Yang-Mills theory, lattice gauge theory and simulations

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Overview

Introduction and physical context

Classical Yang-Mills theory

Lattice gauge theory

Simulating the Glasma in 2+1D

Simulating the Glasma in 2+1D

Relativistic heavy-ion collisions

Relativistic heavy-ion collisions

Heavy-ion collision experiments as a means to study the properties of nuclear matter at extremely high energies

Examples:

- Au+Au at RHIC, BNL with $\sqrt{s_{\mathrm{NN}}}$ up tp $200\,\mathrm{GeV}$.
- Pb+Pb at LHC, CERN with $\sqrt{s_{\mathrm{NN}}}$ up tp $5\,\mathrm{TeV}$.

$\sqrt{s_{\mathrm{NN}}}$ is the collision energy per nucleon pair (protons, neutrons), mostly measured in electron volts eV.

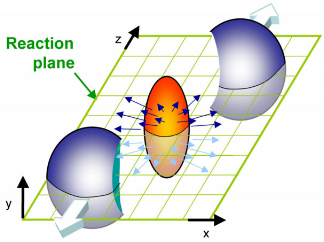
Each nucleon of a gold nucleus ($A = 197$) at RHIC carries $E = 100\,\mathrm{GeV}$ of energy (kinetic + rest mass).

Comparison: The energy $E_0 = m_0 c^2$ due to the rest mass m_0 of a proton is $1\,\mathrm{GeV}$.

$$E^2 = (m_0 c^2)^2 + (pc)^2 = E_0^2 + (pc)^2$$

Relativistic heavy-ion collisions: elliptic flow

Important experimental signature: elliptic flow



[arXiv:nucl-ex/0611012]

- Typically, collisions are off-central
- Directly after the collision: produced matter is "almond" shaped
- Geometric anisotropy of the initial shape of produced matter turns into a momentum anisotropy through collective effects (rescattering)

- Momentum anisotropy measured as the second Fourier coefficient v_2 of the number of particles $n(\vec{p})$ as a function of the azimuthal angle ϕ
- Experimental signature for a strongly-coupled quark gluon plasma (RHIC, 2005)

Relativistic heavy-ion collisions: elliptic flow

Important experimental signature: elliptic flow

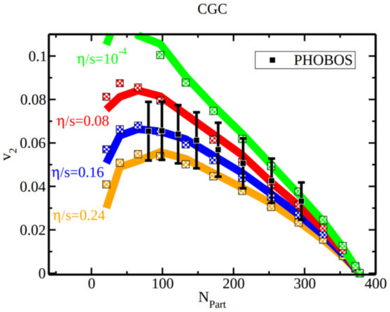


Figure from [arXiv:0804.4015]

N_{part} number of participants $\sim$ measure of how central a collision is

Extract information about system (viscosity η)

Relativistic heavy-ion collisions: theory overview

Theoretical model of heavy-ion collisions:
a chain of models and simulations ("stages")

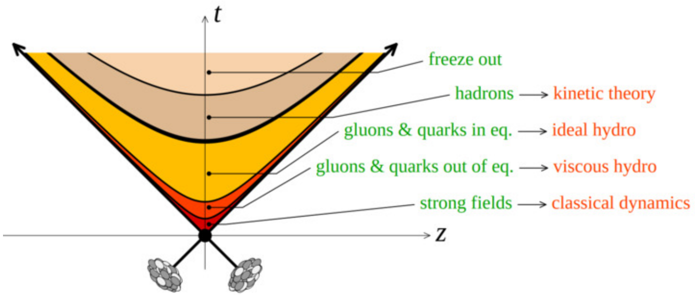
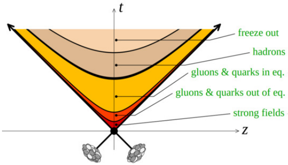


Figure from [arXiv:1110.1544]

Three/four main stages: a) classical Yang-Mills theory,
b) kinetic theory (Boltzmann eqs.) [arXiv:1805.01604]
c) relativistic hydrodynamics, d) kinetic theory

Relativistic heavy-ion collisions: theory overview



- Appropriate length scale: $1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$ (femtometer)
- Nuclear radius $R_{\mathrm{Au}} \approx 7\,\mathrm{fm}$
- Time scale $1\,\mathrm{fm}/c \approx 3 \cdot 10^{-24}\,\mathrm{s}$
- Proper time $\tau^2 = t^2 - z^2$

Rough timeline of a collision:

- Classical Yang-Mills: from $\tau = 0\,\mathrm{fm}/c$ to 0.1 – $1\,\mathrm{fm}/c$
- Hydrodynamics: up to $\tau = 10\,\mathrm{fm}/c$
- Kinetic theory: up to $\tau = 15\,\mathrm{fm}/c$

Afterwards, particles stream freely towards the detector

Each stage is based on theoretical calculations, but there is no full description in terms of quantum chromodynamics (QCD)

Nuclei at high energies

Before the collision: single nuclei

A gold nucleus at rest:

- Spherical, 14 fm diameter
- 197 nucleons (protons and neutrons) + quantum fluctuations
- Very complicated, quantum field theoretical object

A very fast nucleus (RHIC energies):

- "pancake shaped" due to special relativity
- 14 fm diameter in the transverse plane (orthogonal to velocity)
- 0.1 fm longitudinal width, along axis of velocity
- Theoretical description becomes much simpler

Nuclei at high energies

Before the collision: single nuclei

A very fast nucleus (RHIC energies):

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- $14\,\mathrm{fm}$ diameter in the transverse plane (orthogonal to velocity)
- $0.1\,\mathrm{fm}$ longitudinal width, along axis of velocity
- Theoretical description becomes much simpler

Occupation number of gluons becomes very large
$\Rightarrow$ gluons form a coherent state $\sim$ classical color field

Quarks:
carry most of the total momentum of the nucleus
interactions are “frozen” due to time dilation
$\Rightarrow$ effectively classical color charges

Color glass condensate (CGC)

Classical effective theory for high energy quantum chromodynamics

Nuclei are split into two types of degrees of freedom:

- quarks, high momentum gluons: classical color charges
- low momentum gluons: classical color fields

This split uses an arbitrary longitudinal momentum cutoff Λ_c

Requiring that physical observables do not depend on Λ_c yields a set of group renormalization equations called JIMWLK equations.

$\Rightarrow$ If we are only interested in observables evaluated near/at the cutoff Λ_c, an effectively classical treatment is possible

- F. Gelis, "Color Glass Condensate and Glasma", Int. J. Mod. Phys. A 28, 1330001 (2013) [arXiv:1211.3327]

Classical solutions for single nuclei

Yang-Mills with external sources

Yang-Mills eqs. can be extended with an external color current:

$$D_\mu F^{\mu\nu} = J^\nu, \qquad J^\mu : \mathbf{M} \to \mathfrak{su}(N_c),$$

which is gauge-covariantly conserved

$$D_\mu J^\mu = \partial_\mu J^\mu + ig\,[A_\mu, J^\mu] = 0.$$

Gauge transformation of the color current:

$$\begin{aligned} A'_\mu &= \Omega\left(A_\mu + \frac{1}{ig}\partial_\mu\right)\Omega^\dagger \\ F'_{\mu\nu} &= \Omega F_{\mu\nu}\Omega^\dagger \\ \Rightarrow J'^\mu &= \Omega J^\mu \Omega^\dagger \end{aligned}$$

J^0 color charge density, J^i (spatial) color current density

Yang-Mills with external sources

Yang-Mills action can be extended with an external color current:

$$S[A_\mu, J_\mu] = \int d^4x \left(-\frac{1}{2} \operatorname{tr}\left[F_{\mu\nu} F^{\mu\nu}\right] + 2 \operatorname{tr}\left[A_\mu J^\mu\right] \right)$$

The coupling term "$J_\mu A^\mu$" breaks gauge symmetry in the case of a non-Abelian gauge group, but the extrema of S are still gauge invariant.

Note: this problem only arises if J_μ is considered an external source. In the standard model (QCD, electroweak force), the current J_μ is generated by fermionic fields (quarks). The action of QCD is gauge invariant.

Single nucleus solution

Boost invariant (shockwave) approximation:

Speed of nuclei at particle accelerators

- Nuclei at RHIC: $v/c \approx 0.99995$
- Nuclei at LHC: $v/c \approx 0.99999992$

Assume nuclei move at the speed of light.

Longitudinal length contraction:

- Nuclei at RHIC: contracted by $\gamma = 100$
- Nuclei at LHC: contracted by $\gamma = 2500$

Assume nuclei are infinitesimally thin.

Time dilation: interactions in nucleus are frozen

Single nucleus solution

Boost invariant (shockwave) approximation: nuclei move at the speed of light, infinitesimally thin

More appropriate coordinates: light cone coordinates

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}, \qquad x^- = \frac{x^0 - x^3}{\sqrt{2}},$$

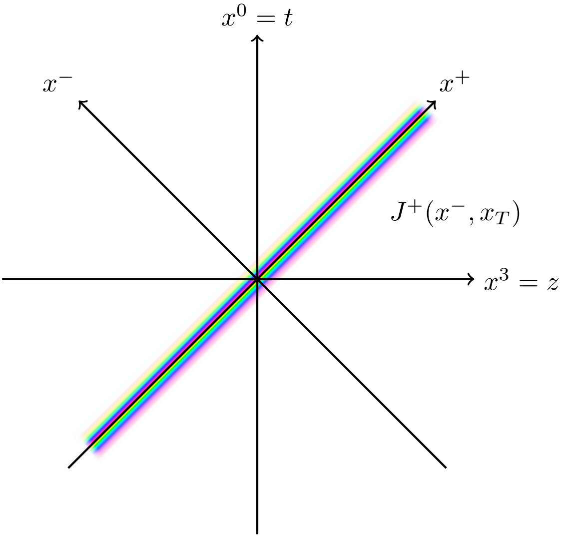
corresponding to 45° axes in the Minkowski diagram.

Assume nucleus is moving along positive x^3 axis.
Color current has only one non-vanishing component:

$$J^+(x^-, x_T) = \delta(x^-)\rho(x_T).$$

where $\rho(x_T)$ describes the distribution of color charges in the transverse plane ($x_T = (x, y)^T$ transverse coordinates)

Single nucleus solution



Single nucleus solution

Color current given by

$$J^+(x^-, x_T) = \delta(x^-)\rho(x_T).$$

Solve Yang-Mills eqs.

$$D_\mu F^{\mu\nu} = J^\nu$$

in Lorenz gauge

$$\partial_\mu A^\mu = 0.$$

One finds:

$$A^+(x^-, x_T) = -\delta(x^-)\Delta_T^{-1}\rho(x_T),$$

where Δ_T is the 2D Laplace operator in the transverse plane and Δ_T^{-1} is the Greens function.

All other components of the gauge field vanish.

Single nucleus solution

Analogous solution for nucleus moving in opposite direction.

Color current

$$J^-(x^+, x_T) = \delta(x^+)\rho(x_T),$$

Gauge field

$$A^-(x^+, x_T) = -\delta(x^+)\Delta_T^{-1}\rho(x_T),$$

Remarkably, the solution $A^\pm$ of the Yang-Mills eqs. only depends linearly on ρ because $J^\pm$ only depends on $x^\mp$ and x_T.

$\Rightarrow$ Interactions stop due to time dilation.

Single nucleus solution

Single nucleus solutions in Lorenz gauge $\partial_\mu A^\mu$:

$$J^\pm(x^\mp, x_T) = \delta(x^\mp)\rho(x_T), \quad A^\pm(x^\mp, x_T) = -\delta(x^\mp)\Delta_T^{-1}\rho(x_T)$$

Solution in light cone gauge ($A^\pm = 0$ for nucleus moving along $x^\mp$): similar to temporal gauge, but along lightlike axes

$$A^i(x^\mp, x_T) = \frac{1}{ig} V(x^\mp, x_T)\partial^i V^\dagger(x^\mp, x_T),$$

with the light like Wilson line V given by

$$V^\dagger(x^\mp, x_T) = \mathcal{P}\exp\left(-ig\int_{-\infty}^{x^\mp} dx'^\mp A^\pm(x'^\mp, x_T)\right)$$

Single nucleus solution

Light cone gauge solution:

The lightlike Wilson line is given by

$$V^\dagger(x^\mp, x_T) = \begin{cases} V^\dagger(x_T) & x^\mp > 0 \\ \mathbf{1} & x^\mp < 0 \end{cases}$$

with $V^\dagger(x_T) = \exp(-ig\Delta_T^{-1}\rho(x_T))$. The transverse gauge field $A^i(x^\mp, x_T)$ has the form of a step function:

$$A^i(x^\mp, x_T) = \frac{1}{ig}\theta(x^\mp)V(x_T)\partial^i V^\dagger(x_T),$$

where θ is the Heaviside step "function"

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Nucleus models

Color current of a nucleus:

$$J^{+}(x^{-}, x_T) = \delta(x^{-})\rho(x_T).$$

How to choose the charge density $\rho(x_T)$?

There is no experimental control over how exactly quarks are distributed in a nucleus when the two nuclei collide.

In the color glass condensate framework, $\rho(x_T)$ is considered a random variable. The distribution of $\rho(x_T)$ is determined by a probability functional $W[\rho]$.

Expectation values of observables are computed using functional integrals. If $A_\mu[\rho]$ is the solution of the Yang-Mills eqs. and $O[A_\mu]$ is a gauge-invariant observable, then the expectation value $\langle O \rangle$ is given by

$$\langle O \rangle = \int \mathcal{D}\rho O[A_\mu[\rho]] W[\rho].$$

Nucleus models

The color glass condensate framework does not predict $W[\rho]$.

The CGC provides a calculation framework in terms of classical Yang-Mills theory and group renormalization eqs. to describe how $W[\rho]$ changes as a function of the cutoff Λ_c (JIMWLK), but no prediction for the form of $W[\rho]$.

$\Rightarrow$ We need models for $W[\rho]$

Earliest, most simple one: McLerran-Venugopalan model (1994)

- L. McLerran, R. Venugopalan, "Computing Quark and Gluon Distribution Functions for Very Large Nuclei", PRD 49 (1994), $\sim$ 1800 citations, [arXiv:hep-ph/9309289]

Nucleus models

Earliest, most simple one: McLerran-Venugopalan model (1994)

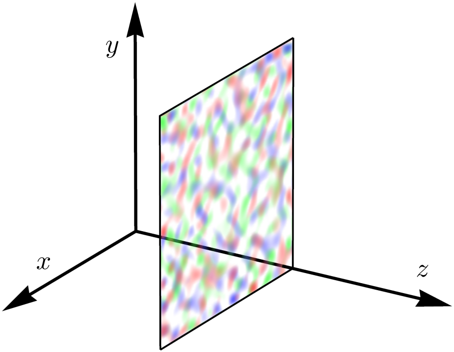
- Assume $W[\rho]$ is Gaussian.
- $W[\rho]$ is determined by specifying mean and covariance:

$$\langle \rho^a(x_T) \rangle = 0$$
$$\left\langle \rho^a(x_T)\rho^b(y_T) \right\rangle = g^2\mu^2\delta^{ab}\delta^{(2)}(x_T - y_T)$$

- Only one model parameter: μ usually given in GeV
  Example: for gold/lead nuclei $\mu \approx 0.5\,\mathrm{GeV}$
- Nuclei assumed to be infinitely large in the transverse plane.
- No finite radius, no inhomogeneous structure, because μ is a constant.

Nucleus models

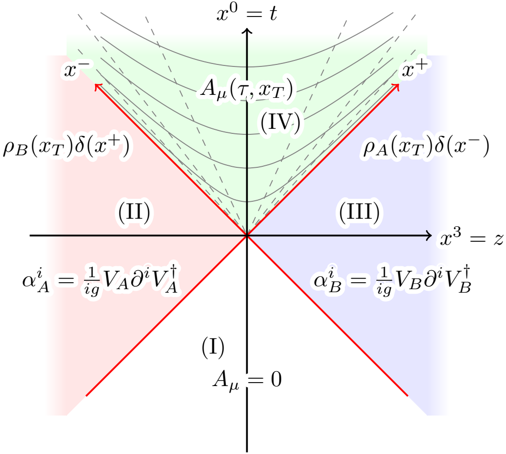
Earliest, most simple one: McLerran-Venugopalan model (1994)



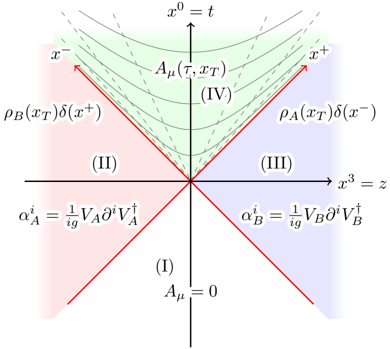
Collisions in the CGC framework

Collisions in the CGC framework

Idea: superimpose the solutions of two single nuclei at some initial time t_0 before the collision. Solve the classical Yang-Mills eqs. up until some later time $t > t_0$ to model a collision.

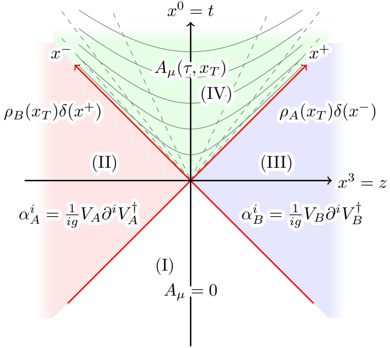


Collisions in the CGC framework



- Nuclei "A" and "B" specified by charge density ρ_A and ρ_B
- Analytic solutions in regions (I), (II) and (III)
- Generally no analytical solutions for arbitrary ρ in region (IV) (forward light cone)
- The field in region (IV) is the "Glasma"
- Analytic solution can be found for the boundary of region (IV) using a matching ansatz (Glasma initial conditions)

Collisions in the CGC framework



- Use appropriate coordinates in region (IV): proper time τ, rapidity η

$$\tau = \sqrt{t^2 - z^2} = \sqrt{2x^+x^-}$$

$$\eta = \frac{1}{2} \ln \left(\frac{x^+}{x^-} \right)$$

- $\tau \to 0^+$ defines the boundary of region (IV)
- Due to the boost-invariant approximation, the solution in (IV) does not depend on η
  $\Rightarrow$ Glasma is effectively 2+1D

Glasma initial conditions

Matching ansatz for all regions:

$$\begin{aligned} A^i(x) &= \theta(x^+)\theta(x^-)\alpha^i(\tau, x_T) \\ &\quad + \theta(-x^+)\theta(x^-)\alpha^i_A(x_T) + \theta(x^+)\theta(-x^-)\alpha^i_B(x_T), \\ A^\eta(x) &= \theta(x^+)\theta(x^-)\alpha^\eta(\tau, x_T), \end{aligned}$$

with $\alpha^i_{A/B} = \frac{1}{ig} V_{A/B} \partial^i V^\dagger_{A/B}$. We use light cone gauge in (II) and (III), Fock-Schwinger gauge in (IV):

$$x^+ A^- + x^- A^+ = 0,$$

which is equivalent to a temporal gauge along proper time τ.

Plug into Yang-Mills equations. Require that coefficients in front of problematic terms ($\delta(x)\delta(x)$, etc.) vanish.

This yields a set of matching conditions at $\tau \to 0^+$ known as the Glasma initial conditions.

Glasma initial conditions

The matching conditions are given by

$$\alpha^i(\tau \to 0^+, x_T) = \alpha^i_A(x_T) + \alpha^i_B(x_T),$$
$$\alpha^\eta(\tau \to 0^+, x_T) = \frac{ig}{2}\left[\alpha^i_A(x_T), \alpha^i_B(x_T)\right],$$

and

$$\partial_\tau \alpha^i(\tau \to 0^+, x_T) = 0,$$
$$\partial_\tau \alpha^\eta(\tau \to 0^+, x_T) = 0.$$

With the gauge fixing condition $A^\tau = 0$ in the forward light cone (IV), we have a fully specified initial value problem.

Glasma initial conditions

Field strengths of nuclei:
purely transverse chromo-electric and -magnetic fields

Field strengths in the Glasma:
(initially) purely longitudinal chromo-electric and -magnetic fields

Equal magnetic and electric contributions to energy (on average)

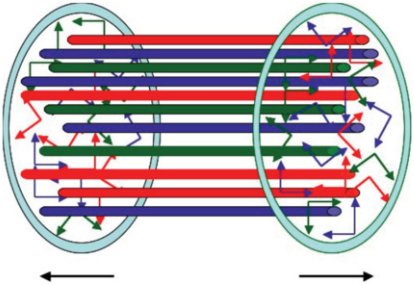


Fig. from [arXiv:1011.3204]

Boost invariant Yang-Mills theory

Next step: formulate numerical scheme for Yang-Mills eqs. in forward light cone in terms of τ and η coordinates.

Boost invariance: fields in forward light cone do not depend on rapidity η
$\Rightarrow$ Drop all terms like $\partial_\eta A_i$ etc.

Boost invariant action

$$S = \int d\tau d^2 x_T d\eta \, \mathrm{tr} \left[\tau F_{\tau i} F_{\tau i} + \frac{1}{\tau} F_{\tau\eta}^2 - \frac{\tau}{2} F_{ij} F_{ij} - \frac{1}{\tau} F_{\eta i} F_{\eta i} \right]$$

Notes:

- Explicit dependence on τ due to use of curvilinear coordinates
- No dependence on η: effectively 2+1D description

Boost invariant Yang-Mills on the lattice

Boost invariant action

$$S = \int d\tau d^2 x_T d\eta \, \mathrm{tr} \left[\tau F_{\tau i} F_{\tau i} + \frac{1}{\tau} F_{\tau\eta}^2 - \frac{\tau}{2} F_{ij} F_{ij} - \frac{1}{\tau} F_{\eta i} F_{\eta i} \right]$$

We use the same procedure as in the 3+1D case with Cartesian coordinates:

Perform discretization of the action:

- Replace integral with sum over lattice sites
- Replace $\mathrm{tr}\left[F_{ij}^2\right]$ terms with corresponding plaquette terms

$\Rightarrow$ Variation yields discrete equations of motion and constraint

Also necessary: discretized Glasma initial conditions [arXiv:hep-ph/9809433]

Boost invariant Yang-Mills on the lattice

Main observable of interest: energy momentum tensor $T_{\mu\nu}$

$$T^{\mu\nu} = F^{a,\mu\rho} F^{a,\nu}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{a,\rho\sigma} F^{a}_{\rho\sigma}$$

Need to discretize $T_{\mu\nu}$ in terms of gauge links and plaquettes

- $T^{\tau\tau}$: energy density
- $T^{i\tau}$: energy flux along transverse axes
- $T^{\eta\tau}$: energy flux along longitudinal axis
- T^{ii} (no sum): transverse pressure densities
- $T^{\eta\eta}$: longitudinal pressure density
- T^{ij} for $i \neq j$, $T^{\eta i}$: shear stress

Boost invariant Yang-Mills on the lattice

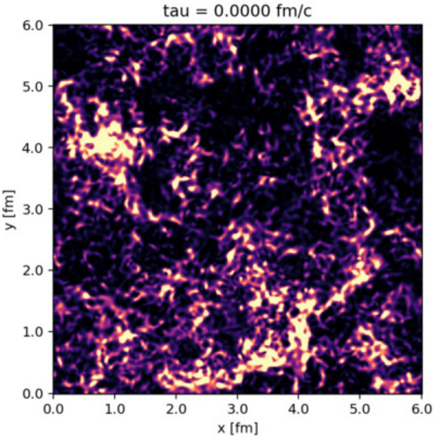
Summary of a typical Glasma simulation:

- Generate initial conditions
  - Pick random samples for charge densities ρ_A and ρ_B for both nuclei using their respective probability functionals $W_A[\rho]$ and $W_B[\rho]$
  - Compute Glasma initial conditions on the lattice
- Solve discretized equations of motion on the lattice starting at $\tau = 0$ up to some final time $\tau_f = 0.1 - 1.0\,\mathrm{fm}/c$
- Compute $T_{\mu\nu}$ as a function of τ and x_T

Perform multiple simulations using random initial condition to approximate the expectation value $\langle T_{\mu\nu} \rangle$ (Monte Carlo sampling)

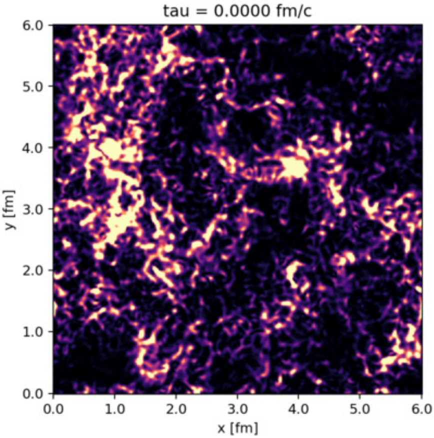
Simulating the Glasma in 2+1D

Random collision event 1: energy density $\varepsilon(\tau, x_T) = T_{\tau\tau}(\tau, x_T)$



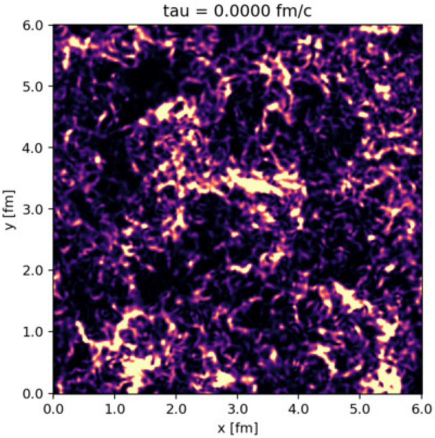
Simulating the Glasma in 2+1D

Random collision event 2: energy density $\varepsilon(\tau, x_T) = T_{\tau\tau}(\tau, x_T)$



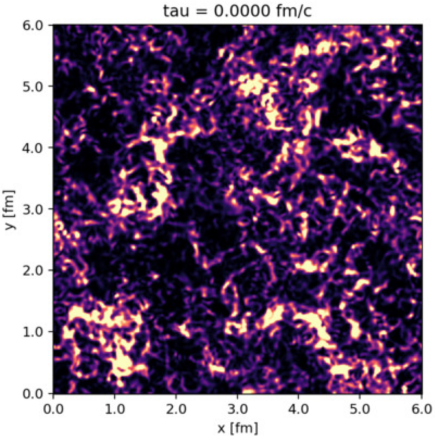
Simulating the Glasma in 2+1D

Random collision event 3: energy density $\varepsilon(\tau, x_T) = T_{\tau\tau}(\tau, x_T)$



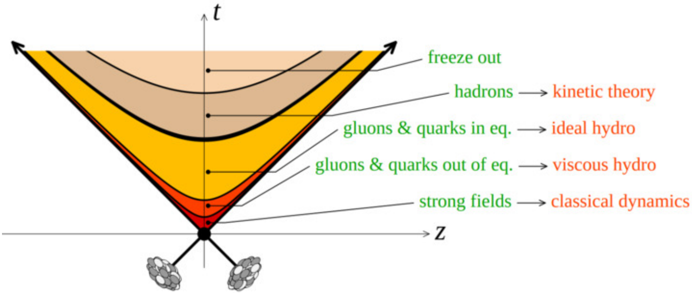
Simulating the Glasma in 2+1D

Random collision event 4: energy density $\varepsilon(\tau, x_T) = T_{\tau\tau}(\tau, x_T)$



Simulating the Glasma in 2+1D

Computing $\langle T_{\mu\nu} \rangle$ at some final time $\tau_f = 0.1 - 1.0\,\mathrm{fm}/c$ provides initial conditions for the next link in the chain of simulations (e.g. hydrodynamical or kinetic theory simulations).



Simulating the Glasma in 2+1D

When does the classical Yang-Mills description become invalid?

As the Glasma expands, the gluon occupation number decreases rapidly. If too low, the coherent state (“effectively classical”) approximation stops being applicable.

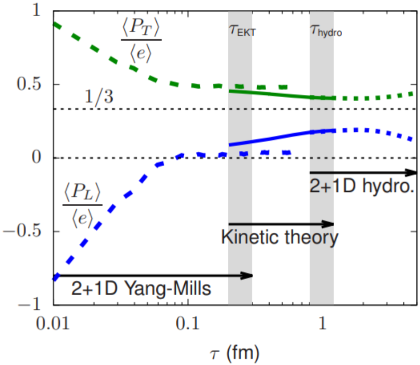


Figure from [arXiv:1805.01604]

Improved nucleus models

Improved nucleus models: transverse details

McLerran-Venugopalan model is too simple:

- No finite radius $\Rightarrow$ cannot model off central collisions
- Variance of random charge densities ρ is the same everywhere $\Rightarrow$ No nucleonic or sub-nucleonic structure

Simple generalization:

$$\left\langle \rho^a(x_T)\rho^b(y_T) \right\rangle = g^2\mu^2(x_T)\delta^{ab}\delta^{(2)}(x_T - y_T),$$

where $\mu^2(x_T)$ is now a function of x_T.

- Let $\mu^2(x_T) \to 0$ outside the nucleus
- Add local variation inside the nucleus (protons, neutrons)

Improved nucleus models: transverse details

Simple generalization:

$$\left\langle \rho^a(x_T)\rho^b(y_T)\right\rangle = g^2\mu^2(x_T)\delta^{ab}\delta^{(2)}(x_T - y_T).$$

Current state-of-the-art: IP-Glasma model

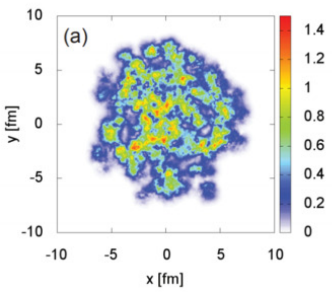


Fig. from [arXiv:1605.07158]

- Sample nucleon positions x_T from a probability density function
- Each nucleon adds an individual contribution to $\mu^2(x_T)$
- Exact form of each contribution is extracted from experimentally measured cross sections of deep inelastic scattering experiments (e.g. proton probed by an electron)

Improved nucleus models: transverse details

Current state of the art: IP-Glasma model

IP-Glasma initial conditions not only describe v_2 (elliptic flow), but also higher coefficients v_n

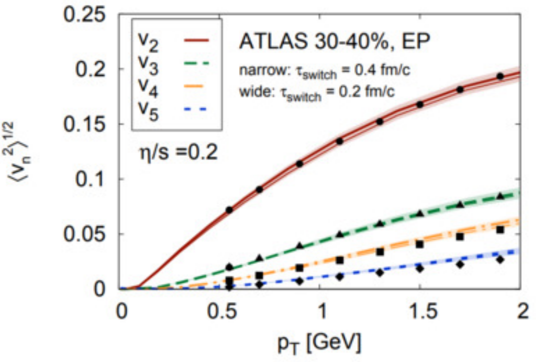
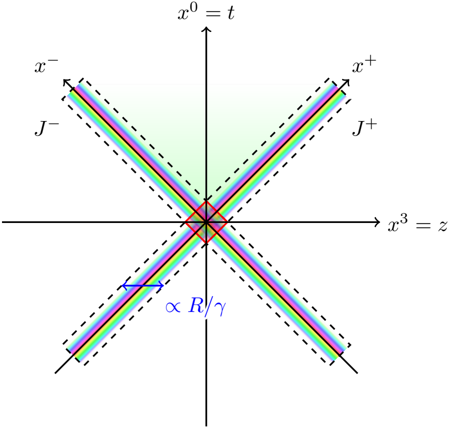


Fig. from [arXiv:1209.6330]

Improved nucleus models: finite width

McLerran-Venugopalan is too simple:

- Width is not actually infinitesimal (only finite collision energy)
- Complicated structure also along longitudinal coordinate z
- Boost invariance is only a rough approximation



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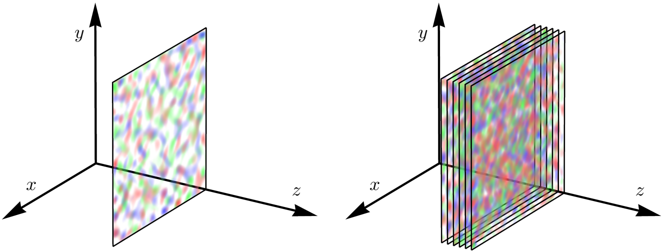

Improved nucleus models: finite width

Finite width along z: breaks boost invariance

$\Rightarrow$ Fields in forward light cone depend on rapidity η

Reduction from 3+1D system to 2+1D does not work anymore.

3+1D simulations required:

- Explicitly include and simulate color currents J^μ
- Have to simulate whole collision, not just forward light cone
- Simulate in laboratory frame (ordinary Cartesian coordinates), instead of τ and η
- Much more computationally demanding:
  - 2+1D simulations: a few minutes per initial condition
  - 3+1D simulations: 1-2 days per initial condition

Improved nucleus models: finite width

3D density plot of energy density $\varepsilon(x)$

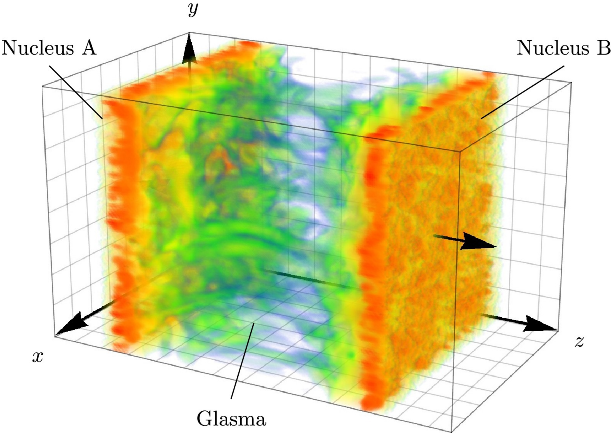

Fig. from [arXiv:1703.00017]

Improved nucleus models: finite width

Comparison of rapidity dependence of $\varepsilon(\tau, x_T, \eta)$ to experimental data from BRAHMS experiment at RHIC using only a very simple modification of the MV model

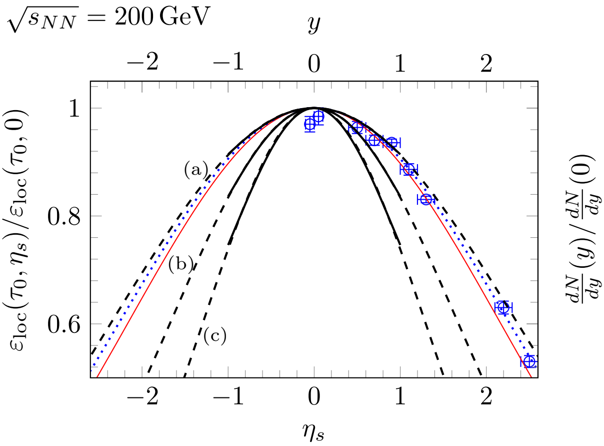

Fig. from [arXiv:1703.00017]

Summary

Simulating the Glasma in 2+1D

- Theoretical description of heavy ion collisions is divided into different stages
- Earliest stages are described by the color glass condensate, which allows for an effective description of the system using classical Yang-Mills theory
- Assuming nuclei to be infinitesimally thin, the collision can be described in 2+1D (boost invariance)
- Yang-Mills eqs. for the Glasma are solved using methods from lattice gauge theory
- Experimental data can only be correctly described using accurate models of nuclei