that have the property $\sum_{j=1}^M \alpha_{j,\ell} = 1$ for each ℓ. Therefore, on R we have

$$\bar{x}(t) + g(t) = \sum_{j=1}^M \alpha_j(t) x_j + \sum_{i=0}^{k-1} N_{m_i}(t) w_i = \sum_\ell N_\ell(t) \Big(\sum_{j=1}^M \alpha_{j,\ell} x_j + \sum_{i=0}^{k-1} \delta_{\ell,m_i} w_i \Big),$$

which, since $\|w\|_\infty \leq \tilde{\varepsilon} \leq \varepsilon$ and $\sum_{j=1}^M \alpha_{j,\ell} = 1$ for each ℓ, implies (iii)(c) on R. □

We now use Lemma 2.1 inductively to prove Theorem 1.10.

Proof of Theorem 1.10. We assume that X does not have the RNP. Then, by Theorem 1.12, the ball $B(0,1/2) \subset X$ contains a non-dentable convex set D satisfying

$$x \in \overline{\operatorname{conv}}\big(D \setminus B(x, 2\delta)\big), \qquad x \in D,$$

for some parameter 2δ. Defining $D_0 = D + B(0, \delta/2)$ and, for $j \geq 1$, $D_j = D_{j-1} + B(0, 2^{-j-1}\delta)$, we use the remark after Definition 1.11 to get that all the sets (D_j) are contained in $B(0,1)$, are convex, and

$$x \in \operatorname{conv}\big(D_j \setminus B(x,\delta)\big), \qquad x \in D_j,\ j \geq 0.$$

We will assume without restriction that $\eta \leq \delta$.

Let $x_{0,1} \in D_0$ be arbitrary and set $f_0 \equiv \mathbb{1}_{[0,1]} \otimes x_{0,1} \in S_{m_0} \otimes X$ on $I_{0,1} := [0,1]$ for $m_0 = 0$. By P_j, we will denote the L^1_X-extension of the orthogonal projection operator onto S_{m_j}, where we assume that $(m_j)_{j=1}^n$ and $(f_j)_{j=1}^n$ with $f_j \in S_{m_j} \otimes X$ for each $j = 1, \ldots, n$ are constructed in such a way that for all $j = 0, \ldots, n$,

(1) $P_{j-1} f_j = f_{j-1}$ if $j \geq 1$;
(2) on all atoms I in $\mathcal{F}_{m_j}$, f_j has the form

$$f_j \equiv \sum_\ell \lambda_\ell \otimes y_\ell, \qquad \text{finite sum,}$$

for functions $\lambda_\ell \in S_{m_j}$ with non-negative B-spline coefficients, $\sum_\ell \lambda_\ell \equiv 1$, and some $y_\ell \in D_j$;
(3) there exists a finite collection of disjoint intervals $(I_{j,i})_i$ that are atoms in $\mathcal{F}_{m_j}$ so that (setting $C_j = \bigcup_i I_{j,i}$)
  (a) for all i, $f_j \equiv x_{j,i} \in D_j$ on $I_{j,i}$,
  (b) $\|f_j - f_{j-1}\|_X \geq \delta$ on $C_j \cap C_{j-1}$ if $j \geq 1$,
  (c) $|C_j \cap C_{j-1} \cap V| \geq (1 - 2^{-j}\eta)|V|$ if $j \geq 1$,
  (d) $|C_j \cap V| \geq (1 - 2^{-j-2}\eta)|V|$,
  (e) $|I_{j,i} \cap V| > 0$ for every i.

We will then perform the construction of m_{n+1}, f_{n+1}, and the collection $(I_{n+1,i})$ of atoms in $\mathcal{F}_{m_{n+1}}$ having properties (1)–(3) for $j = n+1$. Define the collection $\mathscr{C} = \{A \text{ is atom of } \mathcal{F}_{m_n} : |A \cap V| > 0\}$. We will distinguish the two cases $B \in \mathscr{C}_1 := \{A \in \mathscr{C} : A = I_{n,i} \text{ for some } i\}$ and $B \in \mathscr{C}_2 := \mathscr{C} \setminus \mathscr{C}_1$.

Case 1 ($B \in \mathscr{C}_1$). Here, $B = I_{n,i}$ for some i, and we use the fact that on B, $f_n = x_B := x_{n,i} \in D_n$ and write

$$x_B = \sum_{\ell=1}^{M_B} \alpha_{B,\ell} x_{B,\ell}$$

with some positive numbers $(\alpha_{B,\ell})$ satisfying $\sum_\ell \alpha_{B,\ell} = 1$, some $x_{B,\ell} \in D_n$, and $\|x_B - x_{B,\ell}\| \geq \delta$ for any $\ell = 1, \ldots, M_B$. We apply Lemma 2.1 to the interval B with this decomposition and with the parameter $\varepsilon = \eta_n := 2^{-n-3}\eta$. This yields a function $g_B \in S_{K_B} \otimes X$ for some positive integer K_B that has the following properties:

(i) $\int t^\ell g_B(t)\,\mathrm{d}t = 0, \qquad 0 \leq \ell \leq k-1.$
(ii) $\operatorname{supp} g_B \subset \operatorname{int} B$.
(iii) We have a splitting of the collection $\mathscr{A}_B = \{A \subset B : A \text{ is atom in } \mathcal{F}_{K_B}\}$ into $\mathscr{A}_{B,1} \cup \mathscr{A}_{B,2}$ so that
  (a) on each $J \in \mathscr{A}_{B,1}$, $f_n + g_B = x_B + g_B$ is constant on J taking values in $\bigcup_\ell \{x_{B,\ell}\}$;

(b) $|\bigcup_{J\in\mathscr{A}_{B,1}} J\cap V| \geq (1-\eta_n)|B\cap V|$;
(c) on each $J \in \mathscr{A}_{B,2}$, the function $f_n + g_B$ can be written as

$$f_n(t) + g_B(t) = x_B + g_B(t) = \sum_\ell \lambda_{B,\ell}(t) y_{B,\ell}$$

for some functions $\lambda_{B,\ell} \in S_{K_B}$ having non-negative B-spline coefficients with $\sum_\ell \lambda_{B,\ell} \equiv 1$ and $y_{B,\ell} \in \operatorname{conv}\{x_{B,\ell} : 1 \leq j \leq M_B\} + B(0,\eta_n)$.

Case 2 ($B \in \mathscr{C}_2$). On B, f_n is of the form

$$f_n(t) = \sum_{\ell=1}^{M_B} \lambda_\ell(t) y_\ell$$

for some functions $\lambda_\ell \in S_{m_n}$ having non-negative B-spline coefficients with $\sum_\ell \lambda_\ell \equiv 1$ and some $y_\ell \in D_n$. Applying Lemma 2.1 with the parameter $\eta_n = 2^{-n-3}\eta$, we obtain a function $g_B \in S_{K_B} \otimes X$ (for some positive integer K_B) that has the following properties:

(i) $\int t^\ell g_B(t)\,\mathrm{d}t = 0, \qquad 0 \leq \ell \leq k-1.$
(ii) $\operatorname{supp} g_B \subset \operatorname{int} B$.
(iii) We have a splitting of the collection $\mathscr{A}_B = \{A \subset B : A \text{ is atom in } \mathcal{F}_{K_B}\}$ into $\mathscr{A}_{B,1} \cup \mathscr{A}_{B,2}$ so that
  (a) for each $J \in \mathscr{A}_{B,1}$, $f_n + g_B$ is constant on J taking values in $\operatorname{conv}\{y_\ell : 1 \leq \ell \leq M_B\}$,
  (b) $|\bigcup_{J\in\mathscr{A}_{B,1}} J\cap V| \geq (1-\eta_n)|B\cap V|$,
  (c) for each $J \in \mathscr{A}_{B,2}$, the function $f_n + g_B$ can be written as

$$f_n(t) + g_B(t) = \sum_\ell \lambda_{B,\ell}(t) y_{B,\ell}$$

  for some functions $\lambda_{B,\ell} \in S_{K_B}$ having non-negative B-spline coefficients with $\sum_\ell \lambda_{B,\ell} \equiv 1$ and $y_{B,\ell} \in \operatorname{conv}\{y_j : 1 \leq j \leq M_B\} + B(0,\eta_n)$.

Having treated those two cases, we define the index $m_{n+1} := \max\{K_B : B \in \mathscr{C}\}$ and

$$f_{n+1} = f_n + \sum_{B\in\mathscr{C}} g_B.$$

The new collection $(I_{n+1,i})$ is defined to be the decomposition of the set $\bigcup_{B\in\mathscr{C}} \bigcup_{J\in\mathscr{A}_{B,1}} J$ (from the above construction) into $\mathcal{F}_{m_{n+1}}$-atoms after deleting those $\mathcal{F}_{m_{n+1}}$-atoms I with $|I\cap V| = 0$. Since D_n is convex and $\eta \leq \delta$, the corresponding function values of f_{n+1} are contained in $D_n + B(0,\eta_n) \subset D_{n+1}$, and we will enumerate them as $(x_{n+1,i})_i$ accordingly. We additionally set $C_{n+1} := \bigcup_i I_{n+1,i}$.

With these definitions, we will successively show properties (1)–(3) for $j = n+1$. Since the function $g = P_n f_{n+1} \in S_{m_n} \otimes X$ is characterized by the condition

$$\int g(t)s(t)\,\mathrm{d}t = \int f_{n+1}(t)s(t)\,\mathrm{d}t, \qquad s \in S_{m_n},$$

property (1) for $j = n+1$ follows if we show that $\int g_B(t)s(t)\,\mathrm{d}t = 0$ for any $s \in S_{m_n}$ and any $B \in \mathscr{C}$. But this is a consequence of (i) for g_B (in both Cases 1 and 2), since $s \in S_{m_n}$ is a polynomial of order k on B.

Property (2) now is a consequence of (iii) (again for both Cases 1 and 2). We just remark once again that $D_n + B(0,\eta_n) \subset D_{n+1}$ due to $\eta \leq \delta$. Properties (3a), (3b), and (3e) are direct consequences of the construction. Property (3d) follows from (iii)(b) in Cases 1 and 2 since

$$\begin{aligned}|C_{n+1} \cap V| &= \Big|\bigcup_{B\in\mathscr{C}} \bigcup_{J\in\mathscr{A}_{B,1}} J \cap V\Big| = \sum_{B\in\mathscr{C}} \Big|\bigcup_{J\in\mathscr{A}_{B,1}} J\cap V\Big| \\ &\geq (1-\eta_n) \sum_{B\in\mathscr{C}} |B\cap V| = (1-\eta_n)|V|\end{aligned}$$

and $\eta_n = 2^{-n-3}\eta$. For property (3c), we calculate that

$$|C_{n+1} \cap C_n \cap V| \geq (1-\eta_n)|C_n \cap V| \geq (1-\eta_n)(1-2^{-n-2}\eta)|V|$$

by (iii)(b) in Case 1 and by the induction hypothesis. Since $\eta_n = 2^{-n-3}\eta$, we get $(1-\eta_n)(1-2^{-n-2}\eta) \geq 1-2^{-(n+1)}\eta$, and this proves (3c) for $j = n+1$.

Finally, we note that due to (2) and (3)(c), the sequence (m_n), the k-martingale spline sequence (f_n), and the sets $E_n := C_n \cap C_{n-1} \cap V$ have the properties that are desired in the theorem. □

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CHAPTER 10

Martingale inequalities for spline sequences

Positivity
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Positivity

Martingale inequalities for spline sequences

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Abstract
We show that D. Lépingle's $L_1(\ell_2)$-inequality

$$\Big\| \Big(\sum_n \mathbb{E}[f_n | \mathscr{F}_{n-1}]^2 \Big)^{1/2} \Big\|_1 \le 2 \cdot \Big\| \Big(\sum_n f_n^2 \Big)^{1/2} \Big\|_1, \qquad f_n \in \mathscr{F}_n,$$

extends to the case where we substitute the conditional expectation operators with orthogonal projection operators onto spline spaces and where we can allow that f_n is contained in a suitable spline space $\mathscr{S}(\mathscr{F}_n)$. This is done provided the filtration $(\mathscr{F}_n)$ satisfies a certain regularity condition depending on the degree of smoothness of the functions contained in $\mathscr{S}(\mathscr{F}_n)$. As a by-product, we also obtain a spline version of H_1-BMO duality under this assumption.

Keywords Martingale inequalities · Polynomial spline spaces · Orthogonal projection operators

Mathematics Subject Classification 65D07 · 60G42 · 42C10

1 Introduction

This article is part of a series of papers that extend martingale results to polynomial spline sequences of arbitrary order (see e.g. [11,14,16–19,22]). In order to explain those martingale type results, we have to introduce a little bit of terminology: Let k be a positive integer, $(\mathscr{F}_n)$ an increasing sequence of σ-algebras of sets in [0, 1] where each $\mathscr{F}_n$ is generated by a finite partition of [0, 1] into intervals of positive length. Moreover, define the spline space

$$\mathscr{S}_k(\mathscr{F}_n) = \{f \in C^{k-2}[0,1] : f \text{ is a polynomial of order } k \text{ on each atom of } \mathscr{F}_n\}$$

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and let $P_n^{(k)}$ be the orthogonal projection operator onto $\mathscr{S}_k(\mathscr{F}_n)$ with respect to the L_2 inner product on $[0, 1]$ with the Lebesgue measure $|\cdot|$. The space $\mathscr{S}_1(\mathscr{F}_n)$ consists of piecewise constant functions and $P_n^{(1)}$ is the conditional expectation operator with respect to the σ-algebra $\mathscr{F}_n$. Similarly to the definition of martingales, we introduce the following notion: let $(f_n)_{n\geq 0}$ be a sequence of integrable functions. We call this sequence a *k-martingale spline sequence* (adapted to $(\mathscr{F}_n)$) if, for all n,

$$P_n^{(k)} f_{n+1} = f_n.$$

For basic facts about martingales and conditional expectations, we refer to [15].

Classical martingale theorems such as Doob's inequality or the martingale convergence theorem in fact carry over to k-martingale spline sequences corresponding to *arbitrary* filtrations $(\mathscr{F}_n)$ of the above type, just by replacing conditional expectation operators by the projection operators $P_n^{(k)}$. Indeed, we have

(i) (Shadrin's theorem) there exists a constant C_k depending only on k such that

$$\sup_n \|P_n^{(k)} : L_1 \to L_1\| \leq C_k,$$

(ii) (Doob's weak type inequality for splines)
there exists a constant C_k depending only on k such that for any k-martingale spline sequence (f_n) and any $\lambda > 0$,

$$|\{\sup_n |f_n| > \lambda\}| \leq C_k \frac{\sup_n \|f_n\|_1}{\lambda},$$

(iii) (Doob's L_p inequality for splines)
for all $p \in (1, \infty]$ there exists a constant $C_{p,k}$ depending only on p and k such that for all k-martingale spline sequences (f_n),

$$\big\| \sup_n |f_n| \big\|_p \leq C_{p,k} \sup_n \|f_n\|_p,$$

(iv) (Spline convergence theorem)
if (f_n) is an L_1-bounded k-martingale spline sequence, then (f_n) converges almost surely to some L_1-function,

(v) (Spline convergence theorem, L_p-version)
for $1 < p < \infty$, if (f_n) is an L_p-bounded k-martingale spline sequence, then (f_n) converges almost surely and in L_p.

Property (i) was proved by Shadrin in the groundbreaking paper [22]. We also refer to the paper [25] by von Golitschek, who gives a substantially shorter proof of (i). Properties (ii) and (iii) are proved in [19] and properties (iv) and (v) in [14], but see also [18], where it is shown that, in analogy to the martingale case, the validity of (iv) and (v) for all k-martingale spline sequences with values in a Banach space X characterize the Radon–Nikodým property of X (for background information on that material, we refer to the monographs [6,20]).

Here, we continue this line of transferring martingale results to k-martingale spline sequences and extend Lépingle's $L_1(\ell_2)$-inequality [12], which reads

$$\Big\| \Big(\sum_n \mathbb{E}[f_n|\mathscr{F}_{n-1}]^2 \Big)^{1/2} \Big\|_1 \leq 2 \cdot \Big\| \Big(\sum_n f_n^2 \Big)^{1/2} \Big\|_1, \tag{1.1}$$

provided the sequence of (real-valued) random variables f_n is adapted to the filtration $(\mathscr{F}_n)$, i.e. each f_n is $\mathscr{F}_n$-measurable. Different proofs of (1.1) were given by Bourgain [3, Proposition 5], Delbaen and Schachermayer [4, Lemma 1] and Müller [13, Proposition 4.1]. The spline version of inequality (1.1) is contained in Theorem 4.1.

This inequality is an L_1 extension of the following result for $1 < p < \infty$, proved by Stein [24], that holds for *arbitrary* integrable functions f_n:

$$\Big\| \Big(\sum_n \mathbb{E}[f_n|\mathscr{F}_{n-1}]^2 \Big)^{1/2} \Big\|_p \leq a_p \Big\| \Big(\sum_n f_n^2 \Big)^{1/2} \Big\|_p, \tag{1.2}$$

for some constant a_p depending only on p. This can be seen as a dual version of Doob's inequality $\| \sup_\ell |\mathbb{E}[f|\mathscr{F}_\ell]| \|_p \leq c_p \|f\|_p$ for $p > 1$, see [1]. Once we know Doob's inequality for spline projections, which is point (iii) above, the same proof as in [1] works for spline projections if we use suitable positive operators T_n instead of $P_n^{(k)}$ that also satisfy Doob's inequality and dominate the operators $P_n^{(k)}$ pointwise (cf. Sects. 3.1, 3.2).

The usage of those operators T_n is also necessary in the extension of inequality (1.1) to splines. Lépingle's proof of (1.1) rests on an idea by Herz [10] of splitting $\mathbb{E}[f_n \cdot h_n]$ (for f_n being $\mathscr{F}_n$-measurable) by Cauchy–Schwarz after introducing the square function $S_n^2 = \sum_{\ell \leq n} f_\ell^2$:

$$(\mathbb{E}[f_n \cdot h_n])^2 \leq \mathbb{E}[f_n^2/S_n] \cdot \mathbb{E}[S_n h_n^2] \tag{1.3}$$

and estimating both factors on the right hand side separately. A key point in estimating the second factor is that S_n is $\mathscr{F}_n$-measurable, and therefore, $\mathbb{E}[S_n|\mathscr{F}_n] = S_n$. If we want to allow $f_n \in \mathscr{S}_k(\mathscr{F}_n)$, S_n will not be contained in $\mathscr{S}_k(\mathscr{F}_n)$ in general. Under certain conditions on the filtration $(\mathscr{F}_n)$, we will show in this article how to substitute S_n in estimate (1.3) by a function $g_n \in \mathscr{S}_k(\mathscr{F}_n)$ that enjoys similar properties to S_n and allows us to proceed (cf. Sect. 3.4, in particular Proposition 3.4 and Theorem 3.6). As a by-product, we obtain a spline version (Theorem 4.2) of C. Fefferman's theorem [7] on H^1-BMO duality. For its martingale version, we refer to A. M. Garsia's book [8] on Martingale Inequalities.

2 Preliminaries

In this section, we collect all tools that are needed subsequently.

2.1 Properties of polynomials

We will need Remez' inequality for polynomials:

Theorem 2.1 *Let $V \subset \mathbb{R}$ be a compact interval in $\mathbb{R}$ and $E \subset V$ a measurable subset. Then, for all polynomials p of order k (i.e. degree $k-1$) on V,*

$$\|p\|_{L_\infty(V)} \leq \left(4\frac{|V|}{|E|}\right)^{k-1} \|p\|_{L_\infty(E)}.$$

Applying this theorem with the set $E = \{x \in V : |p(x)| \leq 8^{-k+1}\|p\|_{L_\infty(V)}\}$ immediately yields the following corollary:

Corollary 2.2 *Let p be a polynomial of order k on a compact interval $V \subset \mathbb{R}$. Then*

$$\left|\left\{x \in V : |p(x)| \geq 8^{-k+1}\|p\|_{L_\infty(V)}\right\}\right| \geq |V|/2.$$

2.2 Properties of spline functions

For an interval σ-algebra $\mathscr{F}$ (i.e. $\mathscr{F}$ is generated by a finite collection of intervals having positive length), the space $\mathscr{S}_k(\mathscr{F})$ is spanned by a very special local basis (N_i), the so called B-spline basis. It has the properties that each N_i is non-negative and each support of N_i consists of at most k neighboring atoms of $\mathscr{F}$. Moreover, (N_i) is a partition of unity, i.e. for all $x \in [0, 1]$, there exist at most k functions N_i so that $N_i(x) \neq 0$ and $\sum_i N_i(x) = 1$. In the following, we denote by E_i the support of the B-spline function N_i. The usual ordering of the B-splines (N_i)–which we also employ here–is such that for all i, $\inf E_i \leq \inf E_{i+1}$ and $\sup E_i \leq \sup E_{i+1}$.

We write $A(t) \lesssim B(t)$ to denote the existence of a constant C such that for all t, $A(t) \leq CB(t)$, where t denote all implicit and explicit dependencies the expression A and B might have. If the constant C additionally depends on some parameter, we will indicate this in the text. Similarly, the symbols $\gtrsim$ and $\simeq$ are used.

Another important property of B-splines is the following relation between B-spline coefficients and the L_p-norm of the corresponding B-spline expansions.

Theorem 2.3 (B-spline stability, local and global) *Let $1 \leq p \leq \infty$ and $g = \sum_j a_j N_j$. Then, for all j,*

$$|a_j| \lesssim |J_j|^{-1/p}\|g\|_{L_p(J_j)}, \tag{2.1}$$

where J_j is an atom of $\mathscr{F}$ contained in E_j having maximal length. Additionally,

$$\|g\|_p \simeq \|(a_j|E_j|^{1/p})\|_{\ell_p}, \tag{2.2}$$

where in both (2.1) and (2.2), the implied constants depend only on the spline order k.

Observe that (2.1) implies for $g \in \mathscr{S}_k(\mathscr{F})$ and any measurable set $A \subset [0,1]$

$$\|g\|_{L_\infty(A)} \lesssim \max_{j:|E_j \cap A|>0} \|g\|_{L_\infty(J_j)}. \tag{2.3}$$

We will also need the following relation between the B-spline expansion of a function and its expansion using B-splines of a finer grid.

Theorem 2.4 *Let $\mathscr{G} \subset \mathscr{F}$ be two interval σ-algebras and denote by $(N_{\mathscr{G},i})_i$ the B-spline basis of the coarser space $\mathscr{S}_k(\mathscr{G})$ and by $(N_{\mathscr{F},i})_i$ the B-spline basis of the finer space $\mathscr{S}_k(\mathscr{F})$. Then, given $f = \sum_j a_j N_{\mathscr{G},j}$, we can expand f in the basis $(N_{\mathscr{F},i})_i$*

$$\sum_j a_j N_{\mathscr{G},j} = \sum_i b_i N_{\mathscr{F},i},$$

where for each i, b_i is a convex combination of the coefficients a_j with $\operatorname{supp} N_{\mathscr{G},j} \supseteq \operatorname{supp} N_{\mathscr{F},i}$.

For those results and more information on spline functions, in particular B-splines, we refer to [21] or [5].

2.3 Spline orthoprojectors

We now use the B-spline basis of $\mathscr{S}_k(\mathscr{F})$ and expand the orthogonal projection operator P onto $\mathscr{S}_k(\mathscr{F})$ in the form

$$Pf = \sum_{i,j} a_{ij} \Big(\int_0^1 f(x) N_i(x)\,\mathrm{d}x \Big) \cdot N_j \tag{2.4}$$

for some coefficients (a_{ij}). Denoting by E_{ij} the smallest interval containing both supports E_i and E_j of the B-spline functions N_i and N_j respectively, we have the following estimate for a_{ij} [19]: there exist constants C and $0 < q < 1$ depending only on k so that for each interval σ-algebra $\mathscr{F}$ and each i, j,

$$|a_{ij}| \le C \frac{q^{|i-j|}}{|E_{ij}|}. \tag{2.5}$$

2.4 Spline square functions

Let $(\mathscr{F}_n)$ be a sequence of increasing interval σ-algebras in $[0,1]$ and we assume that each $\mathscr{F}_{n+1}$ is generated from $\mathscr{F}_n$ by the subdivision of exactly one atom of $\mathscr{F}_n$ into two atoms of $\mathscr{F}_{n+1}$. Let P_n be the orthogonal projection operator onto $\mathscr{S}_k(\mathscr{F}_n)$. We denote $\Delta_n f = P_n f - P_{n-1} f$ and define the spline square function

$$Sf = \Big(\sum_n |\Delta_n f|^2 \Big)^{1/2}.$$

We have Burkholder's inequality for the spline square function, i.e. for all $1 < p < \infty$ [16], the L_p-norm of the square function Sf is comparable to the L_p-norm of f:

$$\|Sf\|_p \simeq \|f\|_p, \qquad f \in L_p \tag{2.6}$$

with constants depending only on p and k. Moreover, for $p = 1$, it is shown in [9] that

$$\|Sf\|_1 \simeq \sup_{\varepsilon \in \{-1,1\}^{\mathbb{Z}}} \|\sum_n \varepsilon_n \Delta_n f\|_1, \qquad Sf \in L_1, \tag{2.7}$$

with constants depending only on k and where the proof of the $\lesssim$-part only uses Khintchine's inequality whereas the proof of the $\gtrsim$-part uses fine properties of the functions $\Delta_n f$.

2.5 $L_p(\ell_q)$-spaces

For $1 \leq p, q \leq \infty$, we denote by $L_p(\ell_q)$ the space of sequences of measurable functions (f_n) on $[0, 1]$ so that the norm

$$\|(f_n)\|_{L_p(\ell_q)} = \Big(\int_0^1 \Big(\sum_n |f_n(t)|^q\Big)^{p/q} \,\mathrm{d}t\Big)^{1/p}$$

is finite (with the obvious modifications if $p = \infty$ or $q = \infty$). For $1 \leq p, q < \infty$, the dual space (see [2]) of $L_p(\ell_q)$ is $L_{p'}(\ell_{q'})$ with $p' = p/(p-1)$, $q' = q/(q-1)$ and the duality pairing

$$\langle (f_n), (g_n)\rangle = \int_0^1 \sum_n f_n(t) g_n(t) \,\mathrm{d}t.$$

Hölder's inequality takes the form $|\langle (f_n), (g_n)\rangle| \leq \|(f_n)\|_{L_p(\ell_q)} \|(g_n)\|_{L_{p'}(\ell_{q'})}$.

3 Main results

In this section, we prove our main results. Section 3.1 defines and gives properties of suitable positive operators that dominate our (non-positive) operators $P_n = P_n^{(k)}$ pointwise. In Sect. 3.2, we use those operators to give a spline version of Stein's inequality (1.2). A useful property of conditional expectations is the tower property $\mathbb{E}_{\mathscr{G}}\mathbb{E}_{\mathscr{F}} f = \mathbb{E}_{\mathscr{G}} f$ for $\mathscr{G} \subset \mathscr{F}$. In this form, it extends to the operators (P_n), but not to the operators T from Sect. 3.1. In Sect. 3.3 we prove a version of the tower property for those operators. Section 3.4 is devoted to establishing a duality estimate using a spline square function, which is the crucial ingredient in the proofs of the spline versions of both Lépingle's inequality (1.1) and H_1-BMO duality in Sect. 4.

3.1 The positive operators T

As above, let $\mathscr{F}$ be an interval σ-algebra on $[0, 1]$, (N_i) the B-spline basis of $\mathscr{S}_k(\mathscr{F})$, E_i the support of N_i and E_{ij} the smallest interval containing both E_i and E_j. Moreover, let q be a positive number smaller than 1. Then, we define the linear operator $T = T_{\mathscr{F},q,k}$ by

$$Tf(x) := \sum_{i,j} \frac{q^{|i-j|}}{|E_{ij}|} \langle f, \mathbb{1}_{E_i} \rangle \mathbb{1}_{E_j}(x) = \int_0^1 K(x,t) f(t) \, \mathrm{d}t,$$

where the kernel $K = K_T$ is given by

$$K(x,t) = \sum_{i,j} \frac{q^{|i-j|}}{|E_{ij}|} \mathbb{1}_{E_i}(t) \cdot \mathbb{1}_{E_j}(x).$$

We observe that the operator T is selfadjoint (w.r.t the standard inner product on L_2) and

$$k \le K_x := \int_0^1 K(x,t) \, \mathrm{d}t \le \frac{2(k+1)}{1-q}, \qquad x \in [0,1], \tag{3.1}$$

which, in particular, implies the boundedness of the operator T on L_1 and L_∞:

$$\|Tf\|_1 \le \frac{2(k+1)}{1-q} \|f\|_1, \qquad \|Tf\|_\infty \le \frac{2(k+1)}{1-q} \|f\|_\infty.$$

Another very important property of T is that it is a positive operator, i.e. it maps non-negative functions to non-negative functions and that T satisfies Jensen's inequality in the form

$$\varphi(Tf(x)) \le K_x^{-1} T\big(\varphi(K_x \cdot f)\big)(x), \qquad f \in L_1, x \in [0,1], \tag{3.2}$$

for convex functions φ. This is seen by applying the classical Jensen inequality to the probability measure $K(t,x)\,\mathrm{d}t/K_x$.

Let $\mathscr{M}f$ denote the Hardy–Littlewood maximal function of $f \in L_1$, i.e.

$$\mathscr{M}f(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \, \mathrm{d}y,$$

where the supremum is taken over all subintervals of $[0, 1]$ that contain the point x. This operator is of weak type $(1, 1)$, i.e.

$$|\{\mathscr{M}f > \lambda\}| \le C\lambda^{-1} \|f\|_1, \qquad f \in L_1, \lambda > 0$$

for some constant C. Since trivially we have the estimate $\|\mathscr{M} f\|_\infty \le \|f\|_\infty$, by Marcinkiewicz interpolation, for any $p > 1$, there exists a constant C_p depending only on p so that

$$\|\mathscr{M} f\|_p \le C_p \|f\|_p.$$

For those assertions about $\mathscr{M}$, we refer to (for instance) [23].

The significance of T and $\mathscr{M}$ at this point is that we can use formula (2.4) and estimate (2.5) to obtain the pointwise bound

$$|Pf(x)| \le C_1(T|f|)(x) \le C_2\mathscr{M}(x), \qquad f \in L_1, x \in [0,1], \tag{3.3}$$

where $T = T_{\mathscr{F},q,k}$ with q given by (2.5), C_1 is a constant that depends only on k and C_2 is a constant that depends only on k and the geometric progression q. But as the parameter $q < 1$ in (2.5) depends only on k, the constant C_2 will also only depend on k.

In other words, (3.3) tells us that the positive operator T dominates the non-positive operator P pointwise, but at the same time, T is dominated by the Hardy–Littlewood maximal function $\mathscr{M}$ pointwise and independently of $\mathscr{F}$.

3.2 Stein's inequality for splines

We now use this pointwise dominating, positive operator T to prove Stein's inequality for spline projections. For this, let $(\mathscr{F}_n)$ be an interval filtration on $[0, 1]$ and P_n be the orthogonal projection operator onto the space $\mathscr{S}_k(\mathscr{F}_n)$ of splines of order k corresponding to $\mathscr{F}_n$. Working with the positive operators $T_{\mathscr{F}_n,q,k}$ instead of the non-positive operators P_n, the proof of Stein's inequality (1.2) for spline projections can be carried over from the martingale case (cf. [1,24]). For completeness, we include it here.

Theorem 3.1 *Suppose that (f_n) is a sequence of arbitrary integrable functions on $[0, 1]$. Then, for $1 \le r \le p < \infty$ or $1 < p \le r \le \infty$,*

$$\|(P_n f_n)\|_{L_p(\ell_r)} \lesssim \|(f_n)\|_{L_p(\ell_r)} \tag{3.4}$$

where the implied constant depends only on p, r and k.

Proof By (3.3), it suffices to prove this inequality for the operators $T_n = T_{\mathscr{F}_n,q,k}$ with q given by (2.5) instead of the operators P_n. First observe that for $r = p = 1$, the assertion follows from Shadrin's theorem ((i) on page 1). Inequality (3.3) and the $L_{p'}$-boundedness of $\mathscr{M}$ for $1 < p' \le \infty$ imply that

$$\Big\| \sup_{1\le n\le N} |T_n f| \Big\|_{p'} \le C_{p',k}\|f\|_{p'}, \qquad f \in L_{p'} \tag{3.5}$$

with a constant $C_{p',k}$ depending on p' and k. Let $1 \le p < \infty$ and $U_N : L_p(\ell_1^N) \to L_p$ be given by $(g_1, \ldots, g_N) \mapsto \sum_{j=1}^N T_j g_j$. Inequality (3.5) implies the boundedness of the adjoint $U_N^*: L_{p'} \to L_{p'}(\ell_\infty^N)$, $f \mapsto (T_j f)_{j=1}^N$ for $p' = p/(p-1)$ by a constant independent of N and therefore also the boundedness of U_N. Since $|T_j f| \le T_j|f|$ by the positivity of T_j, letting $N \to \infty$ implies (3.4) for T_n instead of P_n in the case $r = 1$ and outer parameter $1 \le p < \infty$.

If $1 < r \le p$, we use Jensen's inequality (3.2) and estimate (3.1) to obtain

$$\sum_{j=1}^N |T_j g_j|^r \lesssim \sum_{j=1}^N T_j(|g_j|^r)$$

and apply the result for $r = 1$ and the outer parameter p/r to get the result for $1 \le r \le p < \infty$. The cases $1 < p \le r \le \infty$ now just follow from this result using duality and the self-adjointness of T_j. □

3.3 Tower property of *T*

Next, we will prove a substitute of the tower property $\mathbb{E}_{\mathscr{G}}\mathbb{E}_{\mathscr{F}} f = \mathbb{E}_{\mathscr{G}} f$ ($\mathscr{G} \subset \mathscr{F}$) for conditional expectations that applies to the operators T.

To formulate this result, we need a suitable notion of regularity for σ-algebras which we now describe. Let $\mathscr{F}$ be an interval σ-algebra, let (N_j) be the B-spline basis of $\mathscr{S}_k(\mathscr{F})$ and denote by E_j the support of the function N_j. The *k-regularity parameter* $\gamma_k(\mathscr{F})$ is defined as

$$\gamma_k(\mathscr{F}) := \max_i \max(|E_i|/|E_{i+1}|, |E_{i+1}|/|E_i|),$$

where the first maximum is taken over all i so that E_i and E_{i+1} are defined. The name k-regularity is motivated by the fact that each B-spline support E_i of order k consists of at most k (neighboring) atoms of the σ-algebra $\mathscr{F}$.

Proposition 3.2 (Tower property of T) *Let $\mathscr{G} \subset \mathscr{F}$ be two interval σ-algebras on $[0,1]$. Let $S = T_{\mathscr{G},\sigma,k}$ and $T = T_{\mathscr{F},\tau,k'}$ for some $\sigma, \tau \in (0,1)$ and some positive integers k, k'. Then, for all $q > \max(\tau, \sigma)$, there exists a constant C depending on q, k, k' so that*

$$|STf(x)| \le C \cdot \gamma^k \cdot (T_{\mathscr{G},q,k}|f|)(x), \qquad f \in L_1, x \in [0,1], \tag{3.6}$$

where $\gamma = \gamma_k(\mathscr{G})$ denotes the k-regularity parameter of $\mathscr{G}$.

Proof Let (F_i) be the collection of B-spline supports in $\mathscr{S}_{k'}(\mathscr{F})$ and (G_i) the collection of B-spline supports in $\mathscr{S}_k(\mathscr{G})$. Moreover, we denote by F_{ij} the smallest interval containing F_i and F_j and by G_{ij} the smallest interval containing G_i and G_j.

We show (3.6) by showing the following inequality for the kernels K_S of S and K_T of T (cf. 3.1)

$$\int_0^1 K_S(x,t)K_T(t,s)\,\mathrm{d}t \le C\gamma^k \sum_{i,j} \frac{q^{|i-j|}}{|G_{ij}|} \mathbb{1}_{G_i}(x)\mathbb{1}_{G_j}(s), \qquad x, s \in [0,1] \tag{3.7}$$

for all $q > \max(\tau, \sigma)$ and some constant C depending on q, k, k'. In order to prove this inequality, we first fix $x, s \in [0, 1]$ and choose i such that $x \in G_i$ and ℓ such that $s \in F_\ell$. Moreover, based on ℓ, we choose j so that $s \in G_j$ and $G_j \supset F_\ell$. There are at most $\max(k, k')$ choices for each of the indices i, ℓ, j and without restriction, we treat those choices separately, i.e. we only have to estimate the expression

$$\sum_{m,r} \frac{\sigma^{|m-i|}\tau^{|r-\ell|}|G_m \cap F_r|}{|G_{im}||F_{\ell r}|}.$$

Since, for each r, there are also at most $k + k' - 1$ indices m so that $|G_m \cap F_r| > 0$ (recall that $\mathscr{G} \subset \mathscr{F}$), we choose one such index $m = m(r)$ and estimate

$$\Sigma = \sum_r \frac{\sigma^{|m(r)-i|}\tau^{|r-\ell|}|G_{m(r)} \cap F_r|}{|G_{i,m(r)}||F_{\ell r}|}.$$

Now, observe that for any parameter choice of r in the above sum,

$$G_{i,m(r)} \cup F_{\ell r} \supseteq (G_{ij} \backslash G_j) \cup G_i$$

and therefore, since also $G_{m(r)} \cap F_r \subset G_{i,m(r)} \cap F_{\ell r}$,

$$\Sigma \le \frac{2}{|(G_{ij} \backslash G_j) \cup G_i|} \sum_r \sigma^{|m(r)-i|}\tau^{|r-\ell|},$$

which, using the k-regularity parameter $\gamma = \gamma_k(\mathscr{G})$ of the σ-algebra $\mathscr{G}$ and denoting $\lambda = \max(\tau, \sigma)$, we estimate by

$$\begin{aligned}\Sigma &\le \frac{2\gamma^k}{|G_{ij}|} \sum_m \lambda^{|m-i|} \sum_{r:m(r)=m} \lambda^{|r-\ell|} \lesssim \frac{\gamma^k}{|G_{ij}|} \sum_m \lambda^{|i-m|+|m-j|} \\ &\lesssim \frac{\gamma^k}{|G_{ij}|}\big(|i-j|+1\big)\lambda^{|i-j|},\end{aligned}$$

where the implied constants depend on λ, k, k' and the estimate $\sum_{r:m(r)=m} \lambda^{|r-\ell|} \lesssim \lambda^{|m-j|}$ used the fact that, essentially, there are more atoms of $\mathscr{F}$ between F_r and F_ℓ (for r as in the sum) than atoms of $\mathscr{G}$ between G_m and G_j. Finally, we see that for any $q > \lambda$,

$$\Sigma \lesssim C\gamma^k \frac{q^{|i-j|}}{|G_{ij}|}$$

for some constant C depending on q, k, k', and, as $x \in G_i$ and $s \in G_j$, this shows inequality (3.7). □

As a corollary of Proposition 3.2, we have

Corollary 3.3 *Let (f_n) be functions in L_1. We denote by P_n the orthogonal projection onto $\mathscr{S}_k(\mathscr{F}_n)$ and by P_n' the orthogonal projection onto $\mathscr{S}_{k'}(\mathscr{F}_n)$ for some positive integers k, k'. Moreover, let T_n be the operator $T_{\mathscr{F}_n,q,k}$ from (3.3) dominating P_n pointwise.*

Then, for any integer n and for any $1 \le p \le \infty$,

$$\Big\| \sum_{\ell \ge n} P_n\big((P_{\ell-1}' f_\ell)^2\big)\Big\|_p \lesssim \Big\| \sum_{\ell \ge n} T_n\big((P_{\ell-1}' f_\ell)^2\big)\Big\|_p \lesssim \gamma_k(\mathscr{F}_n)^k \cdot \Big\| \sum_{\ell \ge n} f_\ell^2 \Big\|_p,$$

where the implied constants only depend on k and k'.

We remark that by Jensen's inequality and the tower property, this is trivial for conditional expectations $\mathbb{E}(\cdot|\mathscr{F}_n)$ instead of the operators P_n, T_n, $P_{\ell-1}'$ even with an absolute constant on the right hand side.

Proof We denote by T_n the operator $T_{\mathscr{F}_n,q,k}$ and by T_n' the operator $T_{\mathscr{F}_n,q',k'}$, where the parameters $q, q' < 1$ are given by inequality (3.3) depending on k and k' respectively. Setting $U_n := T_{\mathscr{F}_n,\max(q,q')^{1/2},k}$, we perform the following chain of inequalities, where we use the positivity of T_n and (3.3), Jensen's inequality for $T_{\ell-1}'$, the tower property for $T_n T_{\ell-1}'$ and the L_p-boundedness of U_n, respectively:

$$\begin{aligned}
\Big\| \sum_{\ell \ge n} T_n\big((P_{\ell-1}' f_\ell)^2\big)\Big\|_p &\lesssim \Big\| \sum_{\ell \ge n} T_n\big((T_{\ell-1}' |f_\ell|)^2\big)\Big\|_p \\
&\lesssim \Big\| \sum_{\ell \ge n} T_n\big(T_{\ell-1}' f_\ell^2\big)\Big\|_p \\
&\le \|T_n(T_{n-1}' f_n^2)\|_p + \Big\| \sum_{\ell > n} T_n\big(T_{\ell-1}' f_\ell^2\big)\Big\|_p \\
&\lesssim \|f_n^2\|_p + \gamma_k(\mathscr{F}_n)^k \cdot \Big\| \sum_{\ell > n} U_n(f_\ell^2)\Big\|_p \\
&\lesssim \gamma_k(\mathscr{F}_n)^k \cdot \Big\| \sum_{\ell \ge n} f_\ell^2 \Big\|_p,
\end{aligned}$$

where the implied constants only depend on k and k'. □

3.4 A duality estimate using a spline square function

In order to give the desired duality estimate contained in Theorem 3.6, we need the following construction of a function $g_n \in \mathscr{S}_k(\mathscr{F}_n)$ based on a spline square function.

Proposition 3.4 *Let (f_n) be a sequence of functions with $f_n \in \mathscr{S}_k(\mathscr{F}_n)$ for all n and set*

$$X_n := \sum_{\ell \le n} f_\ell^2.$$

Then, there exists a sequence of non-negative functions $g_n \in \mathscr{S}_k(\mathscr{F}_n)$ *so that for each n,*

(1) $g_n \leq g_{n+1}$,
(2) $X_n^{1/2} \leq g_n$
(3) $\mathbb{E}g_n \lesssim \mathbb{E}X_n^{1/2}$, *where the implied constant depends on k and on* $\sup_{m\leq n} \gamma_k(\mathscr{F}_m)$.

For the proof of this result, we need the following simple lemma.

Lemma 3.5 *Let* c_1 *be a positive constant and let* $(A_j)_{j=1}^N$ *be a sequence of atoms in* $\mathscr{F}_n$. *Moreover, let* $\ell : \{1, \dots, N\} \to \{1, \dots, n\}$ *and, for each* $j \in \{1, \dots, N\}$, *let* B_j *be a subset of an atom* D_j *of* $\mathscr{F}_{\ell(j)}$ *with*

$$|B_j| \geq c_1 \sum_{\substack{i:\ell(i)\geq\ell(j),\\ D_i\subseteq D_j}} |A_i|. \tag{3.8}$$

Then, there exists a map φ *on* $\{1, \dots, N\}$ *so that*

(1) $|\varphi(j)| = c_1|A_j|$ *for all* j,
(2) $\varphi(j) \subseteq B_j$ *for all* j,
(3) $\varphi(i) \cap \varphi(j) = \emptyset$ *for all* $i \neq j$.

Proof Without restriction, we assume that the sequence (A_j) is enumerated such that $\ell(j+1) \leq \ell(j)$ for all $1 \leq j \leq N-1$. We first choose $\varphi(1)$ as an arbitrary (measurable) subset of B_1 with measure $c_1|A_1|$, which is possible by assumption (3.8). Next, we assume that for $1 \leq j \leq j_0$, we have constructed $\varphi(j)$ with the properties

(1) $|\varphi(j)| = c_1|A_j|$,
(2) $\varphi(j) \subseteq B_j$,
(3) $\varphi(j) \cap \cup_{i<j}\varphi(i) = \emptyset$.

Based on that, we now construct $\varphi(j_0 + 1)$. Define the index sets $\Gamma = \{i : \ell(i) \geq \ell(j_0 + 1), D_i \subseteq D_{j_0+1}\}$ and $\Lambda = \{i : i \leq j_0 + 1, D_i \subseteq D_{j_0+1}\}$. Since we assumed that ℓ is decreasing, we have $\Lambda \subseteq \Gamma$ and by the nestedness of the σ-algebras $\mathscr{F}_n$, we have for $i \leq j_0 + 1$ that either $D_i \subset D_{j_0+1}$ or $|D_i \cap D_{j_0+1}| = 0$. This implies

$$\begin{aligned}
\Big|B_{j_0+1}\setminus \bigcup_{i\leq j_0} \varphi(i)\Big| &= |B_{j_0+1}| - \Big|B_{j_0+1} \cap \bigcup_{i\leq j_0} \varphi(i)\Big| \\
&\geq c_1 \sum_{i\in\Gamma} |A_i| - \Big|D_{j_0+1} \cap \bigcup_{i\leq j_0} \varphi(i)\Big| \\
&\geq c_1 \sum_{i\in\Lambda} |A_i| - \Big|\bigcup_{i\in\Lambda\setminus\{j_0+1\}} \varphi(i)\Big| \\
&\geq c_1 \sum_{i\in\Lambda} |A_i| - \sum_{i\in\Lambda\setminus\{j_0+1\}} c_1|A_i| = c_1|A_{j_0+1}|.
\end{aligned}$$

Therefore, we can choose $\varphi(j_0 + 1) \subseteq B_{j_0+1}$ that is disjoint to $\varphi(i)$ for any $i \leq j_0$ and $|\varphi(j_0 + 1)| = c_1|A_{j_0+1}|$ which completes the proof. □

Proof of Proposition 3.4 Fix n and let $(N_{n,j})$ be the B-spline basis of $\mathscr{S}_k(\mathscr{F}_n)$. Moreover, for any j, set $E_{n,j} = \operatorname{supp} N_{n,j}$ and $a_{n,j} := \max_{\ell \le n} \max_{r: E_{\ell,r} \supset E_{n,j}} \|X_\ell\|^{1/2}_{L_\infty(E_{\ell,r})}$ and we define $\ell(j) \le n$ and $r(j)$ so that $E_{\ell(j),r(j)} \supseteq E_{n,j}$ and $a_{n,j} = \|X_{\ell(j)}\|^{1/2}_{L_\infty(E_{\ell(j),r(j)})}$. Set

$$g_n := \sum_j a_{n,j} N_{n,j} \in \mathscr{S}_k(\mathscr{F}_n)$$

and it will be proved subsequently that this g_n has the desired properties.

PROPERTY (1): In order to show $g_n \le g_{n+1}$, we use Theorem 2.4 to write

$$g_n = \sum_j a_{n,j} N_{n,j} = \sum_j \beta_{n,j} N_{n+1,j},$$

where $\beta_{n,j}$ is a convex combination of those $a_{n,r}$ with $E_{n+1,j} \subseteq E_{n,r}$, and thus

$$g_n \le \sum_j \Big(\max_{r: E_{n+1,j} \subseteq E_{n,r}} a_{n,r} \Big) N_{n+1,j}.$$

By the very definition of $a_{n+1,j}$, we have

$$\max_{r: E_{n+1,j} \subseteq E_{n,r}} a_{n,r} \le a_{n+1,j},$$

and therefore, $g_n \le g_{n+1}$ pointwise, since the B-splines $(N_{n+1,j})_j$ are nonnegative functions.

PROPERTY (2): Now we show that $X_n^{1/2} \le g_n$. Indeed, for any $x \in [0, 1]$,

$$g_n(x) = \sum_j a_{n,j} N_{n,j}(x) \ge \min_{j: E_{n,j} \ni x} a_{n,j} \ge \min_{j: E_{n,j} \ni x} \|X_n\|^{1/2}_{L_\infty(E_{n,j})} \ge X_n(x)^{1/2},$$

since the collection of B-splines $(N_{n,j})_j$ forms a partition of unity.

PROPERTY (3): Finally, we show $\mathbb{E} g_n \lesssim \mathbb{E} X_n^{1/2}$, where the implied constant depends only on k and on $\sup_{m \le n} \gamma_k(\mathscr{F}_m)$. By B-spline stability (Theorem 2.3), we estimate the integral of g_n by

$$\mathbb{E} g_n \lesssim \sum_j |E_{n,j}| \cdot \|X_{\ell(j)}\|^{1/2}_{L_\infty(E_{\ell(j),r(j)})}, \tag{3.9}$$

where the implied constant only depends on k. In order to continue the estimate, we next show the inequality

$$\|X_\ell\|_{L_\infty(E_{\ell,r})} \lesssim \max_{s: |E_{\ell,r} \cap E_{\ell,s}| > 0} \|X_\ell\|_{L_\infty(J_{\ell,s})}, \tag{3.10}$$

where by $J_{\ell,s}$ we denote an atom of $\mathscr{F}_\ell$ with $J_{\ell,s} \subset E_{\ell,s}$ of maximal length and the implied constant depends only on k. Indeed, we use Theorem 2.3 in the form of (2.3) to get ($f_m \in \mathscr{S}_k(\mathscr{F}_\ell)$ for $m \leq \ell$)

$$\begin{aligned}\|X_\ell\|_{L_\infty(E_{\ell,r})} &\leq \sum_{m\leq\ell} \|f_m\|^2_{L_\infty(E_{\ell,r})} \\ &\lesssim \sum_{m\leq\ell} \sum_{s:|E_{\ell,s}\cap E_{\ell,r}|>0} \|f_m\|^2_{L_\infty(J_{\ell,s})} = \sum_{s:|E_{\ell,s}\cap E_{\ell,r}|>0} \sum_{m\leq\ell} \|f_m\|^2_{L_\infty(J_{\ell,s})}.\end{aligned} \tag{3.11}$$

Now observe that for atoms I of $\mathscr{F}_\ell$, due to the equivalence of p-norms of polynomials (cf. Corollary 2.2),

$$\sum_{m\leq\ell} \|f_m\|^2_{L_\infty(I)} \lesssim \sum_{m\leq\ell} \frac{1}{|I|}\int_I f_m^2 = \frac{1}{|I|}\int_I X_\ell \leq \|X_\ell\|_{L_\infty(I)},$$

which means that, inserting this in estimate (3.11),

$$\|X_\ell\|_{L_\infty(E_{\ell,r})} \lesssim \sum_{s:|E_{\ell,s}\cap E_{\ell,r}|>0} \|X_\ell\|_{L_\infty(J_{\ell,s})},$$

and, since there are at most k indices s so that $|E_{\ell,s} \cap E_{\ell,r}| > 0$, we have established (3.10).

We define $K_{\ell,r}$ to be an interval $J_{\ell,s}$ with $|E_{\ell,r} \cap E_{\ell,s}| > 0$ so that

$$\max_{s:|E_{\ell,r}\cap E_{\ell,s}|>0} \|X_\ell\|_{L_\infty(J_{\ell,s})} = \|X_\ell\|_{L_\infty(K_{\ell,r})}.$$

This means, after combining (3.9) with (3.10), we have

$$\mathbb{E}g_n \lesssim \sum_j |J_{n,j}| \cdot \|X_{\ell(j)}\|^{1/2}_{L_\infty(K_{\ell(j),r(j)})}. \tag{3.12}$$

We now apply Lemma 3.5 with the function ℓ and the choices

$$\begin{aligned}A_j &= J_{n,j}, \qquad D_j = K_{\ell(j),r(j)}, \\ B_j &= \Big\{t \in D_j : X_{\ell(j)}(t) \geq 8^{-2(k-1)}\|X_{\ell(j)}\|_{L_\infty(D_j)}\Big\}.\end{aligned}$$

In order to see Assumption (3.8) of Lemma 3.5, fix the index j and let i be such that $\ell(i) \geq \ell(j)$. By definition of $D_i = K_{\ell(i),r(i)}$, the smallest interval containing $J_{n,i}$ and D_i contains at most $2k-1$ atoms of $\mathscr{F}_{\ell(i)}$ and, if $D_i \subset D_j$, the smallest interval containing $J_{n,i}$ and D_j contains at most $2k-1$ atoms of $\mathscr{F}_{\ell(j)}$. This means that, in particular, $J_{n,i}$ is a subset of the union V of $4k$ atoms of $\mathscr{F}_{\ell(j)}$ with $D_j \subset V$. Since each atom of $\mathscr{F}_n$ occurs at most k times in the sequence (A_j), there exists a constant c_1 depending on k and $\sup_{u\leq \ell(j)} \gamma_k(\mathscr{F}_u) \leq \sup_{u\leq n} \gamma_k(\mathscr{F}_u)$ so that

$$|D_j| \geq c_1 \sum_{\substack{i:\ell(i)\geq \ell(j)\\ D_i\subset D_j}} |A_i|,$$

which, since $|B_j| \geq |D_j|/2$ by Corollary 2.2, shows that the assumption of Lemma 3.5 holds true and we get a function φ so that $|\varphi(j)| = c_1|J_{n,j}|/2$, $\varphi(j) \subset B_j$, $\varphi(i)\cap\varphi(j) = \emptyset$ for all i, j. Using these properties of φ, we continue the estimate in (3.12) and write

$$\begin{aligned}
\mathbb{E}g_n &\lesssim \sum_j |J_{n,j}| \cdot \|X_{\ell(j)}\|_{L_\infty(D_j)}^{1/2} \leq 8^{k-1}\cdot \sum_j \frac{|J_{n,j}|}{|\varphi(j)|}\int_{\varphi(j)} X_{\ell(j)}^{1/2} \\
&= \frac{2}{c_1}\cdot 8^{k-1}\cdot \sum_j \int_{\varphi(j)} X_{\ell(j)}^{1/2} \\
&\lesssim \sum_j \int_{\varphi(j)} X_n^{1/2} \leq \mathbb{E}X_n^{1/2},
\end{aligned}$$

with constants depending only on k and $\sup_{u\leq n}\gamma_k(\mathscr{F}_u)$. □

Employing this construction of g_n, we now give the following duality estimate for spline projections (for the martingale case, see for instance [8]). The martingale version of this result is the essential estimate in the proof of both Lépingle's inequality (1.1) and the H^1-BMO duality.

Theorem 3.6 *Let $(\mathscr{F}_n)$ be such that $\gamma := \sup_n \gamma_k(\mathscr{F}_n) < \infty$ and $(f_n)_{n\geq 1}$ a sequence of functions with $f_n \in \mathscr{S}_k(\mathscr{F}_n)$ for each n. Additionally, let $h_n \in L_1$ be arbitrary. Then, for any N,*

$$\sum_{n\leq N} \mathbb{E}[|f_n\cdot h_n|] \lesssim \sqrt{2}\cdot \mathbb{E}\Big[\Big(\sum_{\ell\leq N} f_\ell^2\Big)^{1/2}\Big]\cdot \sup_{n\leq N} \|P_n\Big(\sum_{\ell=n}^{N} h_\ell^2\Big)\|_\infty^{1/2},$$

where the implied constant is the same constant that appears in (3) of Proposition 3.4 and therefore only depends on k and γ.

Proof Let $X_n := \sum_{\ell\leq n} f_\ell^2$ and $f_\ell \equiv 0$ for $\ell > N$ and $\ell \leq 0$. By Proposition 3.4, we choose an increasing sequence (g_n) of functions with $g_0 = 0$, $g_n \in \mathscr{S}_k(\mathscr{F}_n)$ and the properties $X_n^{1/2} \leq g_n$ and $\mathbb{E}g_n \lesssim \mathbb{E}X_n^{1/2}$ where the implied constant depends only on k and γ. Then, apply Cauchy–Schwarz inequality by introducing the factor $g_n^{1/2}$ to get

$$\sum_n \mathbb{E}[|f_n \cdot h_n|] = \sum_n \mathbb{E}\left[\left|\frac{f_n}{g_n^{1/2}} \cdot g_n^{1/2} h_n\right|\right]$$
$$\leq \left[\sum_n \mathbb{E}[f_n^2/g_n]\right]^{1/2} \cdot \left[\sum_n \mathbb{E}[g_n h_n^2]\right]^{1/2}.$$

We estimate each of the factors on the right hand side separately and set

$$\Sigma_1 := \sum_n \mathbb{E}[f_n^2/g_n], \qquad \Sigma_2 := \sum_n \mathbb{E}[g_n h_n^2].$$

The first factor is estimated by the pointwise inequality $X_n^{1/2} \leq g_n$:

$$\Sigma_1 = \mathbb{E}\left[\sum_n \frac{f_n^2}{g_n}\right] \leq \mathbb{E}\left[\sum_n \frac{f_n^2}{X_n^{1/2}}\right]$$
$$= \mathbb{E}\left[\sum_n \frac{X_n - X_{n-1}}{X_n^{1/2}}\right] \leq 2\mathbb{E}\sum_n (X_n^{1/2} - X_{n-1}^{1/2}) = 2\mathbb{E}X_N^{1/2}.$$

We continue with Σ_2:

$$\begin{aligned}
\Sigma_2 &= \mathbb{E}\left[\sum_{\ell=1}^N g_\ell h_\ell^2\right] = \mathbb{E}\left[\sum_{\ell=1}^N \sum_{n=1}^{\ell} (g_n - g_{n-1}) h_\ell^2\right] \\
&= \mathbb{E}\left[\sum_{n=1}^N (g_n - g_{n-1}) \cdot \sum_{\ell=n}^N h_\ell^2\right] \\
&= \mathbb{E}\left[\sum_{n=1}^N P_n(g_n - g_{n-1}) \cdot \sum_{\ell=n}^N h_\ell^2\right] \\
&= \mathbb{E}\left[\sum_{n=1}^N (g_n - g_{n-1}) \cdot P_n\left(\sum_{\ell=n}^N h_\ell^2\right)\right] \\
&\leq \mathbb{E}\left[\sum_{n=1}^N (g_n - g_{n-1})\right] \cdot \sup_{1\leq n\leq N} \left\| P_n\left(\sum_{\ell=n}^N h_\ell^2\right)\right\|_\infty,
\end{aligned}$$

where the last inequality follows from $g_n \geq g_{n-1}$. Noting that by the properties of g_n, $\mathbb{E}\big[\sum_{n=1}^N (g_n - g_{n-1})\big] = \mathbb{E}g_N \lesssim \mathbb{E}X_N^{1/2}$ and combining the two parts Σ_1 and Σ_2, we obtain the conclusion. □

4 Applications

We give two applications of Theorem 3.6, (i) D. Lépingle's inequality and (ii) an analogue of C. Fefferman's H_1-BMO duality in the setting of splines. Once the results from Sect. 3 are known, the proofs of the subsequent results proceed similarly to their martingale counterparts in [8,12] by using spline properties instead of martingale properties.

4.1 Lépingle's inequality for splines

Theorem 4.1 *Let k, k' be positive integers. Let $(\mathscr{F}_n)$ be an interval filtration with $\sup_n \gamma_k(\mathscr{F}_n) < \infty$ and, for any n, $f_n \in \mathscr{S}_k(\mathscr{F}_n)$ and P_n' be the orthogonal projection operator on $\mathscr{S}_{k'}(\mathscr{F}_n)$. Then,*

$$\|(P_{n-1}' f_n)\|_{L_1(\ell_2)} = \left\| \left(\sum_n (P_{n-1}' f_n)^2 \right)^{1/2} \right\|_1 \lesssim \left\| \left(\sum_n f_n^2 \right)^{1/2} \right\|_1 = \|(f_n)\|_{L_1(\ell_2)},$$

where the implied constant depends only on k, k' and $\sup_n \gamma_k(\mathscr{F}_n)$.

We emphasize that the parameters k and k' can be different here, k being the spline order of the sequence (f_n) and k' being the spline order of the projection operators P_{n-1}'. In particular, the constant on the right hand side does not depend on the k'-regularity parameter $\sup_n \gamma_{k'}(\mathcal{F}_n)$.

Proof We first assume that $f_n = 0$ for $n > N$ and begin by using duality

$$\mathbb{E}\left[\left(\sum_n (P_{n-1}' f_n)^2 \right)^{1/2} \right] = \sup_{(H_n)} \mathbb{E}\left[\sum_n (P_{n-1}' f_n) \cdot H_n \right],$$

where sup is taken over all $(H_n) \in L_\infty(\ell_2)$ with $\|(H_n)\|_{L_\infty(\ell_2)} = 1$. By the self-adjointness of P_{n-1}',

$$\mathbb{E}\big[(P_{n-1}' f_n) \cdot H_n\big] = \mathbb{E}\big[f_n \cdot (P_{n-1}' H_n)\big].$$

Then we apply Theorem 3.6 for f_n and $h_n = P_{n-1}' H_n$ to obtain (denoting by P_n the orthogonal projection operator onto $\mathscr{S}_k(\mathscr{F}_n)$)

$$\sum_{n \le N} |\mathbb{E}[f_n \cdot h_n]| \lesssim \mathbb{E}\left[\left(\sum_{\ell \le N} f_\ell^2 \right)^{1/2} \right] \cdot \sup_{n \le N} \left\| P_n \left(\sum_{\ell=n}^{N} (P_{\ell-1}' H_\ell)^2 \right) \right\|_\infty^{1/2}. \quad (4.1)$$

To estimate the right hand side, we note that for any n, by Corollary 3.3,

$$\left\| P_n \left(\sum_{\ell=n}^{N} (P_{\ell-1}' H_\ell)^2 \right) \right\|_\infty \lesssim \left\| \sum_{\ell=n}^{N} H_\ell^2 \right\|_\infty.$$

Therefore, (4.1) implies

$$\mathbb{E}\left[\left(\sum_n (P'_{n-1} f_n)^2\right)^{1/2}\right] = \sup_{(H_n)} \mathbb{E}\left[\sum_n f_n \cdot (P'_{n-1} H_n)\right] \lesssim \mathbb{E}\left[\left(\sum_{\ell \le N} f_\ell^2\right)^{1/2}\right],$$

with a constant depending only on k,k' and $\sup_{n\le N} \gamma_k(\mathscr{F}_n)$. Letting N tend to infinity, we obtain the conclusion. □

4.2 H_1-BMO duality for splines

We fix an interval filtration $(\mathscr{F}_n)_{n=1}^\infty$, a spline order k and the orthogonal projection operators P_n onto $\mathscr{S}_k(\mathscr{F}_n)$ and additionally, we set $P_0 = 0$. For $f \in L_1$, we introduce the notation

$$\Delta_n f := P_n f - P_{n-1} f, \qquad S_n(f) := \left(\sum_{\ell=1}^n (\Delta_\ell f)^2\right)^{1/2}, \qquad S(f) = \sup_n S_n(f).$$

Observe that for $\ell < n$ and $f, g \in L_1$,

$$\mathbb{E}[\Delta_\ell f \cdot \Delta_n g] = \mathbb{E}[P_\ell(\Delta_\ell f) \cdot \Delta_n g] = \mathbb{E}[\Delta_\ell f \cdot P_\ell(\Delta_n g)] = 0. \tag{4.2}$$

Let V be the L_1-closure of $\cup_n \mathscr{S}_k(\mathscr{F}_n)$. Then, the uniform boundedness of P_n on L_1 implies that $P_n f \to f$ in L_1 for $f \in V$. Next, set

$$H_{1,k} = H_{1,k}((\mathscr{F}_n)) = \{f \in V : \mathbb{E}(S(f)) < \infty\}$$

and equip $H_{1,k}$ with the norm $\|f\|_{H_{1,k}} = \mathbb{E}S(f)$. Define

$$\mathrm{BMO}_k = \mathrm{BMO}_k((\mathscr{F}_n)) = \left\{ f \in V : \sup_n \Big\| \sum_{\ell \ge n} T_n\big((\Delta_\ell f)^2\big) \Big\|_\infty < \infty \right\}$$

and the corresponding quasinorm

$$\|f\|_{\mathrm{BMO}_k} = \sup_n \Big\| \sum_{\ell \ge n} T_n\big((\Delta_\ell f)^2\big) \Big\|_\infty^{1/2},$$

where T_n is the operator from (3.3) that dominates P_n pointwise.

Let us now assume $\sup_n \gamma_k(\mathscr{F}_n) < \infty$. In this case we identify, similarly to H_1-BMO-duality (cf. [7,8,10]), BMO_k as the dual space of $H_{1,k}$.

First, we use the duality estimate Theorem 3.6 and (4.2) to prove, for $f \in H_{1,k}$ and $h \in \mathrm{BMO}_k$,

$$\big|\mathbb{E}\big[(P_n f)\cdot(P_n h)\big]\big| \le \sum_{\ell\le n}\mathbb{E}\big[|\Delta_\ell f|\cdot|\Delta_\ell h|\big] \lesssim \|S_n(f)\|_1\cdot\|h\|_{\mathrm{BMO}_k}.$$

This estimate also implies that the limit $\lim_n \mathbb{E}\big[(P_n f)\cdot(P_n h)\big]$ exists and satisfies

$$\big|\lim_n \mathbb{E}\big[(P_n f)\cdot(P_n h)\big]\big| \lesssim \|f\|_{H_{1,k}}\cdot\|h\|_{\mathrm{BMO}_k}.$$

On the other hand, similarly to the martingale case (see [8]), given a continuous linear functional L on $H_{1,k}$, we extend L norm-preservingly to a continuous linear functional Λ on $L_1(\ell_2)$, which, by Sect. 2.5, has the form

$$\Lambda(\eta) = \mathbb{E}\left[\sum_\ell \sigma_\ell\eta_\ell\right], \qquad \eta\in L_1(\ell_2)$$

for some $\sigma\in L_\infty(\ell_2)$. The k-martingale spline sequence $h_n = \sum_{\ell\le n}\Delta_\ell\sigma_\ell$ is bounded in L_2 and therefore, by the spline convergence theorem ((v) on page 2), has a limit $h\in L_2$ with $P_n h = h_n$ and which is also contained in BMO_k. Indeed, by using Corollary 3.3, we obtain $\|h\|_{\mathrm{BMO}_k} \lesssim \|\sigma\|_{L_\infty(\ell_2)} = \|\Lambda\| = \|L\|$ with a constant depending only on k and $\sup_n \gamma_k(\mathscr{F}_n)$. Moreover, for $f\in H_{1,k}$, since L is continuous on $H_{1,k}$,

$$\begin{aligned} L(f) = \lim_n L(P_n f) &= \lim_n \Lambda\big((\Delta_1 f,\dots,\Delta_n f,0,0,\dots)\big) \\ &= \lim_n \sum_{\ell=1}^n \mathbb{E}[\sigma_\ell\cdot\Delta_\ell f] = \lim_n \mathbb{E}\big[(P_n f)\cdot(P_n h)\big]. \end{aligned}$$

This yields the following

Theorem 4.2 *If $\sup_n \gamma_k(\mathscr{F}_n) < \infty$, the linear mapping*

$$u:\mathrm{BMO}_k\to H_{1,k}^*, \qquad h\mapsto \big(f\mapsto \lim_n \mathbb{E}\big[(P_n f)\cdot(P_n h)\big]\big)$$

is bijective and satisfies

$$\|u(h)\|_{H_{1,k}^*} \simeq \|h\|_{\mathrm{BMO}_k},$$

where the implied constants depend only on k and $\sup_n \gamma_k(\mathscr{F}_n)$.

Remark 4.3 We close with a few remarks concerning the above result and we assume that $(\mathscr{F}_n)$ is a sequence of increasing interval σ-algebras with $\sup_n \gamma_k(\mathscr{F}_n) < \infty$.

(1) By Khintchine's inequality, $\|Sf\|_1 \lesssim \sup_{\varepsilon \in \{-1,1\}^{\mathbb{Z}}} \|\sum_n \varepsilon_n \Delta_n f\|_1$. Based on the interval filtration $(\mathscr{F}_n)$, we can generate an interval filtration $(\mathscr{G}_n)$ that contains $(\mathscr{F}_n)$ as a subsequence and each $\mathscr{G}_{n+1}$ is generated from $\mathscr{G}_n$ by dividing exactly one atom of $\mathscr{G}_n$ into two atoms of $\mathscr{G}_{n+1}$. Denoting by $P_n^{\mathscr{G}}$ the orthogonal projection operator onto $\mathscr{S}_k(\mathscr{G}_n)$ and $\Delta_j^{\mathscr{G}} = P_j^{\mathscr{G}} - P_{j-1}^{\mathscr{G}}$, we can write

$$\sum_n \varepsilon_n \Delta_n f = \sum_n \varepsilon_n \sum_{j=a_n}^{a_{n+1}-1} \Delta_j^{\mathscr{G}} f$$

for some sequence (a_n). By using inequalities (2.7) and (2.6) and writing $(S^{\mathscr{G}} f)^2 = \sum_j |\Delta_j^{\mathscr{G}} f|^2$, we obtain for $p > 1$

$$\|Sf\|_1 \lesssim \|S^{\mathscr{G}} f\|_1 \leq \|S^{\mathscr{G}} f\|_p \lesssim \|f\|_p.$$

This implies $L_p \subset H_{1,k}$ for all $p > 1$ and, by duality, $\mathrm{BMO}_k \subset L_p$ for all $p < \infty$.

(2) If $(\mathscr{F}_n)$ is of the form that each $\mathscr{F}_{n+1}$ is generated from $\mathscr{F}_n$ by splitting exactly one atom of $\mathscr{F}_n$ into two atoms of $\mathscr{F}_{n+1}$ and under the condition $\sup_n \gamma_{k-1}(\mathscr{F}_n) < \infty$ (which is stronger than $\sup_n \gamma_k(\mathscr{F}_n) < \infty$), it is shown in [9] that

$$\|Sf\|_1 \simeq \|f\|_{H_1},$$

where H_1 denotes the atomic Hardy space on $[0, 1]$, i.e. in this case, $H_{1,k}$ coincides with H_1.

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