Lecture notes

Sobolev spaces

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Contents

CHAPTER 1

Introduction

The theory of Sobolev spaces give the basis for studying the existence of solutions (in the weak sense) of partial differential equations (PDEs). As motivation for this theory we give a short introduction on second order elliptic partial differential equations, but without going deeper into the PDE-theory. For more information about the analytic and numerical theory see [**1**], [**4**], [**6**] and [**9**].

Boundary value problems for second-order ordinary differential equations. Classical formulation: Find a function $u : [0, 1] \to \mathbb{R}$ such that

$$\begin{aligned} -u''(x) + b(x)u'(x) + c(x)u(x) &= f(x), \qquad x \in \Omega := (0, 1), \\ u(0) = u(1) &= 0, \end{aligned} \tag{1.1}$$

with given continuous coefficient functions b, c and given continuous right-hand side f. A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies (1.1) is called *classical solution.*

Variational formulation Let $v : [0, 1] \to \mathbb{R}$ a so-called test function. We can multiply (1.1) with a test function v and integrate over the interval

$$\int_\Omega \big(- u''(x) + b(x)u'(x) + c(x)u(x) \big)\, v(x)dx = \int_\Omega f(x)v(x)dx. \tag{1.2}$$

Every solution of (1.1) is a solution of (1.2) (for every test function v). On the other hand if a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies equation (1.2) for every test function v then u satisfies the differential equation (1.1).

With integration by parts we can rewrite (1.2) as

$$-u'(x)v(x)\Big|_0^1 + \int_\Omega u'(x)v'(x)dx + \int_\Omega \big(b(x)u'(x) + c(x)u(x) \big)\, v(x)dx = \int_\Omega f(x)v(x)dx\,. \tag{1.3}$$

Using the boundary conditions $v(0) = v(1) = 0$, equation (1.3) yields

$$\int_\Omega u'(x)v'(x)dx + \int_\Omega \big(b(x)u'(x) + c(x)u(x) \big)\, v(x)dx = \int_\Omega f(x)v(x)dx\ . \tag{1.4}$$

Derivatives occur in equation (1.4) only in terms of the form

$$\int_0^1 w'(x)\varphi(x)dx, \tag{1.5}$$

where $\varphi : \overline{\Omega} \to \mathbb{R}$ is sufficiently smooth and $\varphi|_{\partial\Omega} = 0$. Equation (1.5) is for $w \in C^1$ equal to

$$-\int_0^1 w(x)\varphi'(x)dx. \tag{1.6}$$

The existence of the integral in (1.6) is given for $w \in L^1_{\text{loc}}(\Omega)$. The problem is, that if $w \notin C^1$, then w' in equation (1.5) has no meaning. This leads us to the definition of the weak derivative of w.

DEFINITION 1.1. Suppose $w, \tilde{w} \in L^1_{\text{loc}}(\Omega)$. We say $\tilde{w}$ is the weak derivative of w, written $w' = \tilde{w}$ provided

$$\int_\Omega w(x)\varphi'(x)\,dx = -\int_\Omega \tilde{w}(x)\varphi(x)\,dx \quad \text{for all test functions } \varphi \in C_c^\infty(\Omega). \tag{1.7}$$

Therefore, the existence of the integral expressions in equation (1.4) (for $b, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$) is guaranteed for $u, u', v, v' \in L^2(\Omega)$. This suggests the Sobolev space

$$H^1(\Omega) = \{w \in L^1_{\text{loc}}(\Omega) : w, w' \in L^2(\Omega)\}.$$

To incorporate the boundary values of $u, v \in H^1$ we need the Sobolev space H^1_0. Note that as in L^2 pointwise evaluation in H^1 does not make sense. Hence, we need the trace theorem (Theorem 5.1) in order to be able to assign "boundary values" along $\partial\Omega$ to a function in the Sobolev space.

DEFINITION 1.2. We say $u \in H^1_0(\Omega)$ is a *weak solution* of (1.2) if

$$B(u, v) = \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega), \tag{1.8}$$

where $\langle \cdot, \cdot\rangle$ denotes the inner product in $L^2(\Omega)$ and

$$B(u, v) := \int_\Omega u'(x)v'(x)dx + \int_\Omega \big(b(x)u'(x) + c(x)u(x)\big)\,v(x)dx, \quad u, v \in H^1_0(U).$$

The identity (1.8) is called variational formulation of (1.2).

Second order elliptic partial differential equations. The 1-dimensional example brings us to the theory of weak solutions for a greater class of differential equations - second order elliptic partial differential equations. Let $U \subseteq \mathbb{R}^n$ be open and bounded. We consider the following boundary value problem on U.

$$\left.\begin{aligned} Lu &= f \qquad \text{in } U,\\ u &= 0 \qquad \text{on } \partial U, \end{aligned}\right\} \tag{1.9}$$

where $f : U \to \mathbb{R}$ is given and $u : \overline{U} \to \mathbb{R}$ is the unknown function. L denotes a second-order elliptic partial differential operator having either the form

$$Lu = -\sum_{i,j=1}^n a^{i,j}(x)u_{x_i x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u \tag{1.10}$$

or else

$$Lu = -\sum_{i,j=1}^n (a^{i,j}(x)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u, \tag{1.11}$$

for given coefficient functions $(a^{i,j})_{i,j=1}^n, (b^i)_{i=1}^n, c$. We assume the symmetry condition

$$a^{i,j} = a^{j,i} \qquad \text{for all } i, j = 1, \ldots, n. \tag{1.12}$$

The differential operator L is in *divergence form* if it is given by equation (1.11).

EXAMPLE 1.3. Let $a^{i,j} = \delta_{i,j}$, $b^i = 0$ and $c = 0$. Then $Lu = -\Delta u$.

We consider the problem (1.9) for L given by

$$Lu = -\operatorname{div}(a(x)\nabla u(x)), \tag{1.13}$$

where $a \in L^\infty(U)$ and $a > 0$. Note that if $a \in C^1(U)$, then $Lu = -a\Delta u - \nabla u \nabla a$. Since a is positive, we have that L is elliptic partial differential operator.

We assume for the moment that u is a classical solution of (1.9). We multiply (1.9) by a smooth test function $v \in C_c^\infty(U)$, integrate over U and apply integration by parts to the left-hand side to get

$$\int_U a(x)\nabla u(x)\nabla v(x)\,dx = \int_U f(x)v(x)\,dx. \tag{1.14}$$

Note that there are no boundary terms since $v = 0$ on ∂U. By approximation we obtain that the same identity holds when we replace $v \in C_c^\infty(U)$ by any $v \in H_0^1(U)$. The left hand-side of (1.14) makes sense if $u \in H^1(U)$. We choose the Sobolev space to incorporate the boundary condition from (1.9), hence we consider $u \in H_0^1(U)$. This leads to the definition of a weak solution u of (1.9).

DEFINITION 1.4. We say $u \in H_0^1(U)$ is a *weak solution* of problem (1.9) with L given by (1.13) if

$$B(u,v) = \langle f, v\rangle \qquad \forall v \in H_0^1(U), \tag{1.15}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(U)$ and

$$B(u,v) := \int_U a(x)\nabla u(x)\nabla v(x)\,dx, \quad u, v \in H_0^1(U).$$

More generally,

DEFINITION 1.5. We say $u \in H_0^1(U)$ is a *weak solution* of problem (1.9) with L given by (1.11) if

$$B(u,v) = \langle f, v\rangle \qquad \forall v \in H_0^1(U), \tag{1.16}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(U)$ and

$$B(u,v) := \int_U \sum_{i,j=1}^n a^{i,j} u_{x_i} v_{x_i} + \sum_{i=1}^n b^i u_{x_i} v + cuv\,dx, \quad u, v \in H_0^1(U).$$

REMARK 1.6. The identity (1.16) is called *variational formulation* of (1.9).

THEOREM 1.7.
(1) *Let u be a classical solution of (1.9). Let B and $\langle f,\cdot\rangle$ in Definition 1.5 be bounded on $H_0^1(U)$. Then u is a weak solution of (1.9).*
(2) *Let f be continuous, u a weak solution of (1.9) and $u \in C^2(U) \cap C(\overline{U})$. Then u is a classical solution of (1.9).*

Existence of weak solutions The central theorem in the theory of existence and uniqueness of weak solutions is the following.

THEOREM 1.8 (Lax-Milgram). *Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ a bilinear form satisfying the conditions*

(1) *there exists a constant $c_1 > 0$ such that for all $u, v \in H$*

$$|B(u,v)| \le c_1 \|u\|_H \|v\|_H$$

(2) *there exists a constant $c_2 > 0$ such that for all $u \in H$*

$$\|u\|_H^2 \le c_2 B(u,u).$$

Let $f : H \to \mathbb{R}$ be linear and bounded. Then there exists a unique $u \in H$ such that

$$B(u,v) = f(v), \qquad \text{for all } v \in H. \tag{1.17}$$

The following existence theorem for weak solutions of our boundary value problem is based on the *Lax-Milgram Theorem* applied to the Hilbert space $H_0^1(U)$.

THEOREM 1.9 (Existence Theorem for a weak solution). *There is a number $\gamma \ge 0$ such that for each $\mu \ge \gamma$ and each function $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of the boundary value problem*

$$\begin{aligned} Lu + \mu u &= f \qquad && \text{in } U \\ u &= 0 && \text{on } \partial U. \end{aligned}$$

EXAMPLE 1.10. In the case $Lu = -\Delta u$ we have $B(u,v) = \int_U \nabla u(x) \nabla v(x)\, dx$. One can check that $\gamma = 0$ is possible. In the case $Lu = -\operatorname{div}(a(x)\nabla u(x))$, with $a \in L^\infty(U)$, $a \ge c > 0$ the same holds true.

CHAPTER 2

Weak derivatives and Sobolev spaces

2.1. Preliminaries

Notation for derivatives. Let $U \subseteq \mathbb{R}^n$ be open. Assume $u\colon U \to \mathbb{R}$, $x \in U$. Let

$$\frac{\partial u}{\partial x_i}(x) = \lim_{h\to 0} \frac{u(x+he_i)-u(x)}{h},$$

provided the limit exists. We write

$$u_{x_i} = \frac{\partial u}{\partial x_i},\ \frac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j},\ \text{ etc.}$$

A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is called a *multiindex* of order

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Each multiindex α defines a partial differential operator of order $|\alpha|$, given by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u.$$

If $k \in \mathbb{N}_0$, then

$$D^k u = \{D^\alpha u : |\alpha| = k\}$$

is the set of all partial derivatives of order k. If $k = 1$, then

$$Du = \nabla u = (u_{x_1}, \dots, u_{x_n}).$$

Locally integrable functions. The space of *locally integrable functions on* U, denoted by $L^1_{\mathrm{loc}}(U)$, is defined as the set of measurable functions $f : U \to \mathbb{R}$ such that for all compact subsets $K \subseteq U$ the following holds

$$\int_K |f(x)| dx < \infty. \tag{2.1}$$

Remark 2.1. Constant functions, piecewise continuous functions and continuous functions are locally integrable. Every function $f \in L^p(U)$, $1 \le p \le \infty$ is locally integrable.

Test functions. The *space of test functions*, denoted by $C_c^\infty(U)$, is the space of infinitely differentiable functions $\phi\colon U \to \mathbb{R}$ with compact support. Note that the support of ϕ is defined by $\operatorname{supp}\phi = \overline{\{x \in U : \phi(x) \neq 0\}}$.

Example 2.2. The following functions are elements in $C_c^\infty(\mathbb{R})$.

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-|x|}}, & \text{falls } |x| < 1 \\ 0, & \text{falls } |x| \geq 1. \end{cases}$$

$$h(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & \text{falls } |x| < 1 \\ 0, & \text{falls } |x| \geq 1. \end{cases}$$

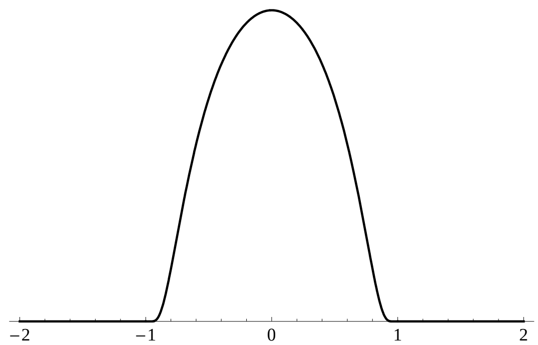


Figure 2.1. $h(x)$

2.2. Weak derivative

Every locally integrable function $u \in L^1_{\mathrm{loc}}(U)$ determines a regular distribution, i.e. a linear and continuous function

$$T_u : C_c^\infty(U) \to \mathbb{R},\ T_u(\phi) = \int_U u(x)\phi(x)dx. \tag{2.2}$$

We also use the notation $\langle T_u, \phi\rangle$ resp. $\langle u, \phi\rangle$ instead of $T_u(\phi)$.

If we assume that $u \in C^1(U)$ then its partial derivatives u_{x_i}, $1 \leq i \leq n$, are continuous and hence, $u_{x_i} \in L^1_{\mathrm{loc}}(U)$. Therefore, u_{x_i} determines a regular distribution given by

$$T_{u_{x_i}} : C_c^\infty(U) \to \mathbb{R},\ T_{u_{x_i}}(\phi) = \int_U u_{x_i}(x)\phi(x)dx. \tag{2.3}$$

Obviously, we have by integration by parts (cf. Theorem 11.15)

$$T_{u_{x_i}}(\phi) = \int_U u_{x_i}(x)\phi(x)dx = -\int_U u(x)\phi_{x_i}(x)dx, \quad \phi \in C_c^\infty(U). \tag{2.4}$$

The left-hand side integral in (2.4) is defined only when u_{x_i} exists a.e. and is in $L^1_{\mathrm{loc}}(U)$, whereas the right-hand side integral is well defined for every $u \in L^1_{\mathrm{loc}}(U)$.

Definition 2.3 (Distributional derivative).

(1) The distributional derivative w.r.t. the i^{th} variable, $1 \leq i \leq n$, of T_u is the distribution (linear and continuous functional on $C_c^\infty(U)$) $\partial_{x_i} T_u$ given by

$$\langle \partial_{x_i} T_u, \phi \rangle = -\langle u, \partial_{x_i} \phi \rangle = -\int_U u(x) \partial_{x_i} \phi(x) dx, \quad \phi \in C_c^\infty(U). \tag{2.5}$$

(2) Let α be a multiindex of order $|\alpha|$. Then the α^{th} distributional derivative is the distribution $D^\alpha T_u$ given by

$$\langle D^\alpha T_u, \phi \rangle = (-1)^{|\alpha|} \int_U u(x) D^\alpha \phi(x) dx, \quad \phi \in C_c^\infty(U). \tag{2.6}$$

If $u \in C^k(U)$ and α a multiindex of order k, then $D^\alpha u$ exists in the classical sense and we have by the integration by parts formula

$$\int_U D^\alpha u(x) \phi(x) dx = (-1)^{|\alpha|} \int_U u(x) D^\alpha \phi(x) dx.$$

$D^\alpha u \in C(U) \subseteq L^1_{\text{loc}}(U)$ defines again a regular distribution given by

$$T_{D^\alpha u}(\phi) = \int_U D^\alpha u(x) \phi(x) dx$$

and obviously,

$$\langle D^\alpha T u, \phi \rangle = T_{D^\alpha u}(\phi), \qquad \text{for all } \phi \in C_c^\infty(U).$$

We say that the classical and the distributional derivative of $u \in L^1_{\text{loc}}(U)$ coincide. What happens if $u \in L^1_{\text{loc}}(U)$, but $u \notin C^k(U)$? This leads us to the definition of the *weak derivative* of u.

Definition 2.4 (Weak derivative). Let $u \in L^1_{\text{loc}}(U)$ and $\alpha \in \mathbb{N}_0^n$ a multiindex. If there exists a $v \in L^1_{\text{loc}}(U)$ such that

$$\int_U v(x) \phi(x) dx = (-1)^{|\alpha|} \int_U u(x) D^\alpha \phi(x) dx \quad \forall \phi \in C_c^\infty(\mathbb{R}^n) \tag{2.7}$$

then v is called the **weak α^{th}- partial derivative** of u, denoted by

$$D^\alpha u = v.$$

In other words, if we are given $u \in L^1_{\text{loc}}(U)$ and if there happens to exist a function $v \in L^1_{\text{loc}}(U)$ satisfying (2.7) for all $\phi \in C_c^\infty(U)$ we say that $D^\alpha u = v$ in the weak sense. If there does not exist such a function v, then u does not possess a weak α^{th}- partial derivative.

Remark 2.5. Classical derivatives are defined pointwise as limit of difference quotients. Weak derivatives, on the other hand, are defined in an integral sense. By changing a function on a set of measure zero we do not affect its weak derivatives.

Lemma 2.6 (Uniqueness). *A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

PROOF. Assume that $v, \tilde{v} \in L^1_{\text{loc}}(U)$ satisfy

$$\int_U u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_U v(x)\phi(x) dx = (-1)^{|\alpha|} \int_U \tilde{v}(x)\phi(x) dx$$

for all $\phi \in C_c^\infty(U)$. This implies

$$\int_U (v - \tilde{v})\phi dx = 0 \qquad \forall \phi \in C_c^\infty(U).$$

Hence, $v - \tilde{v} = 0$ almost everywhere. $\square$

LEMMA 2.7. *Assume that $u \in L^1_{loc}(U)$ has weak derivatives $D^\alpha u$ for $|\alpha| \leq k$. Then for multiindices α, β with $|\alpha| + |\beta| \leq k$ one has*

$$D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta} u.$$

PROOF. Let $\phi \in C_c^\infty(U)$, then also $D^\alpha \phi \in C_c^\infty(U)$. Using the definition of weak derivatives twice we obtain:

$$\begin{aligned} \int_U D^\alpha(D^\beta u)(x)\phi(x) &= (-1)^{|\alpha|} \int_U D^\beta u(x) D^\alpha \phi(x)\, dx \\ &= (-1)^{|\alpha|+|\beta|} \int_U u(x) D^\alpha(D^\beta \phi(x))\, dx \\ &= \int_U D^{\alpha+\beta} u(x)\phi(x)\, dx. \end{aligned}$$

If we change the roles of α and β we obtain

$$\int_U D^\beta(D^\alpha u)(x)\phi(x) = \int_U D^{\alpha+\beta} u(x)\phi(x)\, dx.$$

$\square$

2.3. The Sobolev spaces $W^{k,p}(U)$

Let $U \subseteq \mathbb{R}^n$ open. Let $1 \leq p \leq \infty$ and k be a non-negative integer.

DEFINITION 2.8. The **Sobolev space**

$$W^{k,p}(U)$$

is the space of all locally integrable functions $u : U \to \mathbb{R}$ such that for every multiindex α with $|\alpha| \leq k$ the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^p(U)$.

DEFINITION 2.9. We define the norm of $u \in W^{k,p}(U)$ to be

$$\begin{aligned} \|u\|_{W^{k,p}(U)} &= \Big(\sum_{|\alpha| \leq k} \int_U |D^\alpha u(x)|^p dx \Big)^{\frac{1}{p}}, \quad \text{ if } 1 \leq p < \infty, \\ \|u\|_{W^{k,\infty}(U)} &= \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_{x \in U} |D^\alpha u(x)|. \end{aligned}$$

THEOREM 2.10 (Completeness). *For each $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ the Sobolev space $W^{k,p}(U)$ is a Banach space.*

PROOF. We have to show the following.

(1) $W^{k,p}$ is a normed vector space (exercise).
(2) $W^{k,p}$ is complete.

We show (2). Assume $(u_m)_{m=1}^\infty$ is a Cauchy sequence in $W^{k,p}(U)$. It follows from the definition of the norm on $W^{k,p}(U)$ that $(D^\alpha u_m)_{m=1}^\infty$ is a Cauchy sequence in $L^p(U)$ for each $|\alpha| \leq k$, cf. Remark 2.11. Since $L^p(U)$ is complete, there exist functions u, $u_\alpha \in L^p(U)$ such that

$$\|u_m - u\|_{L^p} \to 0, \qquad \|D^\alpha u_m - u_\alpha\|_{L^p} \to 0, \quad \text{for all } 0 < |\alpha| \leq k.$$

We show that

$$u \in W^{k,p}(U) \quad \text{and} \quad D^\alpha u = u_\alpha, \text{ for all } 0 < |\alpha| \leq k. \tag{2.8}$$

We fix $\phi \in C_c^\infty(U)$. Then

$$\int_U u D^\alpha \phi dx = \lim_{m\to\infty} \int_U u_m D^\alpha \phi dx = (-1)^{|\alpha|} \lim_{m\to\infty} \int_U D^\alpha u_m \phi dx = (-1)^{|\alpha|} \int_U u_\alpha \phi dx.$$

Thus, (2.8) holds and for all $|\alpha| \leq k$

$$D^\alpha u_m \overset{m\to\infty}{\longrightarrow} D^\alpha u \qquad \text{in } L^p(U).$$

Hence,

$$u_m \overset{m\to\infty}{\longrightarrow} u \text{ in } W^{k,p}(U) \ .$$

$\square$

Remark 2.11. $(u_m)_{m=1}^\infty$ is a Cauchy sequence in $W^{k,p}(U)$

$$\begin{aligned}
&\overset{\text{DEF}}{\Longleftrightarrow} \forall \varepsilon > 0\, \exists N\, \forall m,n > N : \|u_m - u_n\|_{W^{k,p}(U)} < \varepsilon \\
&\Longleftrightarrow \forall \varepsilon > 0\, \exists N\, \forall m,n > N : \Big(\sum_{|\alpha|\leq k} \|D^\alpha(u_m - u_n)\|_{L^p(U)}^p\Big)^{\frac{1}{p}} < \varepsilon \\
&\Longleftrightarrow \forall \varepsilon > 0\, \exists N\, \forall m,n > N \ : \ \sum_{|\alpha|\leq k} \|D^\alpha u_m - D^\alpha u_n\|_{L^p(U)}^p < \varepsilon^p \\
&\Longrightarrow \forall \varepsilon > 0\, \exists N\, \forall m,n > N\, \forall \alpha, |\alpha| \leq k \ : \ \|D^\alpha u_m - D^\alpha u_n\|_{L^p(U)} < \varepsilon \\
&\Longrightarrow \forall \alpha, |\alpha| \leq k\, \forall \varepsilon > 0\, \exists N\, \forall m,n > N \ : \ \|D^\alpha u_m - D^\alpha u_n\|_{L^p(U)} < \varepsilon \\
&\Longleftrightarrow \forall \alpha, |\alpha| \leq k \text{ is } (D^\alpha u_m)_{m=1}^\infty \text{ a Cauchy sequence in } L^p(U).
\end{aligned}$$

Remark 2.12. Note that $W^{0,p}(U) = L^p(U)$. For $p = 2$, the norm in Definition 2.9 is induced by the inner product

$$\langle u, v \rangle = \sum_{|\alpha|\leq k} \int_U D^\alpha u(x) D^\alpha v(x) dx. \tag{2.9}$$

Hence, $W^{k,2}(U)$ is a Hilbert space and we write

$$H^k(U) = W^{k,2}(U).$$

Definition 2.13. The subspace $W_0^{k,p}(U) \subseteq W^{k,p}(U)$ is defined by

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{W^{k,p}(U)}.$$

More precisely, $u \in W_0^{k,p}(U)$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_c^\infty(U)$ such that

$$\|u_n - u\|_{W^{k,p}(U)} \to 0.$$

DEFINITION 2.14. Let U, V be open subsets of $\mathbb{R}^n$. We say V is **compactly contained** in U, written

$$V \subset\subset U$$

if $V \subset K \subset U$ and K is compact.

DEFINITION 2.15. We define the space $W^{k,p}_{\text{loc}}(U)$ to be the space of functions $u : U \to \mathbb{R}$ satisfying the following property. Let $V \subset\subset U$, then $u\big|_V \in W^{k,p}(V)$.

Notation. Let $(u_m)_{m=1}^\infty$, $u \in W^{k,p}(U)$

- We say (u_m) converges to u in $W^{k,p}(U)$, written
$$u_m \longrightarrow u \quad \text{in} \;\; W^{k,p}(U), \tag{2.10}$$
provided
$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0. \tag{2.11}$$
- We write
$$u_m \longrightarrow u \quad \text{in} \;\; W^{k,p}_{\text{loc}}(U) \tag{2.12}$$
to mean
$$u_m \longrightarrow u \quad \text{in} \;\; W^{k,p}(V) \tag{2.13}$$
for each $V \subset\subset U$.

LEMMA 2.16. *Let $u \in W^{k,p}(U)$, $\zeta \in C_c^\infty(U)$ and α a multiindex with $|\alpha| \le k$. Then*
(1) $D^\alpha u \in W^{k-|\alpha|,p}(U)$,
(2) $\zeta u \in W^{k,p}(U)$ *and*

$$D^\alpha(\zeta u) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \quad \textit{(Leibniz formula)}. \tag{2.14}$$

PROOF. (1) follows from the definition of the Sobolev space. We show (2). We know that $u \in L^p$ and $D^\alpha u \in L^p$ for all $|\alpha| \le k$. Hence, $\zeta u \in L^p$ and $D^\alpha(\zeta u)$ given by (2.14) is in L^p. Therefore, $\zeta u \in W^{k,p}$. We prove the Leibniz formula by induction. Let $\zeta, \phi \in C_c^\infty(U)$ and $|\alpha| = 1$. let $1 \le i \le n$. Then, by the definition of the weak derivative

$$\int_U (\zeta u)_{x_i} \phi dx = -\int_U \zeta u \phi_{x_i} dx. \tag{2.15}$$

By the chain rule (classical Leibniz formula for the first derivative) we have

$$(\zeta\phi)_{x_i} = \zeta_{x_i}\phi + \zeta\phi_{x_i}. \tag{2.16}$$

Combining (2.15) and (2.16) yields

$$\int_U (\zeta u)_{x_i} \phi \, dx = \int_U \zeta_{x_i} u \, \phi \, dx - \int_U (\zeta\phi)_{x_i} u \, dx.$$

Note that $\zeta\phi \in C_c^\infty$. Again by the definition of the weak derivative we have

$$\begin{aligned}\int_U (\zeta u)_{x_i}\phi\,dx &= \int_U \zeta_{x_i} u\,\phi\,dx + \int_U \zeta\phi\,u_{x_i}\,dx\\ &= \int_U (\zeta_{x_i}u + u_{x_i}\zeta)\phi\,dx.\end{aligned}$$

Therefore, $(\zeta u)_{x_i} = \zeta_{x_i}u + u_{x_i}\zeta$. Now let $|\alpha| = l+1$, then $\alpha = \beta+\gamma$ with $|\beta| = l$ and $|\gamma| = 1$. By the definition of the weak derivative we have

$$\int_U \zeta u\,D^\alpha\phi\,dx = \int_U \zeta u\,D^\beta(D^\gamma\phi)dx = (-1)^{|\beta|}\int_U D^\beta(\zeta u)D^\gamma\phi\,dx. \tag{2.17}$$

Using the induction hypothesis and again the definition of the weak derivative we obtain

$$\begin{aligned}(-1)^{|\beta|}\int_U D^\beta(\zeta u)D^\gamma\phi\,dx. &= (-1)^{|\beta|}\int_U \left[\sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^\sigma\zeta D^{\beta-\sigma}u\right]D^\gamma\phi\,dx\\ &= (-1)^{|\beta|+|\gamma|}\int_U D^\gamma\left[\sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^\sigma\zeta D^{\beta-\sigma}u\right]\phi\,dx\\ &= (-1)^{|\alpha|}\int_U \left[\sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^\gamma\left(D^\sigma\zeta D^{\beta-\sigma}u\right)\right]\phi\,dx\\ &= (-1)^{|\alpha|}\int_U \left[\sum_{\sigma\leq\beta}\binom{\beta}{\sigma}(D^{\sigma+\gamma}\zeta D^{\beta-\sigma}u + D^\sigma\zeta D^{\beta-\sigma+\gamma}u)\right]\phi\,dx.\end{aligned} \tag{2.18}$$

Note that the last equality in (2.18) is due to the induction basis ($|\gamma| = 1$). Let $\rho = \sigma+\gamma$. Then we can rewrite the sum in the right hand side of (2.18) as

$$\begin{aligned}\sum_{\sigma\leq\beta}\binom{\beta}{\sigma}(D^{\sigma+\gamma}\zeta D^{\beta-\sigma}u + D^\sigma\zeta D^{\beta-\sigma+\gamma}u) &= \sum_{\rho\leq\beta+\gamma}\binom{\beta}{\rho-\gamma}D^\rho\zeta D^{\alpha-\rho}u + \sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^\sigma\zeta D^{\alpha-\sigma}u\\ &= \sum_{\sigma\leq\alpha}\binom{\beta}{\sigma-\gamma}D^\sigma\zeta D^{\alpha-\sigma}u + \sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^\sigma\zeta D^{\alpha-\sigma}u\\ &= \sum_{\sigma\leq\beta}\left[\binom{\beta}{\sigma-\gamma}+\binom{\beta}{\sigma}\right]D^\sigma\zeta D^{\alpha-\sigma}u + D^\alpha\zeta\,u\end{aligned} \tag{2.19}$$

Note that by the definition of the binomial coefficients for multiindices we have

$$\binom{\alpha}{\sigma} = \binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma}. \tag{2.20}$$

Combining the equations (2.17), (2.18), (2.19) and (2.20) yields

$$\int_U \zeta u\,D^\alpha\phi\,dx = (-1)^{|\alpha|}\int_U \left[\sum_{\sigma\leq\alpha}\binom{\alpha}{\sigma}D^\sigma\zeta D^{\alpha-\sigma}u\right]\phi\,dx.$$

□

LEMMA 2.17. *Let $u \in W^{k,p}(U)$ and $V \subseteq U$ open, then $u \in W^{k,p}(V)$.*

PROOF. Obvious consequence of the definition. □

2.4. Examples

Weak derivatives.

EXAMPLE 2.18. Let $n = 1$, $U = (0, 2)$ and

$$u(x) = \begin{cases} x, & \text{if } 0 < x \leq 1, \\ 1, & \text{if } 1 < x < 2. \end{cases}$$

We define

$$v(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } 1 < x < 2. \end{cases}$$

and show that for all $\phi \in C_c^\infty(U)$

$$\int_0^2 u(x)\phi'(x)dx = -\int_0^2 v(x)\phi(x)dx$$

holds.

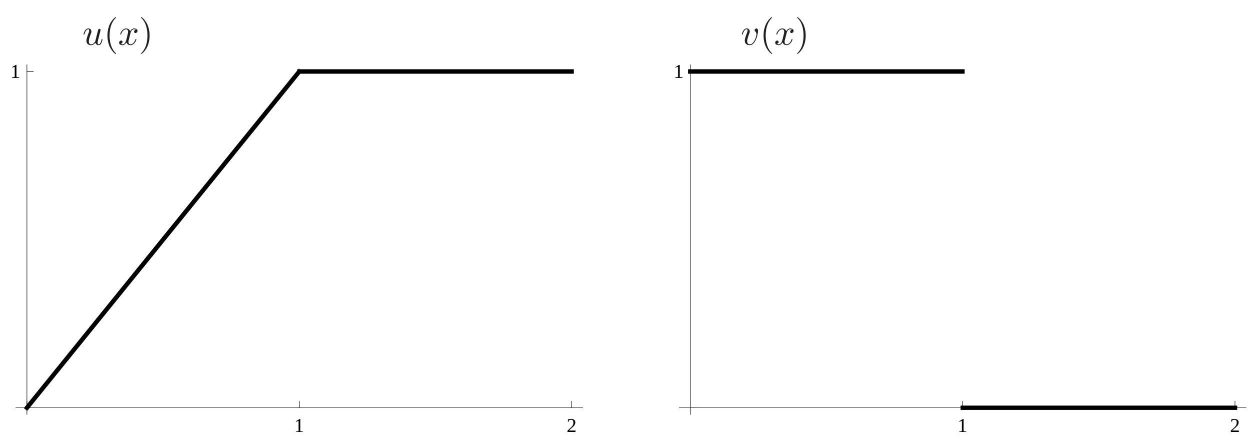


FIGURE 2.2. $u(x)$ and $v(x)$

$$
\begin{aligned}
\int_0^2 u(x)\phi'(x)dx &= \int_0^1 u(x)\phi'(x)dx + \int_1^2 u(x)\phi'(x)dx \\
&= \int_0^1 x\phi'(x)dx + \int_1^2 1\phi'(x)dx \\
&= x\phi(x)\Big|_0^1 - \int_0^1 \phi(x)dx + \int_1^2 \phi'(x)dx \\
&= \phi(1) - \int_0^1 \phi(x)dx + \underbrace{\phi(2)}_{0} - \phi(1) \\
&= -\int_0^1 \phi(x)dx = -\int_0^2 v(x)\phi(x)\,dx.
\end{aligned}
$$

Example 2.19. Let $n = 1$, $U = (0, 2)$ and

$$
u(x) = \begin{cases} x, & \text{if } 0 < x \leq 1, \\ 2, & \text{if } 1 < x < 2. \end{cases}
$$

In order to check, that u does not have a weak derivative we have to show that there does not exist any function $v \in L^1_{\mathrm{loc}}(U)$ satisfying

$$
\int_0^2 u(x)\phi'(x)dx = -\int_0^2 v(x)\phi(x)dx, \tag{2.21}
$$

for all $\phi \in C_c^\infty(U)$. Assume there exists a $v \in L^1_{\mathrm{loc}}(U)$ satisfying (2.21). Then

$$
\begin{aligned}
-\int_0^2 v(x)\phi(x)dx &= \int_0^2 u(x)\phi'(x)dx = \int_0^1 x\phi'(x) + 2\int_1^2 \phi'(x)dx \\
&= x\phi(x)\Big|_0^1 - \int_0^1 \phi(x)dx + 2\big(\phi(2) - \phi(1)\big) = -\phi(1) - \int_0^1 \phi(x)dx
\end{aligned} \tag{2.22}
$$

is valid for all $\phi \in C_c^\infty(U)$. We choose a sequence $(\phi_m)_{m=1}^\infty$ of smooth functions satisfying

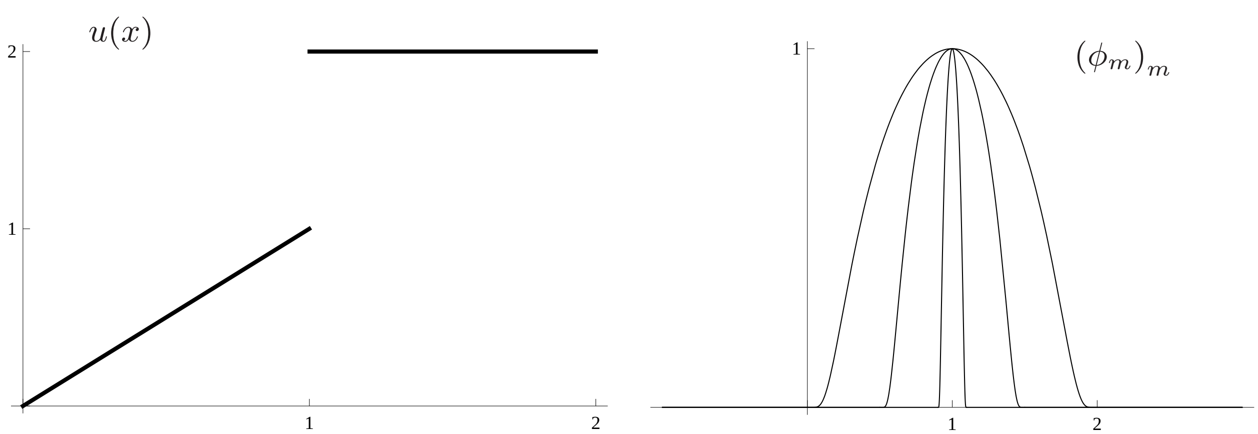


Figure 2.3. $u(x)$ and some elements of the sequence $\phi_m(x)$

$$0 \leq \phi_m \leq 1, \quad \phi_m(1) = 1 \quad \text{and} \quad \phi_m(x) \overset{m\to\infty}{\longrightarrow} 0 \quad \forall x \neq 1.$$

Replacing ϕ by ϕ_m in (2.22) yields

$$1 = \phi_m(1) = \int_0^2 v(x)\phi_m(x)dx - \int_0^1 \phi_m(x)dx.$$

We take the limit for $m \to \infty$:

$$1 = \lim_{m\to\infty} \phi_m(1) = \lim_{m\to\infty} \left[\int_0^2 v(x)\phi_m(x)dx - \int_0^1 \phi_m(x)dx \right] = 0.$$

Elements in Sobolev spaces. Note that if $n = 1$ and U is an open interval in $\mathbb{R}$, then $u \in W^{1,p}(U)$ if and only if u equals a.e. an absolutely continuous function whose derivative (which exists a.e.) and the function itself belong to $L^p(U)$ (Exercise). Such a simple characterization is however only available for $n = 1$.

In general a discontinuous and/or unbounded function can belong to a Sobolev space.

Example 2.20. Let $U = \{x \in \mathbb{R}^n : |x| < 1\} =: B(0,1)$. We fix $\gamma > 0$ and consider the function

$$u(x) = |x|^{-\gamma} = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{-\gamma}{2}}, \qquad x \in U,\, x \neq 0.$$

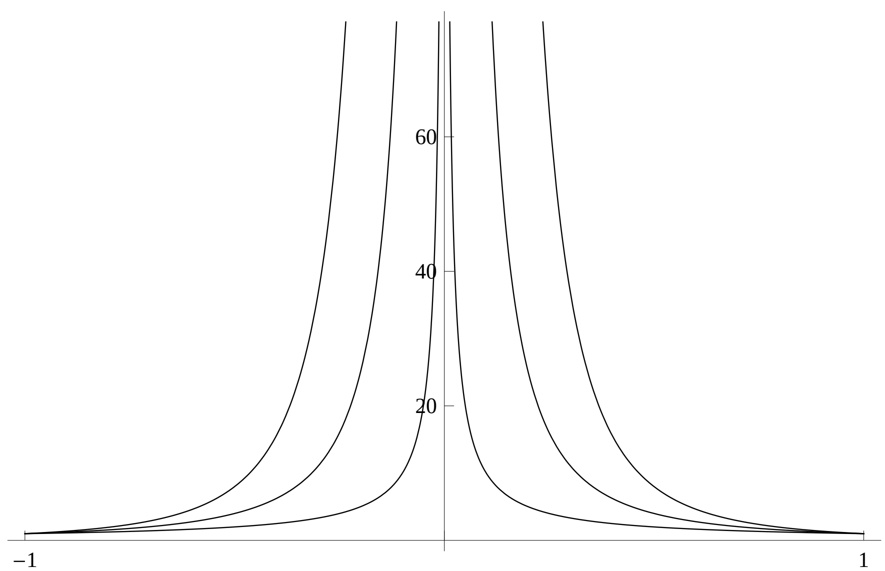


Figure 2.4. $|x|^{-\gamma}$

Statement: $u \in W^{1,p}(U) \Leftrightarrow \gamma < \frac{n-p}{p}$

Proof. Note that $u \in C^1(U \setminus \{0\})$. For $x \in U \setminus \{0\}$ we have

$$u_{x_i}(x) = \frac{-\gamma x_i}{|x|^{\gamma+2}}, \qquad 1 \leq i \leq n. \tag{2.23}$$

Therefore,

$$|Du(x)| = |\nabla u(x)| = \left(\sum_{i=1}^{n} |u_{x_i}|^2\right)^{\frac{1}{2}} = \frac{\gamma}{|x|^{\gamma+1}}$$

and by Corollary (11.17),

$$\begin{aligned}\int_U |Du(x)|dx = \gamma \int_U |x|^{-\gamma-1}dx = C(n)\gamma \int_0^1 r^{-\gamma-1}r^{n-1}dr = C(n)\gamma \int_0^1 r^{-\gamma-2+n}dr \\ = \begin{cases} \frac{C(n)\gamma}{-\gamma-1+n}, & \text{if } -\gamma-1+n>0 \\ \infty, & \text{otherwise.}\end{cases}\end{aligned} \tag{2.24}$$

Hence, if $\gamma + 1 < n$, then $|Du| \in L^1(U)$. Analogously we have $u \in L^1(U)$, if $\gamma + 1 < n$.

On the open set $U \setminus \{0\}$ the function u has weak derivatives $Du = (u_{x_1}, \ldots, u_{x_n})$ and they coincide with the classical derivatives. We show that under certain circumstances u_{x_i} define weak derivatives on the entire domain U.

Let $\phi \in C_c^\infty(U)$ and $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$ for a fixed $\varepsilon > 0$ then (by the integration-by-parts formula (Theorem 11.15))

$$\int_{U\setminus B_\varepsilon} u\phi_{x_i}\, dx = -\int_{U\setminus B_\varepsilon} u_{x_i}\phi\, dx + \int_{\partial B_\varepsilon} u\phi\nu^i dS$$

with $\nu^i(x) = \frac{-x_i}{|x|}$ so that $\nu = (\nu^1 \ldots \nu^n)$ is the inward pointing normal on ∂B_ε and dS is the spherical measure on the surface of the ball B_ε. The following holds.

$$\left|\int_{\partial B_\varepsilon} u\phi\nu^i dS\right| \le \|\phi\|_\infty \int_{\partial B_\varepsilon} \varepsilon^{-\gamma}dS \le C\varepsilon^{-\gamma-1+n} \stackrel{\varepsilon\to 0}{\longrightarrow} 0, \quad \text{if } \gamma+1<n \tag{2.25}$$

$$\lim_{\varepsilon\to 0}\int_{U\setminus B_\varepsilon} u\phi_{x_i}dx = \int_U \lim_{\varepsilon\to 0}\chi_{U\setminus B_\varepsilon}u\phi_{x_i}dx = \int_U u\phi_{x_i}dx \tag{2.26}$$

$$\lim_{\varepsilon\to 0}\int_{U\setminus B_\varepsilon} u_{x_i}\phi dx = \int_U u_{x_i}\phi dx. \tag{2.27}$$

With (2.25),(2.26) and (2.27) it follows that

$$\int_U u\phi_{x_i}dx = -\int_U u_{x_i}\phi dx \qquad \forall \phi \in C_c^\infty(U),$$

and the locally integrable function u_{x_i} defined in (2.23) is in fact the weak derivative of u on the entire domain U.

By an analogous calculation as in (2.24) we have that

$$u, |Du| \in L^p(U) \Leftrightarrow (\gamma+1)p < n.$$

Consequently, $u \in W^{1,p}(U)$ if and only if $\gamma < \frac{n-p}{p}$. In particular $u \notin W^{1,p}(U)$ for each $p \ge n$. $\square$

EXAMPLE 2.21. Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\{r_k : k \in \mathbb{N}\} = U \cap \mathbb{Q}^n$. This forms a dense subset in U. ($\{r_k\}_{k=1}^\infty$ is dense in U $\Leftrightarrow$ for each $u \in U$ there exist a subsequence $\{r_{k_l}\}_{l=1}^\infty$ such that $\lim r_{k_l} \to u$ in U.)

For $(\gamma+1)p < n$ we define

$$u_k(x) = 2^{-k}|x - r_k|^{-\gamma} \qquad \in W^{1,p}(U)$$

and set

$$u(x) = \sum_{k=1}^{\infty} 2^{-k}|x - r_k|^{-\gamma}.$$

Then $u \in W^{1,p}(U)$ and is unbounded on each open subset of U.

CHAPTER 3

Approximation in Sobolev spaces

In order to study the deeper properties of Sobolev spaces, without returning continually to the definition of weak derivatives, we need procedures for approximating a function in a Sobolev space by smooth functions. These approximation procedures allow us to consider smooth functions and then extend the statements to functions in the Sobolev space by density arguments.

We have to prove that smooth functions are in fact dense in $W^{k,p}(U)$. The method of mollifiers provides the tool.

3.1. Smoothing by convolution

Definition 3.1.

(1) Let $\eta \in C^\infty(\mathbb{R}^n)$ be given by

$$\eta(x) = \begin{cases} C\, e^{1/(|x|^2-1)}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

with constant $C > 0$ chosen such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.

(2) For each $\varepsilon > 0$ we define

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

We call η the *standard mollifier*.

Remark 3.2.

(1) $\eta \geq 0$ and $\eta \in C_c^\infty(\mathbb{R}^n)$.

(2) The functions η_ε are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)dx = 1 \quad \text{and} \quad \operatorname{supp}\eta_\varepsilon \subseteq B(0,\varepsilon).$$

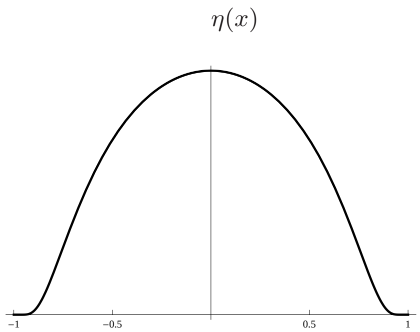
Remark 3.3. There are other examples of mollifiers, e.g.

$$\nu(t) = \begin{cases} \cos(\pi|t|^2) + 1, & \text{if } |t| < 1, \\ 0, & \text{if } |t| \geq 1. \end{cases}$$

Definition 3.4. Let $U \subseteq \mathbb{R}^n$ be open and $\varepsilon > 0$. Let

$$U_\varepsilon = \{x \in U : d(x, \partial U) > \varepsilon\} = \{x \in U : \overline{B(x,\varepsilon)} \subseteq U\},$$

where $B(x,\varepsilon) = \{y \in \mathbb{R}^n : |x-y| < \varepsilon\}$.



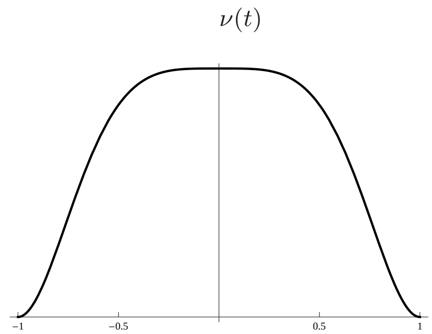


FIGURE 3.1. Standard mollifier and cos-mollifier

Let $f \in L^1_{\text{loc}}(U)$. Then we define for all $x \in U_\varepsilon$

$$f^\varepsilon(x) := f * \eta_\varepsilon(x) = \int_U f(y)\eta_\varepsilon(x-y)dy = \int_{B(0,\varepsilon)} f(x-y)\eta_\varepsilon(y)dy \tag{3.1}$$

f^ε is the mollification of f in U_ε. The mollification of a function $f \in L^1_{loc}(U)$ results from the concept of convolution.

Convolution. Let f, g be measurable functions on $\mathbb{R}^n$. The convolution $f * g$ is defined by

$$f * g\,(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

for all $x \in \mathbb{R}^n$ such that the integral exists.

PROPOSITION 3.5. *Assume that all considered integrals exist. Then*

(1) $f * g = g * f$,
(2) $f * g * h = f * (g * h)$,
(3) $\operatorname{supp}(f * g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$.

THEOREM 3.6.

(1) *If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, $g \in L^1(\mathbb{R}^n)$ then $f * g$ exists a.e. and $f * g \in L^p(\mathbb{R}^n)$. In fact $\|f * g\|_p \le \|f\|_p \cdot \|g\|_1$.*
(2) *Let $1 \le p \le \infty$ and $1 = \frac{1}{p} + \frac{1}{q}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f * g$ exists on $\mathbb{R}^n$ and $|f * g\,(x)| \le \|f\|_p \|g\|_q$ for every $x \in \mathbb{R}^n$.*
(3) *Let $1 \le p, q, r \le \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f * g$ exists a.e. and lies in $L^r(\mathbb{R}^n)$. In fact $\|f * g\|_r \le \|f\|_p \cdot \|g\|_q$. (Young's inequality)*
(4) *Let $f, g \in C_c(\mathbb{R}^n)$. Then $f * g$ exists for every $x \in \mathbb{R}$ and $f * g \in C_c(\mathbb{R}^n)$.*
(5) *If $f \in L^1$, $g \in C^k$ and $D^\alpha g$ is bounded for all multiindices α with $|\alpha| \le k$. Then $f * g \in C^k$ and $D^\alpha(f * g) = f * D^\alpha g$ for $|\alpha| \le k$.*

Mollification Properties.

LEMMA 3.7. *Let $U_\varepsilon \subseteq U$ and f^ε be as in Definition 3.4. Then $f^\varepsilon \in C^\infty(U_\varepsilon)$ and for every multiindex α and $x \in U_\varepsilon$ we have*

$$D^\alpha f^\varepsilon(x) = \int_U D^\alpha_x \eta^\varepsilon(x-y)f(y)dy,$$

where D_x^α denotes the partial derivatives with respect to the variable $x = (x_1, \ldots, x_n)$.

PROOF. Exercise. $\square$

PROOF. Fix $x \in U_\varepsilon$ and h so small, that $x + e_i h \in U_\varepsilon$. Then

$$\frac{f^\varepsilon(x + e_i h) - f^\varepsilon(x)}{h} = \int_U \left(\frac{\eta^\varepsilon(x + e_i h - y) - \eta^\varepsilon(x - y)}{h} \right) f(y) dy.$$

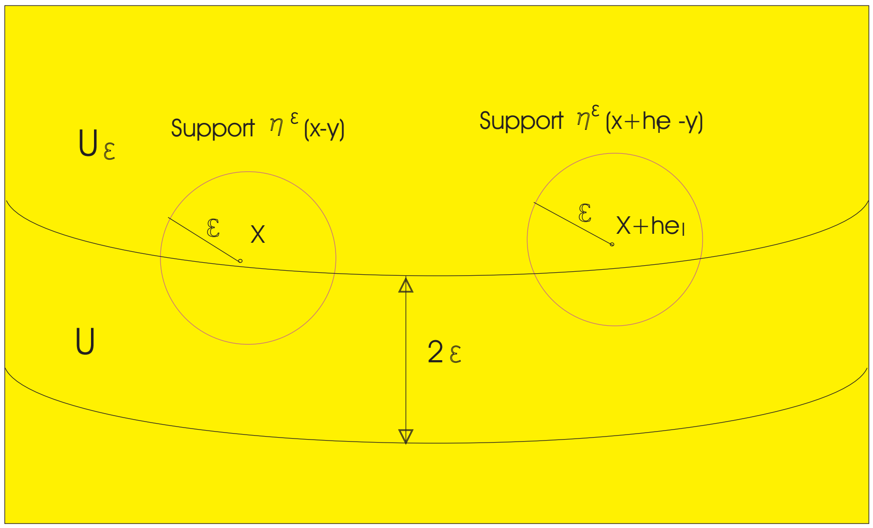


FIGURE 3.2. Supports of η^ε

The support of

$$y \longrightarrow \left(\frac{\eta^\varepsilon(x + e_i h - y) - \eta^\varepsilon(x - y)}{h} \right)$$

is compact in U (cf. figure 3.2). Therefore,

$$\int_U \left(\frac{\eta^\varepsilon(x + e_i h - y) - \eta^\varepsilon(x - y)}{h} \right) f(y) dy = \int_V \left(\frac{\eta^\varepsilon(x + e_i h - y) - \eta^\varepsilon(x - y)}{h} \right) f(y) dy$$

for some $V \subset\subset U$. By the Heine-Cantor theorem and the mean value theorem we have that

$$\lim_{h \to 0} \sup_{y \in V} \left| \frac{\eta^\varepsilon(x + e_i h - y) - \eta^\varepsilon(x - y)}{h} - \frac{\partial \eta^\varepsilon}{\partial_{x_i}}(x - y) \right| = 0.$$

Therefore,

$$\begin{aligned}
&\left| \frac{1}{h} \int_V \left(\eta^\varepsilon(x + e_i h - y) - \eta^\varepsilon(x - y)\right) f(y)\, dy - \int_U \frac{\partial \eta^\varepsilon}{\partial x_i}(x - y) f(y) dy \right| \\
&\le \sup_{y \in V} \left| \frac{\eta^\varepsilon(x + e_i h - y) - \eta^\varepsilon(x - y)}{h} - \frac{\partial \eta^\varepsilon}{\partial_{x_i}}(x - y) \right| \int_U |f(y)| dy \\
&\xrightarrow{h \to 0} 0.
\end{aligned}$$

Hence,
$$\frac{\partial}{\partial_{x_i}} f^\varepsilon(x) = \lim_{h\to 0} \frac{f^\varepsilon(x+e_i h) - f^\varepsilon(x)}{h} = \int_U \partial_{x_i}\eta^\varepsilon(x-y)f(y)dy \ .$$
By the same argument one obtains that for every multiindex α, $D^\alpha f^\varepsilon$ exists and
$$D^\alpha f^\varepsilon(x) = \int_U D^\alpha \eta^\varepsilon(x-y)f(y)dy \qquad (x \in U_\varepsilon) \ .$$
$\square$

Corollary 3.8. *Let $U_\varepsilon \subseteq U$ be as in Definition 3.4. Assume that $f \in L^1_{loc}(U)$ admits a weak derivative $D^\alpha f$ for some multiindex α. Then*
$$D^\alpha(f * \eta_\varepsilon)(x) = \eta_\varepsilon * D^\alpha f\,(x), \qquad \textit{for all } x \in U_\varepsilon.$$

Note that the derivative of the mollification $D^\alpha(f*\eta_\varepsilon)$ exists in the classical sense.

Proof. We have by Lemma 3.7 and by the definition of the weak derivative
$$\begin{aligned} D^\alpha(f*\eta_\varepsilon)(x) &= \int_U D^\alpha_x \eta^\varepsilon(x-y)f(y)dy \\ &= (-1)^{|\alpha|}\int_U D^\alpha_y \eta^\varepsilon(x-y)f(y)dy \\ &= (-1)^{|\alpha|+|\alpha|}\int_U \eta^\varepsilon(x-y)D^\alpha f(y)dy \\ &= \int_U \eta^\varepsilon(x-y)D^\alpha f(y)dy \\ &= \eta^\varepsilon * D^\alpha f\,(x). \end{aligned}$$
$\square$

Theorem 3.9 (Convergence of the mollification).

(1) $f \in L^1_{loc}(U) \Longrightarrow f^\varepsilon(x) \xrightarrow{\varepsilon\to 0} f(x)$ *for almost every $x \in U$, i.e.*
$$\left|\{x \in U : \lim_{\varepsilon\to 0} f^\varepsilon(x) \neq f(x)\}\right| = 0.$$
(2) $f \in C(U)$ *and $K \subset U$ compact. Then*
$$\sup_{z\in K}|f(z) - f^\varepsilon(z)| \xrightarrow{\varepsilon\to 0} 0.$$
(3) $f \in L^p_{loc}(U)$, $1 \le p < \infty$, *and $K \subset U$ compact. Then*
$$\|f - f^\varepsilon\|_{L^p(K)} \xrightarrow{\varepsilon\to 0} 0.$$

Proof. (1) We use Lebesgue's differentiation theorem (Theorem 11.20), which asserts that
$$\lim_{r\to 0} \fint_{B(x,r)} |f(y)-f(x)|dy = \lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f(x)|dy = 0$$

for a.e. $x \in U$. We fix such an point x (Lebesgue point). Then

$$|f^\varepsilon(x) - f(x)| \overset{(*)}{=} \left| \int_{B(x,\varepsilon)} \eta^\varepsilon(x-y) f(y) dy - f(x) \right| \overset{(**)}{=} \left| \int_{B(x,\varepsilon)} \eta^\varepsilon(x-y) \left(f(y) - f(x) \right) dy \right|$$
$$\leq \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\Big(\frac{x-y}{\varepsilon}\Big) |f(y) - f(x)| \, dy \overset{(***)}{\leq} \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |f(y) - f(x)| \, dy$$
$$\overset{(****)}{=} C \fint_{B(x,\varepsilon)} |f(y) - f(x)| dy \xrightarrow{\varepsilon \to 0} 0, \quad \text{for a.e. } x \in U.$$

$(*)$ $\operatorname{supp} \eta_\varepsilon(x - \cdot) \subseteq B(x, \varepsilon)$.
$(**)$ $\int_{\mathbb{R}^n} \eta^\varepsilon(x-y) dy = 1$.
$(***)$ $0 \leq \eta \leq 1$
$(****)$ $|B(x,\varepsilon)| = \varepsilon^n |B(0,1)|$.

Hence,

$$\left| \{ x \in \mathbb{R}^n : \lim_{\varepsilon \to 0} f^\varepsilon(x) \neq f(x) \} \right| = 0.$$

(2) Let $K \subset U$ compact. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ we have $K \subset U_\varepsilon$. Hence, $f^\varepsilon(x)$ is well defined for all $x \in K$. By the same argument as before we have

$$|f(x) - f^\varepsilon(x)| \leq C \fint_{B(x,\varepsilon)} |f(x) - f(y)| \, dy \leq C \sup_{y \in B(x,\varepsilon)} |f(x) - f(y)|.$$

We have that $f \in C(U)$ and $K \subseteq U$ is compact. Hence, f is uniformly continuous on K, i.e.

$$\forall \eta > 0 \, \exists \, \varepsilon > 0 \, \forall x, y \in K : \, |x - y| < \varepsilon \implies |f(x) - f(y)| < \eta.$$

Summarizing we have

$$\forall \eta > 0 \, \exists \, \varepsilon_0 > 0 \, \forall \varepsilon < \varepsilon_0 \, \forall x \in K : \, |f(x) - f^\varepsilon(x)| \leq C \, \eta.$$

(3) Let $K \subset U$ compact. Then there exists an open subset W of U with $W \subset\subset U$ and an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and for all $x \in K$ we have that $B(x, \varepsilon) \subseteq U_\varepsilon \subseteq W$. Let $0 < \varepsilon < \varepsilon_0$. Then

$$\|f^\varepsilon\|_{L^p(K)} \leq \|f\|_{L^p(W)}.$$

Indeed by Hölder's inequality with $\frac{1}{q} + \frac{1}{p} = 1$ we have

$$| \, f^\varepsilon(x) \, | = \left| \int_{B(x,\varepsilon)} \eta^\varepsilon(x-y) f(y) dy \right|$$
$$\leq \int_{B(x,\varepsilon)} |f(y)| \, \eta^\varepsilon(x-y)^{\frac{1}{p}} \eta^\varepsilon(x-y)^{\frac{1}{q}} dy$$
$$\leq \left(\int_{B(x,\varepsilon)} \eta^\varepsilon(x-y) |f(y)|^p dy \right)^{\frac{1}{p}} \underbrace{\left(\int_{B(x,\varepsilon)} \eta^\varepsilon(x-y) dy \right)^{\frac{1}{q}}}_{=1}.$$

Therefore,

$$\begin{aligned}
\|f^\varepsilon\|^p_{L^p(K)} = \int_K \mid f^\varepsilon(x) \mid^p dx &\le \int_K \left(\int_{B(x,\varepsilon)} \eta^\varepsilon(x-y) \mid f(y) \mid^p dy \right) dx \\
&= \int_K \int_W \eta^\varepsilon(x-y) \mid f(y) \mid^p dy\, dx \\
&= \int_W \mid f(y) \mid^p \int_K \eta^\varepsilon(x-y) dx dy \\
&\le \|f\|^p_{L^p(W)}.
\end{aligned}$$

W is compactly contained in U. Hence, $C(W)$ is dense in L^p, i.e.

$$\forall\, f \in L^p(W)\, \forall\, \delta > 0\, \exists\, g \in C(W) : \|f - g\|_{L^p(W)} \le \delta.$$

Fix $\delta > 0$ and choose $g \in C(W)$ such that $\|f - g\|_{L^p} < \delta$. Then

$$\begin{aligned}
\|f^\varepsilon - f\|_{L^p(K)} &\le \|f^\varepsilon - g^\varepsilon\|_{L^p(K)} + \|g^\varepsilon - g\|_{L^p(K)} + \|g - f\|_{L^p(K)} \\
&\le \|f - g\|_{L^p(W)} + \|g^\varepsilon - g\|_{L^p(K)} + \|f - g\|_{L^p(W)} \\
&\le 2\delta + \|g^\varepsilon - g\|_{L^p(K)}.
\end{aligned}$$

By (2) we have that

$$\|g^\varepsilon - g\|_{L^p(K)} \le \mid K \mid^{\frac{1}{p}} \sup_{x \in K} |g^\varepsilon(x) - g(x)| \overset{\varepsilon \to 0}{\longrightarrow} 0.$$

Summarizing we have

$$\forall\, K \subset U \text{ compact } \forall \eta > 0\, \exists\, \varepsilon_0 > 0\, \forall\, \varepsilon < \varepsilon_0 \;:\; \|f^\varepsilon - f\|_{L^p(K)} \le \eta.$$

□

3.2. Partition of unity

In the following section we use the method of mollification to construct partitions of unity. We will use these results in the following proofs to obtain global properties from local ones.

Lemma 3.10. *Let K be a compact subset of $\mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ open such that $K \subset U$. Then there exists a function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ on K and $\operatorname{supp} \psi \subset U$.*

Proof. Let

$$K_\varepsilon = \{x \in \mathbb{R}^n : d(x, K) \le \varepsilon\}, \quad \varepsilon > 0.$$

Let $\varepsilon > 0$ small enough that $K_{3\varepsilon} \subset U$.

We set

$$\psi(x) = \eta^\varepsilon * 1_{K_{2\varepsilon}}(x) = \int_{\mathbb{R}^n} \eta^\varepsilon(x-y) 1_{K_{2\varepsilon}}(y) dy, \qquad x \in \mathbb{R}^n.$$

By the above properties of mollification we know that $\psi \in C^\infty$, $\psi \equiv 1$ on K and

$$\operatorname{supp} \psi \subseteq \overline{\operatorname{supp} \eta^\varepsilon + K_{2\varepsilon}} \subseteq K_{3\varepsilon} \subset U.$$

□

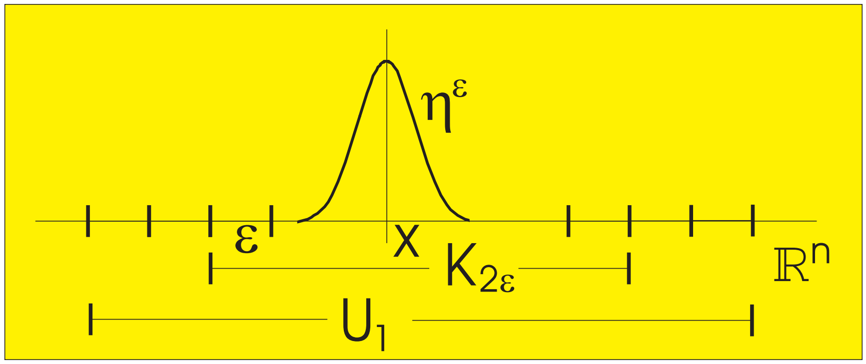


FIGURE 3.3. Partition of unity

LEMMA 3.11. *Let $K \subset \mathbb{R}^n$ be compact with $K \subset \bigcup_{j=1}^k V_j$, where $(V_i)_{i=1}^k$ is a sequence of open sets in $\mathbb{R}^n$. Then there exists a sequence $(K_i)_{i=1}^k$ of compact sets in $\mathbb{R}^n$ such that $K_j \subset V_j$ and $K \subseteq \bigcup_{j=1}^k K_j$.*

PROOF. Since K is bounded we can assume that V_j is bounded. Let

$$V_{j,n} = \{x \in V_j : d(x, \partial V_j) > \frac{1}{n}\}.$$

Then $\overline{V_{j,n}} \subset V_j$, for all $n \in \mathbb{N}$ and each sequence $(V_{j,n})_{n\in\mathbb{N}}$ is increasing. The sets $(V_{j,n})_{j=1,n\in\mathbb{N}}^k$ form an open cover of K. K is compact. Hence, there exists an $N \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{j=1}^k V_{j,N} \subseteq \bigcup_{j=1}^k \overline{V_{j,N}} =: \bigcup_{j=1}^k K_j.$$

$\square$

THEOREM 3.12. *Let $U \subseteq \mathbb{R}^n$ be bounded and $U \subset\subset \bigcup_{i=1}^k V_i$, where $(V_i)_{i=1}^k$ is a sequence of open sets in $\mathbb{R}^n$. There exists a sequence of smooth functions ξ_i, $1 \leq i \leq k$, such that $0 \leq \xi_i \leq 1$, $\operatorname{supp} \xi_i \subset V_i$ and $\sum \xi_i \equiv 1$ on U.*

We call the sequence $(\xi_i)_{i=1}^k$ a **smooth partition of unity subordinate to the open sets** $(V_i)_{i=1}^k$.

PROOF. Let $K \subseteq \mathbb{R}^n$ be compact such that $U \subset K \subset \bigcup_{j=1}^k V_j$. According to Lemma 3.11 there exist compact subsets $K_j \subset V_j$ such that $U \subset K \subset \bigcup_{j=1}^k K_j$. According to Lemma 3.10 for each $j \in \{1, \ldots, k\}$ there exists a function $\psi_j \in C_c^\infty(\mathbb{R}^n)$ satisfying $0 \leq \psi_j \leq 1$, $\psi_j \equiv 1$ on K_j and $\operatorname{supp} \xi_j \subset V_j$. Let

$$\xi_1 = \psi_1,\ \xi_2 = \psi_2(1-\psi_1), \ldots,\ \xi_k = \psi_k(1-\psi_1)\ldots(1-\psi_{k-1}).$$

Then we have that $0 \leq \xi_j \leq 1$ and $\xi_j \in C_c^\infty(V_j)$ for all $1 \leq j \leq k$. Furthermore,

$$1 - \sum_{j=1}^k \xi_j = 1 - [\psi_1 + \psi_2(1-\psi_1) + \ldots + (1-\psi_{k-1})] \tag{3.2}$$

$$= (1-\psi_1)(1-\psi_2)\ldots(1-\psi_k). \tag{3.3}$$

For each point $x \in U$ there is at least one factor $(1-\psi_j)$ that vanishes. Hence, the product equals zero on U and thus,

$$\sum_{j=1}^{k} \xi_j \equiv 1 \quad \text{ on } U.$$

$\square$

THEOREM 3.13. *Let $U \subseteq \mathbb{R}^n$ be open with locally finite open cover $(V_i)_{i=1}^{\infty}$, i.e. $U \subseteq \bigcup_{i=1}^{\infty} V_i$ and for every $x \in U$ there exist only finitely many V_i such that $x \in V_i$. Then there exists a sequence of smooth functions $(\xi_i)_{i\in\mathbb{N}}$ such that $0 \le \xi_i \le 1$, $\operatorname{supp} \xi_i \subset V_i$ and $\sum \xi_i = 1$ on U.*

We call the sequence $(\xi_i)_{i=1}^{\infty}$ a **smooth partition of unity subordinate to the locally finite cover** $(V_i)_{i=1}^{\infty}$.

PROOF. According to Lemma 3.14 we can choose an open cover $(W_i)_{i=1}^{\infty}$ of U such that $\overline{W_i} \subseteq V_i$. Then, analogously to Theorem 3.12 the statement holds. Note that by the local finiteness of the cover we have that for every $x \in U$ there are only finitely many ξ_i such that $x \in \operatorname{supp} \xi_i$. Hence, $\sum \xi_i(x)$ is finite for every $x \in U$. $\square$

LEMMA 3.14. *Let $U \subseteq \mathbb{R}^n$ be open with open cover $(V_i)_{i=1}^{\infty}$. Then there exists an open cover $(W_i)_{i=1}^{\infty}$ of U such that $\overline{W_i} \subseteq V_i$ for all $i \in \mathbb{N}$.*

PROOF. Let $A = U \setminus \bigcup_{i=2}^{\infty} V_i$. Then $A \subset V_1$ and A is closed in U. There exists an open set W_1 such that $A \subset W_1 \subset \overline{W_1} \subset V_1$. The collection $(W_1, V_2, \dots)$ forms a cover of U. Let W_1,...,W_{k1} be open sets such that $\{W_1, ..., W_{k1}, V_k, V_{k+1}, ...\}$ covers U. Let $A = U \setminus \left(\bigcup_{i=1}^{k-1} W_i \cup \bigcup_{i=1}^{\infty} V_{k+i}\right)$. A is closed in U. There exists an open set W_k such that $A \subset W_k \subset \overline{W_k} \subset V_k$. Then $\{W_1, \dots, W_k, V_{k+1}, \dots\}$ is an open cover of U. $\square$

Using the method of mollification and the partition of unity we will show in the following that functions in a Sobolev space can be approximated by smooth functions. We start with local approximation (convergence on $W^{k,p}_{\mathrm{loc}}(U)$), then we extend this idea to global approximation (convergence on $W^{k,p}(U)$) and finally, requiring restrictions on the boundary ∂U, we will obtain approximation by functions belonging to $C^\infty(\overline{U})$, and not just $C^\infty(U)$.

3.3. Local approximation by smooth functions

Let $U \subseteq \mathbb{R}^n$ open. Remember that for $\varepsilon > 0$

$$U_\varepsilon = \{x \in U \,:\, d(x, \partial U) > \varepsilon\}.$$

THEOREM 3.15. *Let $u \in W^{k,p}(U)$, $1 \le p < \infty$. Let $\varepsilon > 0$ and set*

$$u^\varepsilon(x) = (\eta^\varepsilon * u)(x), \qquad x \in U_\varepsilon,$$

where η^ε is the mollifier defined in Definition 3.1. Then

(1) $u^\varepsilon \in C^\infty(U_\varepsilon)$ *for all* $\varepsilon > 0$,
(2) $u^\varepsilon \xrightarrow{\varepsilon \to 0} u$ *in* $W^{k,p}_{loc}(U)$.

PROOF. $u \in W^{k,p}(U)$, therefore $u \in L^1_{\text{loc}}(U)$. Hence, (1) has already be shown in Lemma 3.7. Corollary 3.8 yields that for all $|\alpha| \leq k$

$$D^\alpha u^\varepsilon(x) = \eta_\varepsilon * D^\alpha u\,(x), \qquad \text{for all } x \in U_\varepsilon. \tag{3.4}$$

we have to show that for all $V \subset\subset U$

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - u\|_{W^{k,p}(V)} = 0 \Longleftrightarrow \forall\, |\alpha| \leq k : \lim_{\varepsilon\to 0} \|D^\alpha u^\varepsilon - D^\alpha u\|_{L^p(V)} = 0.$$

Using (3.4) and Theorem 3.9 we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} \|D^\alpha u^\varepsilon - D^\alpha u\|_{L^p(V)} &= \lim_{\varepsilon\to 0} \|\eta^\varepsilon * D^\alpha u - D^\alpha u\|_{L^p(V)} \\ &= \lim_{\varepsilon\to 0} \|(D^\alpha u)^\varepsilon - D^\alpha u\|^p_{L^p(V)} = 0.\end{aligned}$$

□

3.4. Global approximation by smooth functions

THEOREM 3.16. *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $u \in W^{k,p}(U)$, $1 \leq p < \infty$. Then there exists a sequence $(u_m)_{m\in\mathbb{N}}$ in $C^\infty(U) \cap W^{k,p}(U)$ such that*

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0. \tag{3.5}$$

PROOF. Let

$$U_i = \{x \in U \,:\, d(x, \partial U) > \frac{1}{i}\}, \quad i \in \mathbb{N}.$$

Then $U_i \subseteq U_{i+1}$ and

$$U = \bigcup_{i=1}^{\infty} \{x \in U \,:\, d(x, \partial U) > \frac{1}{i}\}.$$

Let $V_i = U_{i+3} \setminus \overline{U}_i$. Then $\#\{j \in \mathbb{N} : V_i \cap V_j \neq \emptyset\} \leq 3$. Therefore, each $x \in U$ is element of at least one and at most three sets of the family $(V_i)_{i\in\mathbb{N}}$. We choose $V_0 \subset\subset U$ such that

$$U = \bigcup_{i\in\mathbb{N}_0} V_i.$$

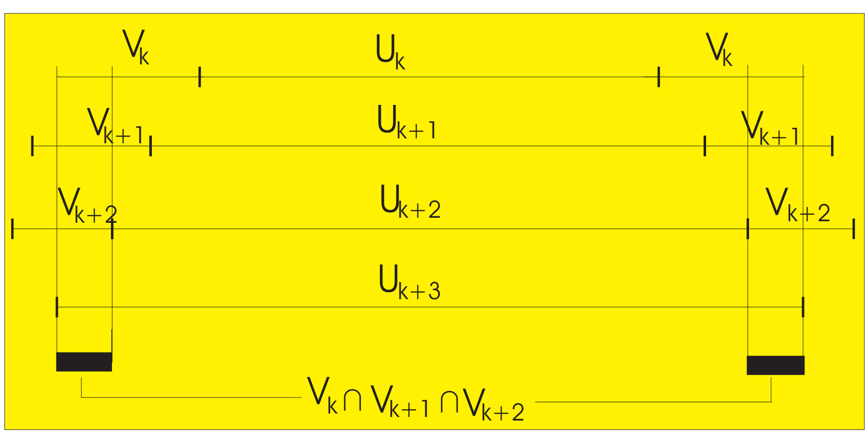


FIGURE 3.4. The families U_k, V_k

Let $(\xi_i)_{i=0}^\infty$ be a smooth partition of unity subordinate to the family of open sets $(V_i)_{i=0}^\infty$, i.e.

$$0 \le \xi_i \le 1,\ \xi_i \in C_c^\infty(V_i),\ \text{for all } i \in \mathbb{N}_0, \quad \sum_{i=0}^\infty \xi_i = 1 \ \text{on U} . \tag{3.6}$$

Let $u \in W^{k,p(U)}$. Then by Lemma 2.16 we have that $\xi_i u \in W^{k,p}(U)$ and $\operatorname{supp} \xi_i u \subset\subset V_i$.

Let $\delta > 0$ be fixed. By Theorem 3.15 we can choose $\varepsilon_i > 0$ such that $u^i = \eta^{\varepsilon_i} * (\xi_i\, u)$ satisfies

$$\|u^i - \xi_i\, u\|_{W^{k,p}(U)} \le \frac{\delta}{2^{i+1}} \tag{3.7}$$

$$\operatorname{supp} u^i \subset W_i = U_{i+4} \setminus \overline{U_i} \supset V_i. \tag{3.8}$$

We define

$$v(x) := \sum_{i=0}^\infty u^i(x), \quad x \in U.$$

$v \in C^\infty(U)$, since for every $x \in U$ we have that $\#\{i \in \mathbb{N}_0 :\ u^i(x) \neq 0\} \le 3$. We have

$$u = u \cdot 1 = \sum_{i=0}^\infty \xi_i u.$$

Therefore,

$$\begin{aligned}\|u - v\|_{W^{k,p}(U)} &= \Big\| \sum_{i=0}^\infty \xi_i u - \sum_{i=0}^\infty u^i \Big\|_{W^{k,p}(U)} \\ &\le \sum_{i=0}^\infty \|\xi_i\, u - u^i\|_{W^{k,p}(U)} \le \sum_{i=0}^\infty \delta\, 2^{-i-1} = \delta.\end{aligned}$$

Note that $\|v\|_{W^{k,p}(U)} \le \|v - u\|_{W^{k,p}(U)} + \|u\|_{W^{k,p}(u)} < \infty$. Summarizing we have that

$$\forall \delta > 0\ \exists v \in W^{k,p}(U) \cap C^\infty(U) : \|u - v\|_{W_{k,p}(U)} < \delta.$$

$\square$

Remark.

(1) The assumption of U to be bounded is not absolutely necessary. The same proof holds for example if $U = \{x \in \mathbb{R}^n :\ x_n > 0\}$. U is unbounded but has boundary $\partial U = \{x_n = 0\}$.

(2) Note that Theorem 3.16 is also true for $U = \mathbb{R}^n$, see [**1**, Theorem 3.16].

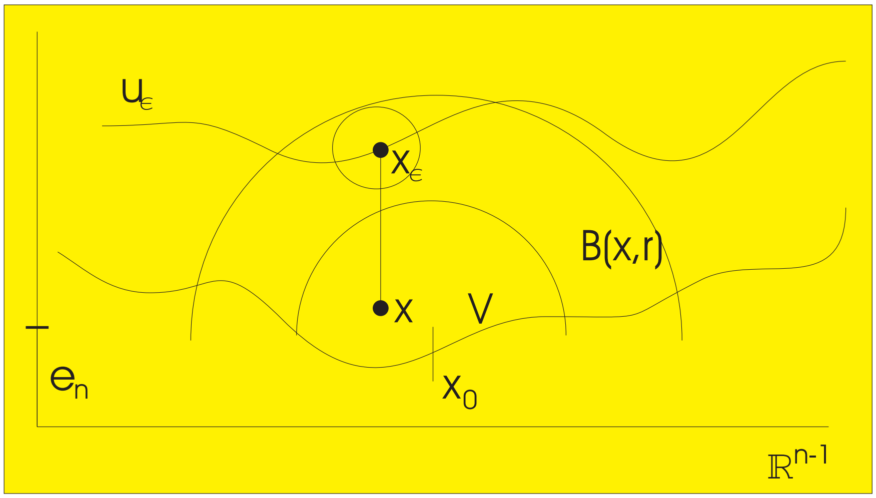
3.5. Global approximation by functions smooth up to the boundary

Theorem 3.17. *Let $U \subseteq \mathbb{R}^n$ be open and bounded and ∂U is C^1. Let $u \in W^{k,p}(U)$, $1 \le p < \infty$. Then there exists a sequence $(u_m)_{m=1}^\infty$ in $C^\infty(\overline{U})$ such that*

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0. \tag{3.9}$$

Proof. **Step 1:** Let $x_0 \in \partial U$. ∂U is C^1, i.e. (cf. Definition 11.12) there exists $r > 0$ and $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R} \in C^1$ such that - upon relabeling and reorienting the coordinate axes if necessary - we have

$$B(x_0, r) \cap U = \{x \in B(x_0, r) : \, x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$



Step 2: Let $V = B(x_0, \frac{r}{2}) \cap U$. Let $0 < \varepsilon << 1$ and $\lambda >> 1$ such that for the shifted point

$$x^\varepsilon = x + \lambda \, \varepsilon \, e_n, \quad x \in \overline{V},$$

where e_n is the n^{th} standard unit vector, the following holds

$$B(x^\varepsilon, \varepsilon) \subseteq U \cap B(x_0, r).$$

Now we define $u_\varepsilon(x) = u(x_\varepsilon)$, $x \in \overline{V}$. This is the function u translated a distance $\varepsilon\lambda$ in e_n direction. The idea is that we have "moved up enough" so that "there is room to mollify within U". We can mollify the function u_ε within the ε - ball (i.e. we can mollify it within U). Let

$$v^\varepsilon(x) := \eta^\varepsilon * u_\varepsilon(x).$$

Clearly, $v^\varepsilon \in C^\infty(\overline{V})$. The mollification v^ε converges towards u in $W^{k,p}(V)$. This is true if and only if for all $|\alpha| \le k$ we have

$$\lim_{\varepsilon \to 0} \|D^\alpha v^\varepsilon - D^\alpha u\|_{L^p(V)} = 0.$$

Indeed,

$$\|D^\alpha v^\varepsilon - D^\alpha u\|_{L^p(V)} \le \|D^\alpha v^\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)} + \|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p(V)} \longrightarrow 0, \quad \text{ when } \varepsilon \to 0.$$

The first term on the right-hand side vanishes for $\varepsilon \to 0$ by the argument of Theorem 3.15. The second term vanishes for $\varepsilon \to 0$ by the fact that translation is continuous in L^p (Exercise).

Step 3: ∂U is C^1. Hence, by definition, for every $x \in \partial U$ there exists $r_x > 0$ and a continuous function $\gamma_x \colon \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that

$$B(x, r_x) \cap U = \{\omega \in B(x, r_x) : \, \omega_n > \gamma_x(\omega_1, \ldots, \omega_{n-1})\}.$$

Therefore, $\{B(x, r_x) : x \in \partial U\}$ forms an open covering of ∂U. ∂U is compact (bounded and closed). Hence, there exists $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in \partial U$ such that

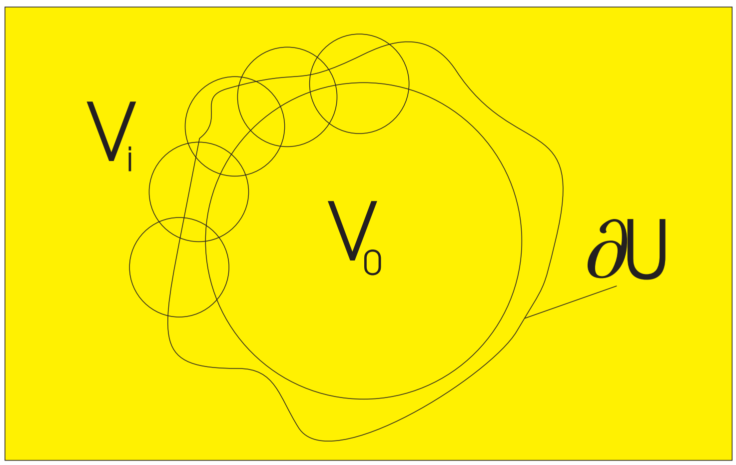
$$\partial U \subseteq \bigcup_{i=1}^{N} B(x_i, \frac{r_i}{2}). \tag{3.10}$$

Let $V_i = B(x_i, \frac{r_i}{2}) \cap U$. Choose $\delta > 0$. Steps 1-2 show that for every $1 \le i \le N$ there exists a function $v_i \in C^\infty(\overline{V}_i)$ with

$$\|v_i - u\|_{W^{k,p}(V_i)} \le \delta.$$

We choose $V_0 \subset\subset U$ such that $U \subseteq \bigcup_{i=0}^{N} V_i$. By Theorem 3.15 we get that there exists a function $v_0 \in C^\infty(\overline{V_0})$ such that

$$\|v_0 - u\|_{W^{k,p}(V_0)} \le \delta. \tag{3.11}$$



Let $(\xi_i)_{i=0}^{N}$ be a smooth partition of unity subordinate to the open cover

$$\{V_0, B(x_1, \frac{r_1}{2}), \ldots, B(x_N, \frac{r_N}{2})\}$$

of $\overline{U}$, i.e. $0 \le \xi_i \le 1$, $\xi_0 \in C_c^\infty(V_0)$, $\xi_i \in C_c^\infty(B(x_i, \frac{r_i}{2}))$, $1 \le i \le N$ and $\sum_{i=0}^{N} \xi_i = 1$ on $\overline{U}$. We define

$$v := \sum_{i=0}^{N} \xi_i v_i \ \in C^\infty(\overline{U}).$$

Then we have for all $|\alpha| \le k$:

$$\|v - u\|_{W^{k,p}(U)} = \left\| \sum_{i=0}^{N} \xi_i (v_i - u) \right\|_{W^{k,p}(U)} \le \sum_{i=0}^{N} \|\xi_i (v_i - u)\|_{W^{k,p}(U)}$$

$$\le C(N, k, p) \sum_{i=0}^{N} \|v_i - u\|_{W^{k,p}(V_i)} \le C\, N\, \delta =: \delta_0.$$

Note that $\|v\|_{W^{k,p}(U)} < \infty$. Summarizing we have that

$$\forall \delta_0 > 0\, \exists v \in C^\infty(\overline{U}) \cap W^{k,p}(U) : \ \|v - u\|_{W^{k,p}(U)} < \delta_0.$$

$\square$

REMARK 3.18.

$$\|\xi_i(v_i-u)\|_{W^{k,p}(V_i)} = \left(\sum_{|\alpha|\le k} \|D^\alpha(\xi_i(v_i-u))\|^p_{L^p(V_i)}\right)^{\frac{1}{p}}$$

and by Leibniz's formula

$$\begin{aligned}
|D^\alpha(\xi_i(v_i-u))(x)| &= \left|\sum_{\beta\le\alpha}\binom{\alpha}{\beta} D^{\alpha-\beta}\xi_i(x)\, D^\beta(v_i-u)(x)\right| \\
&\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left|D^{\alpha-\beta}\xi_i(x)\right|\left|\, D^\beta(v_i-u)(x)\right| \\
&\le \sup_{\beta\le\alpha}\left|D^{\alpha-\beta}\xi_i(x)\right| \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left|\, D^\beta(v_i-u)(x)\right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|D^\alpha(\xi_i(v_i-u))\|^p_{L^p(V_i)} &\le \sup_{x\in V_i}\sup_{\beta\le\alpha}\left|D^{\alpha-\beta}\xi_i(x)\right|^p \left\|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left|\, D^\beta(u-v_i)\right|\right\|^p_{L^p(V_i)} \\
&\le C_i(p,\alpha)^p \sup_{x\in V_i}\sup_{\beta\le\alpha}\left|D^{\alpha-\beta}\xi_i(x)\right|^p \sum_{\beta\le\alpha}\binom{\alpha}{\beta}^p \left\|\, D^\beta(u-v_i)\right\|^p_{L^p(V_i)}.
\end{aligned}$$

Summarizing we have

$$\begin{aligned}
\left(\sum_{|\alpha|\le k}\|D^\alpha(\xi_i(v_i-u))\|^p_{L^p(V_i)}\right)^{\frac{1}{p}} &\le C_i(p,k)\sup_{|\alpha|\le k}\sup_{x\in V_i}|D^\alpha\xi_i(x)| \left(\sum_{|\alpha|\le k}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}^p \left\|\, D^\beta(u-v_i)\right\|^p_{L^p(V_i)}\right)^{\frac{1}{p}} \\
&\le C_i(p,k)\sup_{|\alpha|\le k}\sup_{x\in V_i}|D^\alpha\xi_i(x)| \left(\sum_{|\alpha|\le k}\|\, D^\alpha(u-v_i)\|^p_{L^p(V_i)}\right)^{\frac{1}{p}} \\
&= C_i(p,k,V_i,\xi_i)\left(\sum_{|\alpha|\le k}\|\, D^\alpha(u-v_i)\|^p_{L^p(V_i)}\right)^{\frac{1}{p}}.
\end{aligned}$$

Actually an analogous proof (see [**1**, Theorem 3.18]) gives the following statement

THEOREM 3.19. *Let $U\subseteq\mathbb{R}^n$ be open and let it have the segment property (see* [**1**, p.54]*). Then the set of restrictions to U in $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(U)$, $1\le p<\infty$.*

and the following corollary

COROLLARY 3.20. *Let $1\le p<\infty$. Then*

$$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n). \tag{3.12}$$

CHAPTER 4

Extensions

In general, many properties of $W^{k,p}(U)$ can be inherited from $W^{k,p}(\mathbb{R}^n)$ provided U is "nice". The goal of this section is to extend functions in the Sobolev space $W^{k,p}(U)$ to become functions in the Sobolev space $W^{k,p}(\mathbb{R}^n)$. Indeed, we need a strong theorem. Observe for instance that extending $u \in W^{1,p}(U)$ by setting it zero in $\mathbb{R}^n \setminus U$ will not in general work, as we thereby create such a strong discontinuity along ∂U that the extended function no longer has a weak partial derivative. We must invent a way to extend u that preserves the weak derivatives across ∂U.

THEOREM 4.1 (Extension Theorem). *Let $k \in \mathbb{N}_0$. Let $1 \leq p \leq \infty$. Let $U \subset \mathbb{R}^n$ open and bounded and assume ∂U is C^k. Let $V \subset \mathbb{R}^n$ be open such that $U \subset\subset V$. Then there exists a linear and bounded operator*

$$E : W^{k,p}(U) \longrightarrow W^{k,p}(\mathbb{R}^n)$$

such that for all $u \in W^{k,p}(U)$:

(1) *$Eu = u$ a.e. on U,*
(2) $\operatorname{supp} Eu \subset V$,
(3) $\|E(u)\|_{W^{k,p}(\mathbb{R}^n)} \leq C(p,k,U,V)\, \|u\|_{W^{k,p}(U)}$.

DEFINITION 4.2. We call Eu an extension of u to $\mathbb{R}^n$.

DEFINITION 4.3 (essential support). Let $u \in W^{k,p}(U)$. Then the support of u is given by

$$\operatorname{supp}(u) = U \setminus \bigcup \{V \subseteq U \text{ open} :\ u = 0 \text{ a.e. on } V\}.$$

If necessary we write $\operatorname{ess\,supp}(u)$ (essential support) for the support of $u \in W^{k,p}(U)$ to avoid confusion with the classical support of a continuous function. Note that for a continuous function the essential support and the classical support coincide (Exercise).

PROOF. Let $k = 1$ and $1 \leq p < \infty$.

Step 1: Let $x_0 \in \partial U$. Suppose that ∂U is flat near x_0 and lies in the plane $\{x_n = 0\}$, see figure 4.1.

Then we may assume there exists $\delta > 0$ such that

$$B^+ := U \cap B(x_0, \delta) = B \cap \{x_n > 0\}$$
$$B^- := (\mathbb{R}^n \setminus U) \cap B(x_0, \delta) = B \cap \{x_n \leq 0\},$$

where $B = B(x_0, \delta)$.

Assume that $u \in C^1(\overline{U})$. We define

$$\overline{u}(x) = \begin{cases} u(x), & \text{if } x \in B^+, \\ u(x_1, \dots, x_{n-1}, -x_n) + 4 \cdot u(x_1, \dots, x_{n-1}, -\frac{x_n}{2}), & \text{if } x \in B^-. \end{cases} \tag{4.1}$$

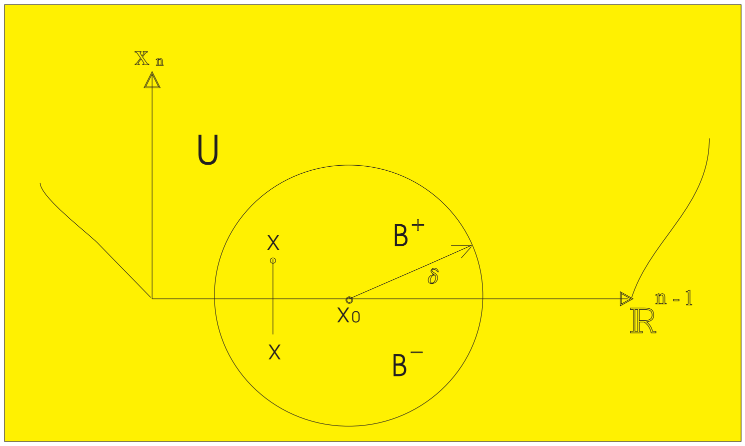


FIGURE 4.1. half-ball at the boundary

This is called a *higher-order reflection* of u from B^+ to B^-.

Step 2: We show that $\overline{u} \in C^1(B)$.

We use the notation $u^- := \overline{u}\big|_{B^-}$ and $u^+ := \overline{u}\big|_{B^+}$. Then we have

$$\lim_{x_n \to 0^-} u^-(x_1, \ldots, x_{n-1}, x_n) = \lim_{x_n \to 0^+} u^+(x_1, \ldots, x_{n-1}, x_n).$$

Obviously, for $1 \leq i \leq n-1$,

$$\frac{\partial u^-(x)}{\partial x_i} = -3\partial_{x_i} u(x_1, \ldots, -x_n) + 4\partial_{x_i} u(x_1, \ldots, -\frac{x_n}{2})$$

Hence, by the above

$$\lim_{x_n \to 0^-} \frac{\partial u^-}{\partial x_i}(x_1, \ldots, x_n) = \frac{\partial u}{\partial x_i}(x_1, \ldots, x_{n-1}, 0) = \lim_{x_n \to 0^+} \frac{\partial u^+}{\partial x_i}(x_1, \ldots, x_n)$$

For $i = n$ we have

$$\frac{\partial}{\partial x_n} u^-(x) = 3\partial_{x_n} u(x_1, \ldots, -x_n) - 2\partial_{x_n} u(x_1, \ldots, -\frac{x_n}{2})$$

and therefore,

$$\lim_{x_n \to 0^-} \frac{\partial u^-}{\partial x_n}(x_1, \ldots, x_n) = \partial_{x_n} u(x_1, \ldots, x_{n-1}, 0) = \lim_{x_n \to 0^+} \frac{\partial u^+}{\partial x_n}(x_1, \ldots, x_n)$$

Summarizing we have for all multiindices $|\alpha| \leq 1$

$$D^\alpha u^+|_{\{x_n=0\}} = D^\alpha u^-|_{\{x_n=0\}}.$$

Step 3 Using these calculations we have

$$\|\overline{u}\|_{W^{1,p}(B)} \leq C_p \|u\|_{W^{1,p}(B^+)}, \tag{4.2}$$

where C_p is some constant that does not depend on u. (Exercise).

Step 4 What happens if ∂U is not flat near x_0? We reduce the general case to the case where ∂U is flat near x_0. We need the assumption that ∂U is C^1. Then for every $x_0 \in \partial U$ there exists an $r > 0$ and a C^1-function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that we have

$$U \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, ..., x_{n-1})\}$$

and

$$\partial U \cap B(x_0, r) = \{x \in B(x_0, r) : x_n = \gamma(x_1, ..., x_{n-1})\}.$$

There exists a bijective C^1-function ϕ and its inverse ψ such that ϕ straightens out ∂U near x_0". Explicitly ϕ resp. ψ are given by

$$\phi : B = B(x_0, r) \longrightarrow W \subseteq \mathbb{R}^n,\ (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n - \gamma(x_1, \ldots, x_{n-1})).$$

$$\psi : W \longrightarrow B,\ (y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_{n-1}, y_n - \gamma(y_1, \ldots, y_{n-1})).$$

These two maps have the following properties:

(1) $\phi(\partial U \cap B) \longrightarrow \{x_n = 0\} \cap W$,
(2) $\phi \circ \psi = Id_B$ and $\psi \circ \phi = Id_W$

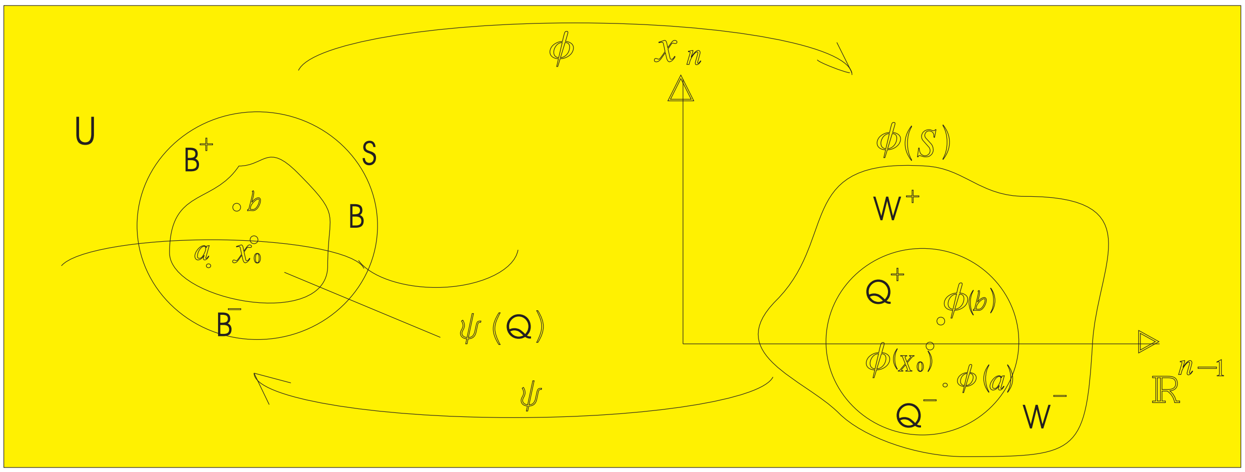


Figure 4.2. Straightening out the boundary

Let $y = \phi(x)$, $x = \psi(y)$ and $u'(y) = u(\psi(y))$ for $y \in W^+$. Then choose a small ball Q around $y_0 = \phi(x_0)$ and use the steps 1-3 to obtain a function $\overline{u}' \in C^1(Q)$ as extension of u' from Q^+ to Q satisfying

$$\|\overline{u'}\|_{W^{1,p}(Q)} \leq C \cdot \|u'\|_{W^{1,p}(Q^+)}.$$

Let $R := \psi(Q)$. We transform back to x-variables, and obtain an extension

$$\overline{u} : R \to \mathbb{R},\ x \mapsto \overline{u'}(\phi(x))$$

satisfying

$$\|\overline{u}\|_{W^{1,p}(R)} \leq C \cdot \|u\|_{W^{1,p}(U)}, \tag{4.3}$$

where C is independent of u.

Step 5: ∂U is compact and C^1. Therefore, there exist finitely many open sets U_i such that

$$\partial U \subseteq \bigcup_{i=1}^{N} U_i.$$

$U \subset\subset V$. Hence, we can arrange that $U_i \subseteq V$ for all i. We choose $U_0 \subset\subset U$ such that

$$\overline{U} \subset \bigcup_{i=0}^{N} U_i \subseteq V.$$

Subordinate to the cover $(U_i)_{i=0}^N$ of $\overline{U}$ there exists a smooth partition of unity, i.e. a sequence of smooth functions $(\xi_i)_{i=0}^N$ such that $0 \leq \xi_i \leq 1$, $\operatorname{supp} \xi_i \subseteq U_i$ and $\sum \xi_i = 1$ on $\overline{U}$.

Note that since $u \in C^1(\overline{U})$ we have that $\xi_i u \in C^1(\overline{U})$. According to Steps 1-4 there exists an extension

$$\overline{\xi_i u} \colon U_i \longrightarrow \mathbb{R},$$

such that

$$\|\overline{\xi_i u}\|_{W^{1,p}(U_i)} \leq C_i \|\xi_i u\|_{W^{1,p}(U)}. \tag{4.4}$$

Note that $\overline{\xi_0 u} = u$, since $\operatorname{supp} \xi_0 \subset U_0 \subset\subset U$. We define

$$\overline{u}(x) = \sum_{i=0}^N \overline{\xi_i u}(x), \qquad x \in \bigcup_{i=0}^N U_i \subset V$$

and $\overline{u} \equiv 0$ for $x \in \mathbb{R}^n \setminus (\bigcup_{i=0}^N U_i)$. Then $\operatorname{supp} \overline{u} \subset V$ and

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(U)} \tag{4.5}$$

for some constant C, independent of u.

Step 6: For $u \in C^1(\overline{U})$ and we can define our extension operator E as follows

$$Eu = \overline{u}. \tag{4.6}$$

The operator is linear and satisfies: $Eu = u$ on U, $\operatorname{supp} Eu \subset V$ and

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(U)}.$$

Step 7: Let $u \in W^{1,p}(U)$ and $1 \leq p < \infty$. We define the extension operator using a density argument. By Theorem 3.17 we can choose a sequence $(u_m)_{m=1}^\infty \in C^1(\overline{U})$ such that

$$\|u_m - u\|_{W^{1,p}(U)} \overset{m\to\infty}{\longrightarrow} 0. \tag{4.7}$$

By the linearity of E and equation (4.5) we have

$$\|E(u_m) - E(u_k)\|_{W^{1,p}(\mathbb{R}^n)} \leq C \cdot \|u_m - u_k\|_{W^{1,p}(U)}.$$

$(u_m)_{m=1}^\infty$ is Cauchy-sequence in $W^{1,p}(U)$. Hence, $(E(u_m))_{m=1}^\infty$ is Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$. Since $W^{1,p}(\mathbb{R}^n)$ is complete there exists a limit $\lim_{m\to\infty} Tu_m$ in $W^{1,p}(\mathbb{R}^n)$ and we can define

$$Eu = \lim_{m\to\infty} Eu_m. \tag{4.8}$$

The operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ defined in (4.8) is well-defined (i.e. does not depend on the choice of the sequence (u_m)) and satisfies the properties of the theorem.

Step 8: The case $p = \infty$ is left to the reader. $\square$

Remark 4.4.

(1) Theorem 4.1 is also true for the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}^+$. This is obtained by Step 1-3 of the proof.
(2) Assume that ∂U is C^2. Then the extension operator E constructed above is also a bounded linear operator from $W^{2,p}(U)$ to $W^{2,p}(\mathbb{R}^n)$.
(3) The above construction does **not** provide an extension for the Sobolev spaces $W^{k,p}(U)$, $k > 2$. This requires a more complicated higher-order reflection technique, see e.g. [**1**, Chapter 4].

CHAPTER 5

Traces

In the following chapter we discuss the possibility of assigning "boundary values" along ∂U to a function $u \in W^{1,p}(U)$, assuming that ∂U is C^1. If $u \in C(\overline{U})$, then u clearly has values on ∂U in the usual sense, but a typical function $u \in W^{1,p}(U)$ is in general not continuous and only defined almost everywhere in U. Since ∂U has n-dimensional Lebesgue-measure zero, there is no direct meaning we can give to the expression "u restricted to the boundary". To resolve this problems we need the *trace operator*.

THEOREM 5.1 (Trace Theorem). *Let $U \subseteq \mathbb{R}^n$ be open, bounded and ∂U is C^1. Let $1 \leq p < \infty$. Then there exists a linear bounded operator*

$$T : W^{1,p}(U) \longrightarrow L^p(\partial U)$$

such that

(1) $Tu = u\big|_{\partial U}$ *for all* $u \in W^{1,p}(U) \cap C(\overline{U})$

(2)
$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$$
for each $u \in W^{1,p}(U)$*, with the constant depending only on p and U.*

DEFINITION 5.2. We call Tu the trace of u on ∂U.

REMARK 5.3.

(1) There is a version of Theorem 5.1 for the Sobolev spaces $W^{k,p}(U)$ for $1 < k < \frac{n}{p}$, see e.g. [**1**, Theorem 5.22].

(2) There does not exist a bounded linear operator
$$T : L^p(U) \longrightarrow L^p(\partial U)$$
such that $T_u = u\big|_{\partial U}$ whenever $u \in C(\overline{U}) \cap L^p(U)$.

PROOF. Let $u \in C^1(\overline{U})$.

Step 1: Let $x_0 \in \partial U$. As in the proof of Theorem 4.1 we assume that ∂U is flat near x_0 lying in the plane $\{x_n = 0\}$. Choose an open ball $B = B(x_0, r)$ such that

$$B^+ := U \cap B = B \cap \{x_n > 0\},$$
$$B^- := (\mathbb{R}^n \setminus U) \cap B = B \cap \{x_n \leq 0\}.$$

Let $\hat{B} = B(x_0, \frac{r}{2})$ and $\Gamma = \hat{B} \cap \partial U$, see figure 5.1. We show that

$$\|u\|_{L^p(\Gamma)} \leq C(p, B^+)\|u\|_{W^{1,p}(B^+)}. \tag{5.1}$$

We choose $\xi \in C_c^\infty(B)$ such that $\xi \geq 0$ on B and $\xi = 1$ on $\hat{B}$. Then $\operatorname{supp} \xi u \subseteq B^+$ and $\xi u = u$ on Γ. Let $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Then, by the fundamental theorem and Hölder's inequality, we have

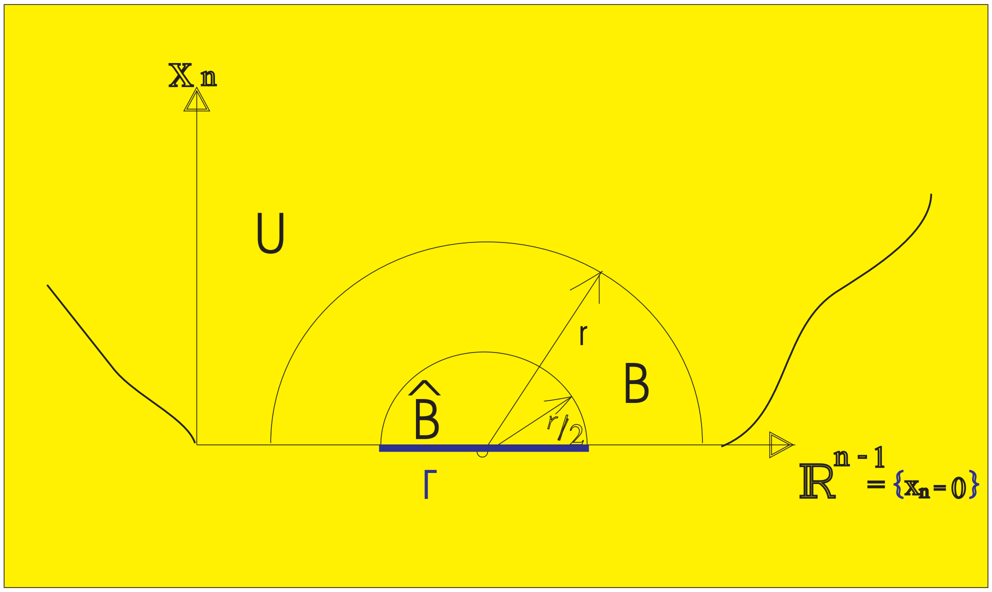


FIGURE 5.1.

$$
\begin{aligned}
|\xi u\,(x',0)|^p &\le \left(\int_0^\infty |(\xi u)_{x_n}(x',t)|dt\right)^p \\
&\le r^{p-1}\int_0^\infty |(\xi u)_{x_n}(x',t)|^p dt \\
&\le r^{p-1}2^{p-1}\int_0^\infty |\xi_{x_n}u(x',t)|^p + |\xi u_{x_n}(x',t)|^p dt.
\end{aligned}
$$

We integrate over Γ and obtain

$$
\begin{aligned}
\int_\Gamma |\xi u\,(x',0)|^p dx' &\le (2r)^{p-1}\int_{\mathbb{R}^{n-1}}\int_0^\infty |\xi_{x_n}u(x',t)|^p + |\xi u_{x_n}(x',t)|^p dt dx' \\
&\le C(p,B^+)\int_{B^+} |u(x)|^p + |u_{x_n}(x)|^p dx \\
&\le C(p,B^+)\|u\|_{W^{1,p}(B^+)}.
\end{aligned}
$$

This yields equation (5.1).

Step 2: Analogously to the proof of Theorem 4.1 we can straighten out the boundary near x_0 if necessary to obtain the setting in Step 1 (cf. figure 4.2). After transforming back to the original setting we obtain the estimate

$$
\int_\Gamma |u|^p\,dS \le C\|u\|^p_{W^{1,p}(U)}, \tag{5.2}
$$

where Γ is some open subset of ∂U containing x_0.

Step 3: ∂U is compact. Therefore, there exist finitely many open subsets Γ_i of ∂U such that

$$\partial U = \bigcup_{i=1}^{N} \Gamma_i$$

and by Step 1 and Step 2 we obtain

$$\|u\|_{L^p(\Gamma_i)} \leq C_i \|u\|_{W^{1,p}(U)}. \tag{5.3}$$

Therefore,

$$\|u\|_{L^p(\partial U)} = \int_{\partial U} |u|^p dx \leq \sum_{i=1}^{N} \int_{\Gamma_i} |u|^p dx = \sum_{i=1}^{N} \|u\|_{L^p(\Gamma_i)} \leq C(N,p)\|u\|_{W^{1,p}(U)}. \tag{5.4}$$

We define

$$Tu = u\big|_{\partial U} \tag{5.5}$$

then our previous estimate implies

$$\|Tu\|_{L^p(\partial U)} \leq C\,\|u\|_{W^{1,p}(U)} \;\;. \tag{5.6}$$

Hence, Theorem 5.1 is proven for $u \in C^1(\overline{U})$.

Step 4: Assume now $u \in W^{1,p}(U)$. Choose $(u_m)_{m=1}^{\infty} \in C^{\infty}(\overline{U}) \cap W^{1,p}(U)$ such that

$$\|u_m - u\|_{W^{1,p}(U)} \stackrel{m\to\infty}{\longrightarrow} 0\,.$$

By the equations (5.5) and (5.6) we have

$$\|T(u_m - u_l)\|_{L^p(\partial U)} \leq C\,\|u_m - u_l\|_{W^{1,p}(U)}.$$

Hence, $(Tu_m)_{m=1}^{\infty}$ is a Cauchy sequence in L^p with limit $\lim_{m\to\infty} Tu_m \in L^p$. We define

$$Tu = \lim_{m\to\infty}(Tu_m).$$

The operator Tu is well defined (does not depend on the choice of the sequence $(u_m)_{m=1}^{\infty}$), bounded and linear.

The sequence $(u_m)_{m=1}^{\infty}$ is constructed from u by smoothing by convolution (Theorem 3.17). If $u \in C(\overline{U}) \cap W^{1,p}(U)$ we have by Theorem 3.9 that (u_m) converges uniformly to u on compact subsets of $\overline{U}$, especially on ∂U. Therefore, $(Tu_m)_{m=1}^{\infty}$ converges uniformly to $u\big|_{\partial U}$ on ∂U. ☐

Of special interest are functions which have trace zero. In the following theorem we examine more closely what it means for a Sobolev function to have zero trace.

Theorem 5.4 (Trace Zero Theorem). *Let $U \subseteq \mathbb{R}^n$ be open. Assume U is bounded and ∂U is C^1. Let $1 \leq p < \infty$ and $u \in W^{1,p}(U)$. Then*

$$Tu \equiv 0 \Leftrightarrow u \in W_0^{1,p}(U)$$

Recall that

$$u \in W_0^{1,p}(U) \Longleftrightarrow \exists \text{ sequence } (u_m)_{m=1}^{\infty} \text{ in } C_c^{\infty} \text{ such that } \; \|u_m - u\|_{W^{1,p}(U)} \stackrel{m\to\infty}{\longrightarrow} 0.$$

PROOF. 1. "$\Leftarrow$": By the definition of the trace operator we have for $u \in W_0^{1,p}(U)$ with $(u_m)_{m=1}^\infty$ as above that

$$\begin{aligned}\|Tu\|_{L^p(\partial U)} &= \|Tu - Tu_m + Tu_m\|_{L^p(\partial U)}\\ &\le \|Tu - Tu_m\|_{L^p(\partial U)} + \|Tu_m\|_{L^p(\partial U)}\\ &= \|Tu - Tu_m\|_{L^p(\partial U)} \to 0.\end{aligned}$$

Therefore, $Tu \equiv 0$ with equality in L^p. We have that

$$W_0^{1,p}(U) \subseteq \{u \in W^{1,p}(U) : Tu \equiv 0\}.$$

2. "$\Rightarrow$": We show $Tu \equiv 0 \Rightarrow u \in W^{1,p}(U)$. By definition $Tu \equiv 0$ if and only if there exists a sequence (u_m) in $C^1(\overline{U})$ such that

$$u_m \to u \text{ in } W^{1,p}(U) \quad \text{ and } \quad \|Tu_m\|_{L^p(\partial U)} \to 0, \quad \text{ if } m \to \infty. \tag{5.7}$$

Using partition of unity and flattening out ∂U as before, we may assume that

$$\begin{aligned}&U = \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}^+ = \mathbb{R}^{n-1} \times \{x \in \mathbb{R}, x > 0\},\\ &u \in W^{1,p}(U) \text{ with compact support in } \overline{\mathbb{R}^n_+}.\end{aligned} \tag{5.8}$$

Recall that

$$\operatorname{supp}(u) = U \setminus \bigcup \{V \subseteq U \text{ open} : u = 0 \text{ a.e. on } V\}.$$

Step 1: Let $x' \in \mathbb{R}^{n-1},\ x_n \in \mathbb{R}^+$. Let (u_m) in $C^1(\overline{U})$ be as in equation (5.7). Then

$$u_m(x', x_n) = u_m(x', 0) + \int_0^{x_n} (u_m)_{x_n}(x', t) dt$$

and by the triangle inequality and inequality (11.2)

$$|u_m(x', x_n)|^p \le C_p \left(|u_m(x', 0)|^p + \left(\int_0^{x_n} |(u_m)_{x_n}(x', t)|\, dt \right)^p \right).$$

By Hölder's inequality we have

$$|u_m(x', x_n)|^p \le C_p \left(|u_m(x', 0)|^p + x_n^{p-1} \int_0^{x_n} |(u_m)_{x_n}(x', t)|^p dt \right)$$

We fix x_n and integrate over $\mathbb{R}^{n-1}$:

$$\int_{\mathbb{R}^{n-1}} |u_m(x', x_n)|^p dx' \le C_p \left(\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p dx' + x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |(u_m)_{x_n}(x', t)|^p dx' dt \right).$$

Let $m \to \infty$. Then by equation (5.7) we have

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \le C_p\, x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |u_{x_n}(x', t)|^p dx' dt. \tag{5.9}$$

Note that

$$\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p dx' = \|Tu_m\|_{L^p(\mathbb{R}^{n-1})} \to 0, \text{ if } m \to \infty.$$

Step 2: Let $\zeta \in C^\infty(\overline{\mathbb{R}^+})$ satisfying $0 \le \zeta \le 1$ and

$$\zeta\big|_{[0,1]} = 1, \qquad \zeta\big|_{\overline{\mathbb{R}^+} \setminus [0,2]} = 0.$$

Let $x = (x', x_n) \in \mathbb{R}^n_+$ and define the function $\zeta_m(x) = \zeta(m x_n)$.

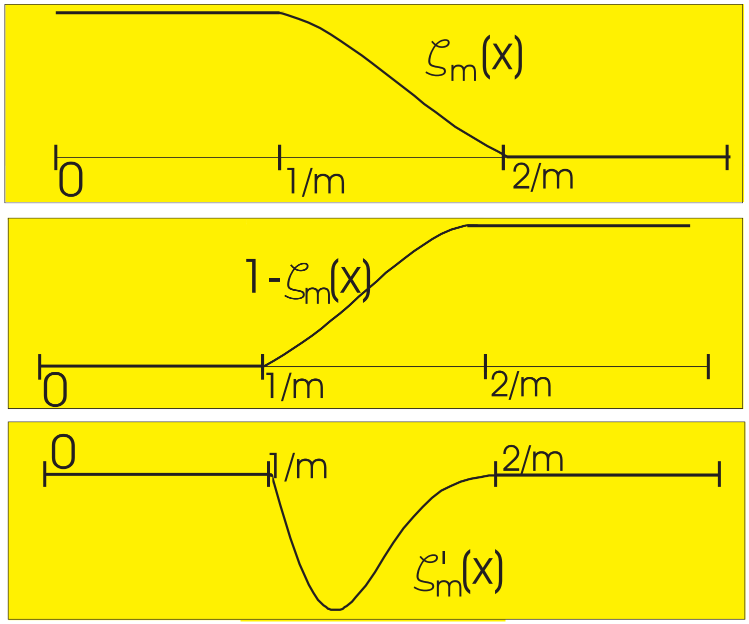


FIGURE 5.2. $\zeta_m(x)$

The function

$$w_m(x) = (1 - \zeta_m(x))u(x)$$

is in $W^{1,p}(\mathbb{R}^n_+)$ and

$$\operatorname{supp} w_m \subseteq \{(x', t) \in \operatorname{supp} u : \, t > \frac{1}{m}\}.$$

We show that

$$\|w_m - u\|_{W^{1,p}(\mathbb{R}^n_+)} \to 0, \quad m \to \infty. \tag{5.10}$$

The weak partial derivatives of w_m are given by

$$(w_m)_{x_n}(x', x_n) = u_{x_n}(x', x_n)(1 - \zeta_m(x', x_n)) - m\zeta'(mx_n)u(x', x_n)$$

and for $1 \leq i < n$

$$(w_m)_{x_i}(x) = (1 - \zeta_m(x))u_{x_i}(x).$$

Hence, for $1 \leq i < n$

$$\begin{aligned}\int_{\mathbb{R}^n_+} |(w_m)_{x_i} - u_{x_i}|^p dx &= \int |(1 - \zeta_m(x))u_{x_i}(x) - u_{x_i}(x)|^p dx \\ &= \int |\zeta_m(x)u_{x_i}(x)|^p dx \overset{m \to \infty}{\longrightarrow} 0.\end{aligned}$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^n_+} |(w_m)_{x_n} - u_{x_n}|^p dx &= \int_{\mathbb{R}^{n-1}} \int_0^\infty |\zeta_m(x',t) u_{x_n}(x',t) + m\, u(x',t)\zeta'(mt)|^p dt dx' \\
&\le C_p \left(\int_{\mathbb{R}^n_+} |\zeta_m u_{x_n}|^p dx + m^p \int_{\mathbb{R}^{n-1}} \int_0^\infty |\zeta'(mt) u(x',t)|^p dt dx' \right) \\
&=: C_p \left(A_m + B_m \right).
\end{aligned}
$$

Since $u \in W^{1,p}(\mathbb{R}^n_+)$ we have that $u_{x_n} \in L^p(\mathbb{R}^n_+)$. By definition $\zeta_m \le 1$ and $\zeta_m \to 0$ for all $x \in \mathbb{R}^n_+$. Therefore, by Lebesgue domination theorem

$$
A_m \to 0, \quad \text{ if } m \to \infty.
$$

We use Step 1 in order to get an estimate for B_m. By equation (5.9) we have

$$
\begin{aligned}
B_m &= m^p \int_{\mathbb{R}^{n-1}} \int_0^{\frac{2}{m}} |\zeta'(mt) u(x',t)|^p dt dx' \\
&\le C_p m^p \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{n-1}} |u(x',t)|^p dx' dt \\
&\le C_p\, m^p \int_0^{\frac{2}{m}} s^{p-1} \left(\int_0^s \int_{\mathbb{R}^{n-1}} |u_{x_n}(x',t)|^p dx' dt \right) ds \\
&\le C_p m^p \left(\int_0^{\frac{2}{m}} s^{p-1} ds \right) \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{n-1}} |u_{x_n}(x',t)|^p dx' dt \\
&= C_p \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{n-1}} |u_{x_n}(x',t)|^p dx' dt \stackrel{m\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Hence,

$$
\int_{\mathbb{R}^n_+} |(w_m)_{x_n} - u_{x_n}|^p dx \to 0, \quad if m \to \infty.
$$

Summarizing we have equation (5.10).

Step 4: We use smoothing by convolution to construct a sequence $(\tilde{u}_m)$ in $C_c^\infty(\mathbb{R}^n_+)$. Let $\varepsilon_m = \frac{1}{m}$. We know that $w_m = 0$ for all $(x', x_n) \in \mathbb{R}^{n-1} \times (0, \frac{1}{m})$ and w_m has compact support in $\mathbb{R}^n_+$. Hence, $\omega_m * \eta_{\varepsilon_m}$ is well defined on $\mathbb{R}^n_+$ and has compact support. Therefore, $\omega * \eta_{\varepsilon_m} \in C_c^\infty(\mathbb{R}^n_+)$ and by Theorem 3.9

$$
\|w_m * \eta_{\varepsilon_m} - w_m\|_{W^{1,p}} \to 0.
$$

Let $\tilde{u}_m = w_m * \eta_{\varepsilon_m}$. Then

$$
\|\tilde{u}_m - u\|_{W^{1,p}} \le \|\tilde{u}_m - w_m\|_{W^{1,p}} + \|w_m - u\|_{W^{1,p}} \to 0, \quad m \to \infty.
$$

$\square$

CHAPTER 6

Sobolev inequalities

In this chapter we prove a class of inequalities of the form

$$\|u\|_X \leq C\|u\|_{W^{k,p}(U)}, \tag{6.1}$$

where X is a Banach space, i.e. we consider the question: "If $u \in W^{k,p}(U)$, does u belong automatically to a certain other Banach space X?" Inequalities of the form (6.1) are called **Sobolev type inequalities**. This kind of estimates give us information on the embeddings of Sobolev spaces into other spaces.

Recall that we say that a Banach space E is *continuously embedded* into another Banach space F, written $E \hookrightarrow F$ if there exists a constant C such that for all $x \in E$

$$\|x\|_F \leq C\|x\|_E. \tag{6.2}$$

This means that the natural inclusion map $i : E \to F$, $x \mapsto x$ is continuous.

We start the investigations with the Sobolev spaces $W^{1,p}(U)$ and will observe that these Sobolev spaces indeed embed into certain other spaces, but which other spaces depends upon whether

$$1 \leq p < n, \tag{6.3}$$

$$p = n, \tag{6.4}$$

$$n < p \leq \infty. \tag{6.5}$$

The case (6.3) is covered by the Gagliardo-Nirenberg-Sobolev inequality, see Section 6.1 and the case (6.5) is covered by the so called Morrey's inequality, see Section (6.5).

6.1. Gagliardo-Nirenberg-Sobolev inequality

Definition 6.1. If $1 \leq p < n$, the Sobolev conjugate of p is

$$p^* = \frac{np}{n-p}.$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \quad \text{and} \quad p* > p. \tag{6.6}$$

Theorem 6.2 (Gagliardo-Nirenberg-Sobolev Inequality). *Let $1 \leq p < n$. There exists a constant C, depending only on n and p such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \tag{6.7}$$

for all $u \in C^1_c(\mathbb{R}^n)$.

Motivation. We first demonstrate that if any inequality of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}, \tag{6.8}$$

for certain constants $C > 0$, $1 \leq q < \infty$ and functions $u \in C_c^\infty(\mathbb{R}^n)$ holds, then the number q cannot be arbitrary. Let $u \in C_c^\infty(\mathbb{R}^n)$, $u \not\equiv 0$ and define for $\lambda > 0$

$$u_\lambda(x) := u(\lambda x) \qquad (x \in \mathbb{R}^n)\ .$$

We assume that (6.8) holds and apply it to u_λ, i.e. there exists a constant C such that for all $\lambda > 0$

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C\|Du_\lambda\|_{L^p(\mathbb{R}^n)}. \tag{6.9}$$

Now

$$\int_{\mathbb{R}^n} |u_\lambda(x)|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \frac{1}{\lambda^n}\int_{\mathbb{R}^n} |u(y)|^q dy$$

and

$$\int_{\mathbb{R}^n} |Du_\lambda(x)|^p dx = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n}\int_{\mathbb{R}^n} |Du(y)|^p dy.$$

Hence, by (6.9) we get

$$\Big(\frac{1}{\lambda^n}\Big)^{\frac{1}{q}}\|u\|_{L^q(\mathbb{R}^n)} \leq C\Big(\frac{\lambda^p}{\lambda^n}\Big)^{\frac{1}{p}}\|Du\|_{L^p(\mathbb{R}^n)}$$

and therefore,

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}}\|Du\|_{L^p(\mathbb{R}^n)}.$$

If $1 - \frac{n}{p} + \frac{n}{q} \neq 0$ we can obtain a contradiction by sending λ to 0 or ∞, depending on whether $1 - \frac{n}{p} + \frac{n}{q} > 0$ or $1 - \frac{n}{p} + \frac{n}{q} < 0$. Thus, if in fact the desired inequality (6.1) holds, we must necessarily have $1 - \frac{n}{p} + \frac{n}{q} = 0$. This implies that $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and therefore, $q = \frac{np}{n-p}$.

Proof. Assume $p = 1$. Note that u has compact support. Therefore, we have for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)dy_i$$

and

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)|dy_i\ .$$

Then

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)|\, dy_i\right)^{\frac{1}{n-1}}.$$

We integrate the above inequality with respect to x_1 and obtain:

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\leq \int_{-\infty}^{\infty}\prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)|\, dy_i\right)^{\frac{1}{n-1}} dx_1\\ &= \left(\int_{-\infty}^{\infty}|Du|\, dy_1\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{i=2}^n\left(\int_{-\infty}^{\infty}|Du|\, dy_i\right)^{\frac{1}{n-1}} dx_1.\end{aligned}$$

Applying the general Hölder inequality (Theorem 11.6) with $p_i = \frac{1}{n-1}$, $i = 1, \ldots, n-1$ we obtain

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \le \left(\int_{-\infty}^{\infty} |Du| \, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, dx_1 \, dy_i \right)^{\frac{1}{n-1}}.$$

Now we integrate with respect to x_2 and obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 &\le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Du| \, dy_1 \right)^{\frac{1}{n-1}} dx_2 \int_{-\infty}^{\infty} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, dx_1 \, dy_i \right)^{\frac{1}{n-1}} dx_2 \\ &= \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, dx_1 \, dy_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1, i \ne 2}^{n} I_i^{\frac{1}{n-1}} dx_2,\end{aligned}$$

where

$$I_1 = \int_{-\infty}^{\infty} |Du| dy_1 \quad \text{and} \quad I_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \ \text{ for } i = 3, \ldots, n.$$

Applying the general Hölder inequality once more we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \le &\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, dx_1 \, dy_2 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, dy_1 \, dx_2 \right)^{\frac{1}{n-1}} \\ &\prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, dx_1 \, dx_2 \, dy_i \right)^{\frac{1}{n-1}}.\end{aligned}$$

We continue by integrating with respect to $x_3, \ldots, x_n$ and and using Hölder's general inequality to obtain finally

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| \, dx_1 \ldots dy_i \ldots dx_n \right)^{\frac{1}{n-1}} = \left(\int_{\mathbb{R}^n} |Du| dx \right)^{\frac{n}{n-1}}. \tag{6.10}$$

This is the Gagliardo-Nirenberg-Sobolev inequality for p=1.

We consider now the case $1 < p < n$. Let $v := |u|^{\gamma}$ for some $\gamma > 1$. We apply (6.10) to v. Then, by Hölder's inequality (Theorem 11.5)

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx \right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n} |D|u|^{\gamma}| \, dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma - 1} |Du| \, dx \\ &\le \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p dx \right)^{\frac{1}{p}}.\end{aligned} \tag{6.11}$$

We choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$. That is, we set

$$\gamma = \frac{p(n-1)}{n-p}$$

in which case $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$. Therefore, (6.11) becomes

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{n-1}{n}} \leq \gamma \left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{\frac{1}{p}}$$

what is equal to

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{n-1}{n} - \frac{p-1}{p}} \leq \gamma \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{\frac{1}{p}}.$$

Note that

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{1}{p^*} \quad \text{and } \gamma = C(n,p).$$

□

6.2. Estimates for $W^{1,p}$ and $W_0^{1,p}$, $1 \leq p < n$

The Gagliardo-Nirenberg-Sobolev inequality (Theorem 6.2) gives the continuous embedding of $W^{1,p}(U)$, $1 \leq p < n$, into the space L^{p^*}, where p^* is the Sobolev conjugate of p.

THEOREM 6.3. *Let $U \subseteq \mathbb{R}^n$ open and bounded and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$ with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}, \tag{6.12}$$

where the constant C depends on n,p and U. In particular, we have for all $1 \leq q \leq p^*$

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}. \tag{6.13}$$

PROOF. The Extension Theorem (Theorem 4.1) yields that there exists an extension $\overline{u} = Eu \in W^{1,p}(\mathbb{R}^n)$ such that $\overline{u} = u$ in U, $\overline{u}$ has compact support and

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \tag{6.14}$$

Because $\overline{u}$ has compact support we know from Theorem 3.15 that there exists a sequence $(u_m)_{m=1}^{\infty}$ of functions in $C_c^{\infty}(\mathbb{R}^n)$ such that

$$u_m \to \overline{u} \quad \text{in } W^{1,p}(\mathbb{R}^n). \tag{6.15}$$

Now according to Theorem 6.2 we have that for all $l, m \geq 1$

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}. \tag{6.16}$$

Thus, by equation (6.15) and (6.16),

$$u_m \to \overline{u} \quad \text{in } L^{p^*}. \tag{6.17}$$

By the Gagliardo-Nirenberg-Sobolev inequality we have

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)}$$

and hence,

$$\|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\overline{u}\|_{L^p(\mathbb{R}^n)} \leq C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)}. \tag{6.18}$$

Therefore, by the properties of the extension $\overline{u}$ we have

$$\|u\|_{L^{p^*}(U)} = \|\overline{u}\|_{L^{p^*}(U)} \leq \|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

Since $|U| < \infty$ we obtain by Hölder's inequality equation (6.13). $\square$

REMARK 6.4. The boundedness of U is not essential. Remark 4.4 gives the statement (6.12) for $U = \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}^+$. Corollary 3.20 gives the statement (6.12) for $U = \mathbb{R}^n$.

THEOREM 6.5. *Let $U \subseteq \mathbb{R}^n$ be open and bonded. Let $u \in W^{1,p}_0(U)$, $1 \leq p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$. The constant depends only on p, q, n and U.

In particular , for all $1 \leq p < n$,

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}.$$

PROOF. Let $u \in W^{1,p}_0$. Then there exists a sequence $(u_m)_{m=1}^\infty$ in $C^\infty_c(U)$ such that $u_m \to u$ in $W^{1,p}(U)$. Now we extend each function u_m to be 0 on $\mathbb{R}^n \setminus \overline{U}$. Analogously to the above proof we get from the Gagliardo-Nirenberg-Sobolev inequality (Theorem 6.2) the following estimate

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}.$$

Since U is bounded we have $|U| < \infty$ and therefore, for every $1 \leq q \leq p^*$ the following estimate holds

$$\|u\|_{L^q(U)} \leq C\|u\|_{L^{p^*}(U)}.$$

$\square$

6.3. Alternative proof of the Gagliardo-Nirenberg-Sobolev inequality

DEFINITION 6.6 (Maximal function). Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then

$$M(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

is the maximal function of f.

THEOREM 6.7 (Hardy-Littlewood maximal inequality). *If $f \in L^1(\mathbb{R}^n)$, then for every $\lambda > 0$*

$$|\{M(f) > \lambda\}| \leq C_1 \frac{\|f\|_1}{\lambda}, \tag{6.19}$$

where C_1 is a constant which depends only on the dimension n. We say that $M(f)$ is of weak type $(1,1)$.

If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, then

$$\|M(f)\|_p \leq C_p \|f\|_p\,, \tag{6.20}$$

where C_p depends only on n and p.

PROOF. See [**10**]. $\square$

REMARK 6.8. The proof of inequality (6.20) is based on a typical interpolation argument: If we have

$$\|M(f)\|_1 \leq C_1 \|f\|_1 \tag{6.21}$$

and

$$\|M(f)\|_\infty \leq C_p \|f\|_\infty \tag{6.22}$$

then it follows that

$$\|M(f)\|_p \leq C_p \|f\|_p \quad 1 < p < \infty.$$

Note that $M(f)$ does not satisfy (6.21), but the weaker estimate (6.19). Hence, we need a stronger interpolation argument: Estimate (6.19) and

$$\|M(f)\|_\infty \leq C \|f\|_\infty$$

yield

$$\|M(f)\|_p \leq C_p \|f\|_p \quad 1 < p < \infty.$$

Definition 6.9 (Riesz potentials). Let $f \in L^1_{loc}(\mathbb{R}^n)$ be a non-negative function. The *Riesz potential* of f of order 1 is given by

$$I_1(f)(x) = (|\cdot|^{1-n} * f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}} dy.$$

(The Riesz potential of order $\alpha > 0$ would be $I_\alpha(f)(x) = (|\cdot|^{\alpha-n} * f)(x)$.)

Proposition 6.10. *If $f \in L^1(\mathbb{R}^n)$, then for all $\lambda > 0$*

$$|\{I_1(|f|) > \lambda\}| \leq C_1(n) \left(\frac{\|f\|_1}{\lambda}\right)^{\frac{n}{n-1}}, \tag{6.23}$$

where $C_1(n)$ is some positive constant that depends only on n.

If $f \in L^p(\mathbb{R}^n)$, $1 < p < n$, then

$$\|I_1(|f|)\|_{\frac{np}{n-p}} \leq C_p(n) \|f\|_p, \tag{6.24}$$

where $C_p(n)$ is some positive constant that depends only on n and p.

Proof. We may assume that $f \geq 0$. Given $\delta > 0$ we divide the integral defining $I_1(f)$ into a good part and a bad part.

$$I_1(f)(x) = \int_{B(x,\delta)} \frac{f(y)}{|x-y|^{n-1}} dy + \int_{\mathbb{R}^n \setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-1}} dy = b_\delta(x) + g_\delta(x). \tag{6.25}$$

We get an estimate for the "good part" $g_\delta(x)$ by Hölder's inequality. For $p > 1$ we have

$$g_\delta(x) = \int_{\mathbb{R}^n \setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-1}} dy \leq \|f\|_p \left(\int_{\mathbb{R}^n \setminus B(x,\delta)} |x-y|^{q(1-n)} dy\right)^{\frac{1}{q}},$$

where $q = \frac{p}{p-1}$. Substituting $z = y - x$ in the second term on the right-hand side and applying integration in polar coordinates (Theorem 11.16) yields

$$\begin{aligned}
\left(\int_{\mathbb{R}^n \setminus B(x,\delta)} |x-y|^{q(1-n)} dy\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}^n \setminus B(0,\delta)} |z|^{q(1-n)} dz\right)^{\frac{1}{q}} \\
&= \left(\int_\delta^\infty \int_{S^{n-1}} r^{q(1-n)+n-1} dS(\omega) dr\right)^{\frac{1}{q}} \\
&= |\partial B(0,1)|^{\frac{p-1}{p}} c(n,p) \delta^{\frac{p-n}{p}}.
\end{aligned}$$

Summarizing we obtain for $p > 1$

$$g_\delta(x) \leq C(n,p) \|f\|_p \delta^{\frac{p-n}{p}}. \tag{6.26}$$

For $p = 1$ we obtain:

$$g_\delta(x) = \int_{\mathbb{R}^n \setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-1}} dy \leq \|f\|_1 \sup_{y \in \mathbb{R}^n \setminus B(x,\delta)} |x-y|^{1-n} \leq \|f\|_1 \delta^{1-n}.$$

The "bad part" $b_\delta(x)$ can be dealt with by a maximal function argument. Let

$$B_j = B(x, 2^{-j}\delta) \quad \text{and} \quad A_j = B_j \setminus B_{j+1}, \quad j \geq 0.$$

The sets A_j, $j \geq 0$ form a partition of the ball $B(x, \delta)$. Since

$$A_j = \{y \in \mathbb{R}^n : 2^{-(j+1)}\delta < |x-y| \leq 2^{-j}\delta\},$$

we have

$$\begin{aligned} b_\delta(x) &= \int_{B(x,\delta)} \frac{f(y)}{|x-y|^{n-1}} dy \\ &= \sum_{j \geq 0} \int_{A_j} \frac{f(y)}{|x-y|^{n-1}} dy \\ &\leq \sum_{j \geq 0} 2^{(j+1)(n-1)} \delta^{1-n} \int_{A_j} f(y) dy \\ &= 2^{n-1}\delta \sum_{j \geq 0} 2^{-j} (2^{-j}\delta)^{-n} \int_{A_j} f(y) dy. \end{aligned} \tag{6.27}$$

Since $f \geq 0$, $|A_j| \leq |B_j|$ and $|B_j| = (2^{-j}\delta)^n |B(0,1)|$, we get the following estimate

$$\sum_{j \geq 0} 2^{-j} (2^{-j}\delta)^{-n} \int_{A_j} f(y) dy \leq c(n) \sum_{j \geq 0} 2^{-j} \frac{1}{|B_j|} \int_{B_j} f(y) dy, \tag{6.28}$$

where $c(n) = |B(0,1)|$. Using the definition of the maximal function (Definition 6.6) we obtain

$$\sum_{j \geq 0} 2^{-j} \frac{1}{|B_j|} \int_{B_j} f(y) dy \leq 2M(f)(x). \tag{6.29}$$

Summarizing the estimates (6.27), (6.28) and (6.29) we obtain

$$b_\delta(x) \leq C(n) \delta M(f)(x). \tag{6.30}$$

Putting the estimates (6.26) and (6.29) into (6.25) yields the upper bound

$$I_1(f)(x) \leq C(n,p) \left(\delta M(f)(x) + \delta^{1-\frac{n}{p}} \|f\|_p \right). \tag{6.31}$$

Observe that the minimum of the right-hand side is attained with $\delta = C(n,p) \left(\frac{\|f\|_p}{M(f)(x)} \right)^{p/n}$. We get, by putting the minimum into equation (6.31), the following

$$I_1(f)(x) \leq C(n,p) \|f\|_p^{\frac{p}{n}} (M(f)(x))^{1-\frac{p}{n}}. \tag{6.32}$$

Hence,

$$\int_{\mathbb{R}^n} |I_1(f)(x)|^{\frac{np}{n-p}} dx \leq C(n,p) \|f\|_p^{\frac{p^2}{n-p}} \int_{\mathbb{R}^n} |M(f)(x)|^p dx.$$

Applying Theorem 6.7 for $p > 1$ yields

$$\int_{\mathbb{R}^n} |I_1(f)(x)|^{\frac{np}{n-p}} dx \leq C(n,p) \|f\|_p^{\frac{np}{n-p}}.$$

This is what we wanted to prove.

The case $p = 1$ is similar. We use the weak conclusion of Theorem 6.7, which asserts that

$$\forall \lambda > 0 : |\{M(f) > \lambda\}| \leq C_1 \frac{\|f\|_1}{\lambda}.$$

Hence, by equation (6.32) we have

$$|\{I_1(f) > \lambda\}| \leq C(n)(\lambda^{-1} \|f\|_1)^{\frac{n}{n-1}}.$$

$\square$

Gagliardo-Nirenberg inequality - another proof. Proposition 6.10 gives an alternative proof for Theorem 6.2.

Proof II Theorem 6.2. Let $u \in C_c^1(\mathbb{R}^n)$ and $1 < p < n$. For every $x \in \mathbb{R}^n$, $s \in \mathbb{R}$ and $\omega \in \partial B(0,1)$ we have

$$\begin{aligned} u(x+s\omega) - u(x) &= \int_0^s \frac{d}{dr} u(x+r\omega) dr \\ &= \int_0^s Du(x+r\omega) \cdot \omega \, dr. \end{aligned} \tag{6.33}$$

Since u has compact support, we have

$$\lim_{s\to\infty} u(x+s\omega) = 0$$

and therefore,

$$u(x) = -\int_0^\infty Du(x+r\omega) \cdot \omega \, dr. \tag{6.34}$$

Integration over $\partial B(0,1)$ yields

$$\int_{\partial B(0,1)} u(x) dS(\omega) = -\int_{\partial B(0,1)} \int_0^\infty Du(x+r\omega) \cdot \omega \, dr dS(\omega). \tag{6.35}$$

The left-hand side equals

$$c(n)u(x), \tag{6.36}$$

where $c(n) = |\partial B(0,1)|$. Using Fubini's theorem we get for the right-hand side of (6.35)

$$\int_{\partial B(0,1)} \int_0^\infty Du(x+r\omega) \cdot \omega \, dr dS = \int_0^\infty \int_{\partial B(0,1)} Du(x+r\omega) \cdot \omega \, dS dr. \tag{6.37}$$

Using the transformation formula in Theorem 11.16 yields

$$\int_0^\infty \int_{\partial B(0,1)} Du(x+r\omega) \cdot \omega \, dS(\omega) dr = \int_{\mathbb{R}^n} \frac{Du(x+z)z}{|z|^n} dz = \int_{\mathbb{R}^n} \frac{Du(y)(y-x)}{|x-y|^n} dy. \tag{6.38}$$

Summarizing the equations (6.35), (6.36), (6.37) and (6.38) we obtain

$$u(x) = C(n) \int_{\mathbb{R}^n} \frac{Du(y)(x-y)}{|x-y|^n} dy \tag{6.39}$$

and therefore,

$$|u(x)| \leq C(n) \int_{\mathbb{R}^n} \frac{|Du(y)|}{|x-y|^{n-1}} dy. \tag{6.40}$$

Hence, by Definition 6.9

$$|u(x)| \leq C(n) I_1(|Du|)(x). \tag{6.41}$$

Proposition 6.10 yields

$$\|I_1(|Du|)\|_{\frac{np}{n-p}} \leq C_p(n) \|Du\|_p . \tag{6.42}$$

Equations (6.41) and (6.42) give the statement

$$\|u\|_{\frac{np}{n-p}} \leq C(n,p) \|Du\|_p .$$

It remains to show the statement for $p=1$. We may assume that u is non-negative. The support of u can be written as union of the sets

$$A_j := \left\{ x \in \mathbb{R}^n : 2^j < u(x) \leq 2^{j+1} \right\} \quad , j \in \mathbb{Z}.$$

We consider the function

$$v_j(x) = \begin{cases} 0, & \text{if } u(x) \leq 2^j \\ u(x) - 2^j, & \text{if } 2^j < u(x) \leq 2^{j+1} \\ 2^j, & \text{if } 2^{j+1} < u(x). \end{cases} \tag{6.43}$$

Since $v_j(x) > 2^{j-1}$ if and only if $u(x) - 2^j > 2^{j-1}$, we obtain

$$\begin{aligned} |A_{j+1}| &= |\left\{ 2^{j+1} < u \leq 2^{j+2} \right\}| \leq |\left\{ u > 2^{j+1} \right\}| = |\left\{ u > 4 \cdot 2^{j-1} \right\}| \\ &\leq |\left\{ u > 3 \cdot 2^{j-1} \right\}| = |\left\{ v_j > 2^{j-1} \right\}|. \end{aligned} \tag{6.44}$$

The function v_j is continuous on $\mathbb{R}^n$ and compactly supported. Hence, by smoothing by convolution we can construct a sequence in $C_c^\infty(\mathbb{R}^n)$, which converges by Theorem 3.9 uniformly to v_j. This approximation argument allows us to apply the potential estimate (6.41) to v_j:

$$|v_j(x)| \leq C(n) I_1(|Dv_j|)(x). \tag{6.45}$$

Equation (6.45) and (6.44) yield

$$\begin{aligned} |A_{j+1}| &\leq |\left\{ v_j > 2^{j-1} \right\}| \\ &\leq |\left\{ I_1(|Dv_j|) > C(n)^{-1} 2^{j-1} \right\}|. \end{aligned}$$

Using the weak estimate (6.23) in Proposition 6.10 for $\lambda = C(n)^{-1} 2^{j-1}$ we get

$$|A_{j+1}| \leq C_1(n) \left(C(n) 2^{-j+1} \int_{\mathbb{R}^n} |Dv_j| dx \right)^{\frac{n}{n-1}} .$$

The definition of v_j yields that the support of Dv_j is contained in A_j and $Dv_j = Du$ on A_j. Hence,

$$|A_{j+1}| \leq C(n) \left(2^{-j} \int_{A_j} |Du| dx \right)^{\frac{n}{n-1}} . \tag{6.46}$$

By the definition of A_j we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx &= \sum_{j\in\mathbb{Z}} \int_{A_j} |u(x)|^{\frac{n}{n-1}} dx \\ &\leq \sum_{j\in\mathbb{Z}} \left(2^{j+1}\right)^{\frac{n}{n-1}} |A_j| \\ &= 2^{\frac{n}{n-1}} \sum_{j\in\mathbb{Z}} \left(2^{j+1}\right)^{\frac{n}{n-1}} |A_{j+1}|. \end{aligned} \tag{6.47}$$

Equation (6.46) yields

$$\begin{aligned} 2^{\frac{n}{n-1}} \sum_{j\in\mathbb{Z}} \left(2^{j+1}\right)^{\frac{n}{n-1}} |A_{j+1}| &\leq C(n) \sum_{j\in\mathbb{Z}} \left(\int_{A_j} |Du(x)| dx \right)^{\frac{n}{n-1}} \\ &\leq C(n) \left(\sum_{j\in\mathbb{Z}} \int_{A_j} |Du(x)| dx \right)^{\frac{n}{n-1}} \\ &= C(n) \left(\int_{\mathbb{R}^n} |Du(x)| dx \right)^{\frac{n}{n-1}} . \end{aligned} \tag{6.48}$$

Summarizing equation (6.47) and (6.48) yields the statement:

$$\|u\|_{\frac{n}{n-1}} \leq C(1,n) \|Du\|_1 .$$

$\square$

6.4. Hölder spaces

Morrey's inequality (Section 6.5) gives the continuous embedding of the Sobolev spaces $W^{1,p}(U)$, $p > n$ into spaces of Hölder continuous functions, the so called Hölder spaces.

Throughout this chapter let $U \subseteq \mathbb{R}^n$ be open, $0 < \gamma \leq 1$.

Definition 6.11 (Hölder continuous). A function $u : U \to \mathbb{R}$ is said to be *Hölder continuous* with exponent γ, if there exists a constant $C > 0$ such that for all $x, y \in U$

$$|u(x) - u(y)| \leq C|x - y|^{\gamma}.$$

For $\gamma = 1$ the function is said to be *Lipschitz continuous* and C is called Lipschitz constant.

Example 6.12. $f(x) = \sqrt{x}$, $x \in [0, 1]$, is Hölder continuous with exponent $\gamma = \frac{1}{2}$, but it is not Lipschitz continuous.

We show that

$$\forall\, x, y \in [0, 1] : \ |f(x) - f(y)| \leq |x - y|^{\frac{1}{2}},$$

i.e.

$$|\sqrt{x} - \sqrt{y}| \leq |x - y|^{\frac{1}{2}}. \tag{6.49}$$

Since $|x-y| = |\sqrt{x}+\sqrt{y}||\sqrt{x}-\sqrt{y}|$ we have

$$|\sqrt{x}-\sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| \leq \frac{\sqrt{|x|+|y|}}{\sqrt{x}+\sqrt{y}}|x-y|^{\frac{1}{2}} \leq \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}|x-y|^{\frac{1}{2}} = |x-y|^{\frac{1}{2}}.$$

Now assume f is Lipschitz continuous, i.e.

$$\exists L > 0\, \forall x, y \in [0,1] :\ |\sqrt{x}-\sqrt{y}| \leq L|x-y|.$$

Let $x = \frac{1}{n^2}$, $y = \frac{1}{n^4}$, then the following holds

$$\exists\, L > 0\, \forall\, n \in \mathbb{N} :\ |\frac{1}{n} - \frac{1}{n^2}| \leq L|\frac{1}{n^2} - \frac{1}{n^4}|.$$

This is equivalent to

$$\exists\, L > 0\, \forall\, n \in \mathbb{N} :\ |n-1| \leq L|1 - \frac{1}{n^2}| \leq L.$$

Such constant L can not exist. Therefore, f is not Lipschitz continuous.

Definition 6.13.

(1) If $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_\infty = \sup_{x \in U} |u(x)|.$$

(2) The γ^{th}- Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{0,\gamma} := \sup_{x \neq y \in U} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

and the γ^{th}- Hölder norm is defined by

$$\|u\|_{0,\gamma} := \|u\|_\infty + [u]_{0,\gamma}.$$

Definition 6.14 (Hölder space). Let $k \in \mathbb{N}_0$ and $0 < \gamma \leq 1$. The Hölder space $C^{k,\gamma}(\overline{U})$ consists of all functions $C^k(\overline{U})$ for which the norm

$$\|u\|_{k,\gamma} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_\infty + \sum_{|\alpha| = k} [D^\alpha u]_{0,\gamma}$$

is finite.

So the Hölder space consists of all the functions that are k-times continuously differentiable and whose k-th partial derivatives are bounded and Hölder continuous.

Theorem 6.15. *$(C^{k,\gamma}(\overline{U}), \|\cdot\|_{k,\gamma})$ is a Banach space.*

Proof. First we need to verify that $\|\cdot\|_{k,\gamma}$ indeed is a norm, so one has to check the norm properties:

(1) $\|u\| = 0 \Rightarrow u = 0$
(2) $\|\lambda u\| = |\lambda| \cdot \|u\|$
(3) $\|u+v\| \leq \|u\| + \|v\|$

which follow directly from $\|u\|_k := \sum_{\alpha\leq k} \|D^\alpha u\|_\infty$ being a norm and the seminorm properties of $[\,\cdot\,]_{0,\gamma}$.
Now let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C^{k,\gamma}(\overline{U})$, i.e.

$$\forall\, \varepsilon > 0\, \exists\, n(\varepsilon) > 0\, \forall\, m, n \geq n(\varepsilon) : \|u_m - u_n\|_{k,\gamma} \leq \varepsilon.$$

Then $(u_n)_{n\in\mathbb{N}}$ also is a Cauchy sequence in $(C^k(\overline{U}), \|\cdot\|_k)$, where

$$\|u\|_k := \sum_{\alpha\leq k} \|D^\alpha u\|_\infty.$$

This is a Banach space. Therefore, there exists a limit $u \in C^k(\overline{U})$.

What is left to show is that for any multiindex α with $|\alpha| \leq k$:

$$\lim_{n\to\infty} [D^\alpha u_n - D^\alpha u]_{0,\gamma} = 0.$$

We know that

$$\begin{aligned}
&\forall x, y \in U, x \neq y\, \forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall m, n \in \mathbb{N} : \\
&|D^\alpha u_n(x) - D^\alpha u_m(x) - D^\alpha u_n(y) + D^\alpha u_m(y)| < \varepsilon |x - y|^\gamma
\end{aligned}$$

and

$$\forall x \in U\, \forall \varepsilon > 0\, \exists \overline{N} \in \mathbb{N}\, \forall n > \overline{N} :\ |D^\alpha u_n(x) - D^\alpha u(x)| < \varepsilon.$$

We fix $x, y \in U$ and $\varepsilon > 0$. Then

$$\begin{aligned}
&|D^\alpha u_n(x) - D^\alpha u(x) - D^\alpha u_n(y) + D^\alpha u(y)| \\
&\quad \leq |D^\alpha u_n(x) - D^\alpha u_m(x) - D^\alpha u_n(y) + D^\alpha u_m(y)| + |D^\alpha u_m(x) - D^\alpha u(x)| \\
&\qquad + |D^\alpha u_m(x) - D^\alpha u(x)| \\
&\quad \leq \varepsilon |x - y|^\gamma + |D^\alpha u_m(x) - D^\alpha u(x)| + |D^\alpha u_m(x) - D^\alpha u(x)|
\end{aligned}$$

Let $m \to \infty$. Then for all $x, y \in U$ and for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$

$$|D^\alpha u_n(x) - D^\alpha u(x) - D^\alpha u_n(y) + D^\alpha u(y)| \leq \varepsilon |x - y|^\gamma.$$

Hence,

$$\lim_{n\to\infty} [D^\alpha u_n - D^\alpha u]_{0,\gamma} = 0$$

and $u \in C^{k,\gamma}(\overline{U})$. □

6.5. Morrey's inequality

THEOREM 6.16 (Morrey's inequality). *Let $n < p \leq \infty$. Then there exists a constant C, depending only on n and p such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}, \tag{6.50}$$

for all $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$, where $\gamma = 1 - \frac{n}{p}$.

PROOF. 1. We show that there exists a constant $C(n)$ such that for any $B(x,r) \subseteq \mathbb{R}^n$

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y)-u(x)|dy \leq C(n)\int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}dy. \tag{6.51}$$

Let $x \in \mathbb{R}^n$, $r > 0$ be fixed. Let $w \in \partial B(0,1)$ and $s < r$. Then

$$|u(x+sw)-u(x)| \leq \int_0^s \left|\frac{d}{dt}u(x+tw)\right|dt = \int_0^s |Du(x+tw)\cdot w|dt = \int_0^s |Du(x+tw)|dt.$$

Hence,

$$\int_{\partial B(0,1)} |u(x+sw)-u(x)|dS(w) \leq \int_{\partial B(0,1)}\int_0^s |Du(x+tw)|dt\, dS(w). \tag{6.52}$$

We apply Fubini to the right hand side and apply integration in polar coordinates (Theorem 11.16) to obtain

$$\begin{aligned}\int_{\partial B(0,1)}\int_0^s |Du(x+tw)|dt\, dS(w) &= \int_0^s\int_{\partial B(0,1)} |Du(x+tw)|dS(w)dt \\ &= \int_{B(x,s)} \frac{|Du(y)|}{|y-x|^{n-1}}dy.\end{aligned}$$

Now, multiplying equation (6.52) by s^{n-1} and integrating from 0 to r with respect to s, yields the inequality:

$$\int_0^r\int_{\partial B(0,1)} |u(x+sw)-u(x)|dS(w)s^{n-1}ds \leq \int_0^r s^{n-1}\int_{B(x,r)}\frac{|Du(y)|}{|y-x|^{n-1}}dy\ ds. \tag{6.53}$$

On the left-hand side of (6.53) we apply integration in polar coordinates to obtain

$$\int_{B(x,r)} |u(v)-u(x)|dv \leq \int_0^r s^{n-1}\,ds\int_{B(x,r)}\frac{|Du(y)|}{|y-x|^{n-1}}\,dy\ = \frac{r^n}{n}\int_{B(x,r)}\frac{|Du(y)|}{|y-x|^{n-1}}\,dy.$$

Note that $|B(x,r)| = r^n|B(0,1)| = r^nC(n)$. Hence, we have

$$\int_{B(x,r)} |u(v)-u(x)|dv \leq C(n)|B(x,r)|\int_{B(x,r)}\frac{|Du(y)|}{|y-x|^{n-1}}\,dy.$$

This is equation (6.51).

2. Now fix $x \in \mathbb{R}^n$. We apply equation (6.51) as follows

$$\begin{aligned}|u(x)| &\leq \frac{1}{|B(x,1)|}\int_{B(x,1)} |u(x)-u(y)|dy + \frac{1}{|B(x,1)|}\int_{B(x,1)} |u(y)|dy \\ &\leq \int_{B(x,1)}\frac{|Du(y)|}{|y-x|^{n-1}}dy + \frac{1}{|B(x,1)|}\int_{B(x,1)} |u(y)|dy \\ &= \int_{B(x,1)}\frac{|Du(y)|}{|y-x|^{n-1}}dy + \int_{B(x,1)} |u(y)|\frac{dy}{|B(x,1)|} \\ &\leq \int_{B(x,1)}\frac{|Du(y)|}{|y-x|^{n-1}}dy + \left(\int_{B(x,1)} |u(y)|^p\frac{dy}{|B(x,1)|}\right)^{\frac{1}{p}}.\end{aligned}$$

The last inequality holds, since $(B(x,1), \frac{dy}{|B(x,1)|})$ is a probability space. We apply Hölder's inequality to the first term on the right-hand side and obtain

$$|u(x)| \leq \left(\int_{B(x,1)} |Du(y)|^p dy\right)^{\frac{1}{p}} \left(\int_{B(x,1)} \frac{1}{|y-x|^{n-1\frac{p}{p-1}}} dy\right)^{\frac{p-1}{p}} + C(n,p)\|u\|_{L^p(B(x,1))}.$$

Hence, by integration in polar coordinates we have

$$\int_{B(x,1)} \frac{1}{|y-x|^{n-1\frac{p}{p-1}}} dy = C(n) \int_0^1 \frac{r^{n-1}}{r^{(n-1)\frac{p}{p-1}}} dr = \int_0^1 r^{(n-1)-(n-1)\frac{p}{p-1}} dr = \int_0^1 r^{-\frac{n-1}{p-1}} dr$$

Since $p > n$, we have $\frac{n-1}{p-1} < 1$. Therefore,

$$\int_0^1 r^{-\frac{n-1}{p-1}} dr = C(n,p) r^{\frac{p-n}{p-1}} \Big|_0^1 = C(n,p).$$

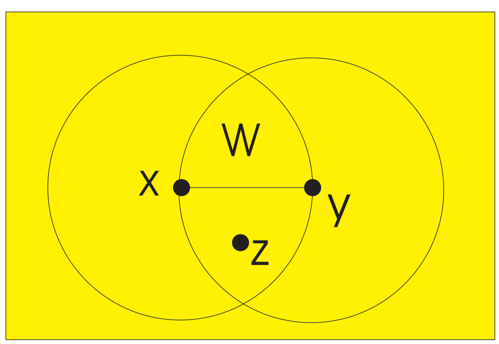
Summarizing we have

$$|u(x)| \leq C(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Since x was arbitrary, we can conclude

$$\sup_{x \in \mathbb{R}^n} |u(x)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{6.54}$$

3. Choose any two points $x, y \in \mathbb{R}^n$ and write $r := |x-y|$. Let $W = B(x,r) \cap B(y,r)$.



Then

$$\begin{aligned}
|u(x) - u(y)| &\leq \frac{1}{|W|} \int_W |u(x) - u(z)| dz + \frac{1}{|W|} \int_W |u(y) - u(z)| dz \\
&\leq \frac{C}{|B(x,r)|} \int_{B(x,r)} |u(x) - u(z)| dz + \frac{C}{|B(y,r)|} \int_{B(y,r)} |u(y) - u(z)| dz =: A + B
\end{aligned}$$

By inequality 6.51 we obtain

$$
\begin{aligned}
A &\leq C \int_{B(x,r)} \frac{|Du(z)|}{|x-z|^{n-1}} dz \\
&\leq \left(\int_{\mathbb{R}^n} |Du(z)|^p dz \right)^{\frac{1}{p}} \left(\int_{B(x,r)} \frac{1}{|x-z|^{(n-1)\frac{p}{p-1}}} dz \right)^{\frac{p-1}{p}} \\
&\leq C(n,p) \|Du\|_{L^p(\mathbb{R}^n)} r^{1-\frac{n}{p}} \\
&= C(n,p) \|Du\|_{L^p(\mathbb{R}^n)} |x-y|^{1-\frac{n}{p}}
\end{aligned}
$$

The same estimate holds for B. Therefore, we have the following estimate

$$
|u(x) - u(y)| \leq C \|Du\|_{L^p(\mathbb{R}^n)} |x-y|^{1-\frac{n}{p}}
$$

which implies

$$
\frac{|u(x) - u(y)|}{|x-y|^{\gamma}} \leq C \|Du\|_{L^p(\mathbb{R}^n)}, \qquad \forall x, y \in \mathbb{R}^n.
$$

Thus,

$$
[u]_{0,\gamma} = \sup_{x \neq y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{\gamma}} \leq C \|Du\|_{L^p} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{6.55}
$$

The inequalities (6.54) and (6.55) yield the statement. □

Remark 6.17. A slight variant of the proof above provides

$$
|u(x) - u(y)| \leq C r^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |Du(z)|^p dz \right)^{\frac{1}{p}}.
$$

for all $u \in C^1(B(x,2r))$, $y \in B(x,r) \subseteq \mathbb{R}^n$, $n < p < \infty$. The estimate is indeed valid if we integrate on the right hand side over $B(x,r)$ instead of $B(x,2r)$, but the proof is a bit trickier.

6.6. Estimates for $W^{1,p}$ and $W_0^{1,p}$, $n < p \leq \infty$

Definition 6.18. We say u^* is a version of a given function u provided

$$
u = u^* \quad \text{a.e.}
$$

Theorem 6.19. *Assume $u \in W^{1,p}(\mathbb{R}^n)$, $n < p \leq \infty$. Then u has a version $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$
\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{6.56}
$$

Proof. Use Corollary 3.20 and follow the proof of Theorem 6.20. □

Theorem 6.20. *Let $U \subseteq \mathbb{R}^n$ open, bounded and suppose ∂U is C^1. Assume $u \in W^{1,p}(U)$, $n < p \leq \infty$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$
\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C \|u\|_{W^{1,p}(U)}. \tag{6.57}
$$

PROOF. According to Theorem 4.1 there exists a compactly supported function $\overline{u} = Eu \in W^{1,p}(\mathbb{R}^n)$ such that $u = \overline{u}$ on U and

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \tag{6.58}$$

Since $\overline{u}$ has compact support, we obtain from Theorem 3.15 the existence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|u_m - \overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \to 0. \tag{6.59}$$

Now according to Theorem 6.16 we have for all $m, l \in \mathbb{N}$

$$\|u_m - u_l\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}. \tag{6.60}$$

$(u_m)_{m=1}^\infty$ converges to $\overline{u}$ in $W^{1,p}(\mathbb{R}^n)$, therefore it is Cauchy sequence in $C^{0,\gamma}(\mathbb{R}^n)$. Since this is a complete Banach space, there exists a function $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ such that

$$\|u_m - u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \to 0. \tag{6.61}$$

Owing to the equations (6.60) and (6.61) we see that $\overline{u} = u^*$ a.e. on $\mathbb{R}^n$, i.e. u^* is a version of $\overline{u}$. Note that $\overline{u} = u$ a.e. on U hence, u^* is a version of u on U.

Theorem 6.16 can be applied to the functions $u_m \in C_c^\infty(\mathbb{R}^n)$, i.e.

$$\|u_m\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$$

and therefore, by the equations (6.60),(6.61) and (6.58) we have

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

By the definition of the norm $\|\cdot\|_{C^{0,\gamma}}$ we have

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq \|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)}.$$

□

REMARK 6.21.

(1) Note that for $p = \infty$ we have the following inequality

$$\|u^*\|_{C^{0,1}(\overline{U})} \leq C\|u\|_{W^{1,\infty}(U)}.$$

Hence, every $u \in W^{1,\infty}(U)$ has a Lipschitz continuous version u^*.

(2) Let $n < p < \infty$. Inequality (6.57) does not hold for $\gamma \in (1 - \frac{n}{p}, 1]$.

(3) If $U \subseteq \mathbb{R}^n$ is bounded then

$$\|u^*\|_{C^{0,\beta}(\overline{U})} \leq C\|u\|_{W^{1,p}(U)}$$

for all $0 < \beta \leq \gamma$.

(4) Let $U \subseteq \mathbb{R}^n$ open and bounded. Assume $u \in W_0^{1,p}(U)$, $n < p \leq \infty$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ for $\gamma = 1 - \frac{n}{p}$, with the estimate

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|Du\|_{L^p}.$$

6.7. General Sobolev inequalities

THEOREM 6.22. *Let $U \subseteq \mathbb{R}^n$ be open and bounded with a C^1 boundary. Assume $u \in W^{k,p}(U)$, $1 \le p < \infty$.*

(1) *If $k < \frac{n}{p}$, then $u \in L^q(U)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ with*

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{k,p}(U)}, \tag{6.62}$$

where the constant C depends only on k, p, n and U.

(2) *If $k > \frac{n}{p}$, then $u \in C^{m,\gamma}$, where $m = k - \lfloor \frac{n}{p} \rfloor - 1$ and*

$$\gamma = \begin{cases} \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer,} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have the estimate

$$\|u\|_{C^{m,\gamma}(\overline{U})} \le C\|u\|_{W^{k,p}(U)}. \tag{6.63}$$

PROOF. **Case 1:** Let $k < \frac{n}{p}$. $u \in W^{k,p}(U)$, then $D^\beta u \in W^{1,p}(U)$ for all $|\beta| \le k-1$ and

$$\left\|D^\beta u\right\|_{W^{1,p}(U)} \le \|u\|_{W^{k,p}(U)}. \tag{6.64}$$

We apply Theorem 6.3 and obtain

$$\left\|D^\beta u\right\|_{L^{p^*}(U)} \le C\left\|D^\beta u\right\|_{W^{1,p}(U)} \le \|u\|_{W^{k,p}(U)}, \quad \text{for all} \quad |\beta| \le k-2. \tag{6.65}$$

Using this equation we obtain

$$\|u\|_{W^{k-1,p^*}(U)} \le C\|u\|_{W^{k,p}(U)}. \tag{6.66}$$

Set $p_1 = p^*$ and apply the same step again to $u \in W^{k-1,p_1}(U)$. Then we obtain

$$\left\|D^\beta u\right\|_{L^{p_1^*}(U)} \le C\|u\|_{W^{k,p}(U)}, \quad \text{for all} \quad |\beta| \le k-2. \tag{6.67}$$

Applying the step k-times yields the estimate

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{k,p}(U)}, \tag{6.68}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.

Case 2: Let $k > \frac{n}{p}$ and $\frac{n}{p}$ is no integer. Choose ℓ such that $\ell < \frac{n}{p} < \ell + 1$, i.e. $\ell = \lfloor \frac{n}{p} \rfloor$. Then $\ell < k$. Since $u \in W^{k,p}(U)$ we have $D^\beta u \in W^{\ell,p}(U)$ for all $|\beta| \le k - \ell$ and

$$\left\|D^\beta u\right\|_{W^{\ell,p}(U)} \le \|u\|_{W^{k,p}(U)}. \tag{6.69}$$

Since $\ell < \frac{n}{p}$ we can apply case 1 and obtain

$$\left\|D^\beta u\right\|_{L^q(U)} \le C\left\|D^\beta u\right\|_{W^{\ell,p}(U)}, \quad \text{for all} \quad |\beta| \le k-\ell,$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\ell}{n}$. Therefore,

$$\left\|D^\beta u\right\|_{L^q(U)} \le C\|u\|_{W^{k,p}(U)}, \quad \text{for all} \quad |\beta| \le k-\ell.$$

This equation yields

$$\|u\|_{W^{k-\ell,q}(U)} \le C\|u\|_{W^{k,p}(U)}. \tag{6.70}$$

Hence, $u \in W^{k-\ell,q}(U)$.

Note that $\frac{1}{q} = \frac{1}{p} - \frac{\ell}{n}$. Hence, $q = \frac{np}{n-\ell p}$. Since $\frac{n}{p} < \ell + 1$, we have $q > n$. We can apply Theorem 6.20 to $D^\beta u \in W^{1,q}$, $|\alpha| \le k - \ell - 1$ and obtain

$$\left\|D^\beta u\right\|_{C^{0,\gamma}(\overline{U})} \le C\left\|D^\beta u\right\|_{W^{1,q}(U)} \le C\|u\|_{W^{k-\ell,q}(U)}, \tag{6.71}$$

where $\gamma = 1 - \frac{n}{q} = \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}$.

Using the equations (6.70) and (6.71) yields

$$\begin{aligned}\|u\|_{C^{k-\lfloor \frac{n}{p} \rfloor - 1,\gamma}} &= \sum_{|\alpha| \le k - \lfloor \frac{n}{p} \rfloor - 1} \|D^\alpha u\|_\infty + \sum_{|\alpha| = k - \lfloor \frac{n}{p} \rfloor - 1} [D^\alpha u]_{0,\gamma} \\ &\le C\|u\|_{W^{k-\ell,q}(U)} \le C\|u\|_{W^{k,p}(U)}.\end{aligned}$$

.

Case 3: Let $k > \frac{n}{p}$ and $\frac{n}{p}$ is an integer. Set $\ell = \frac{n}{p} - 1$. Then $\ell < \frac{n}{p} < k$. Analogously to the case 2 we get that $u \in W^{k-\ell,q}$, where $\frac{1}{q} = \frac{1}{p} - \frac{\ell}{n}$. This implies $q = n$. Since $|U| < \infty$ we have $u \in W^{k-\ell,r}$ for all $r < n$. Hence, $D^\alpha u \in W^{1,r}(U)$ for all $|\alpha| \le k - \ell - 1$. We apply Theorem 6.3 and obtain

$$\begin{aligned}\|D^\alpha u\|_{L^{r^*}} &\le C\|D^\alpha u\|_{W^{1,r}(U)} \\ &\le C\|u\|_{W^{k-\ell,r}(U)} \le C\|u\|_{W^{k-\ell,n}}.\end{aligned}$$

Note that the last constant depends on $|U|$. Therefore, for all $|\alpha| \le k - \ell - 1 = k - \frac{n}{p}$ the functions $D^\alpha u$ are in L^s, $n \le s < \infty$.

Hence, $u \in W^{k-\frac{n}{p},s}(U)$ and $D^\alpha u \in W^{1,s}$ for all $\alpha \le k - \frac{n}{p} - 1$. Let $n < s < \infty$ and apply Theorem 6.20

$$\|D^\alpha u\|_{C^{0,\gamma}(\overline{U})} \le C\|D^\alpha u\|_{W^{1,s}(U)}, \qquad \text{for all} \quad |\alpha| \le k - \frac{n}{p} - 1,$$

where $\gamma = 1 - \frac{n}{s}$. Analogously to the case 2 we obtain

$$u \in C^{k-\frac{n}{p}-1,\gamma}(\overline{U}), \quad 0 < \gamma < 1$$

and

$$\|u\|_{C^{k-\frac{n}{p}-1,\gamma}} \le C\|u\|_{W^{k-\ell,n}(U)} \le C\|u\|_{W^{k,p}(U)}.$$

$\square$

Corollary 6.23. *Let $U \subseteq \mathbb{R}^n$ be open and bounded with a C^1 boundary. Let $j, k \in \mathbb{N}_0$ and $1 \le p < \infty$. Assume $u \in W^{k+j,p}(U)$. If $k < \frac{n}{p}$, then $u \in W^{j,q}(U)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ with*

$$\|u\|_{W^{j,q}(U)} \le C\|u\|_{W^{k+j,p}(U)}, \tag{6.72}$$

where the constant C depends only on k, p, n and U.

6.8. The borderline case

We established Sobolev inequalities for

(1) $k < \frac{n}{p}$ (Gagliardo-Nirenberg-Sobolev inequality)
(2) $k > \frac{n}{p}$ (Morrey's inequality)

What can we say about the case $n = \frac{k}{p}$?

Considering the fact that the Sobolev conjugate index $p^* = \frac{np}{n-p}$ converges to ∞ as $p \to n$, we might expect by the Gagliardo-Nirenberg-Sobolev inequality that $W^{1,n}(U)$ is continuously embedded in $L^\infty(U)$. This is however false for $n > 1$. Let $U = B(0,1) \subseteq \mathbb{R}^n$. The function $u(x) = \log\left(\log\left(1 + \frac{1}{|x|}\right)\right)$ belongs to $W^{1,n}(U)$ but not to $L^\infty(U)$.

LEMMA 6.24. *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $u \in W^{1,n}(U)$. Then there exists a constant C such that*

$$\|u\|_{L^p(U)} \leq C\|u\|_{W^{1,n}(U)}, \tag{6.73}$$

where

$$\begin{cases} p = \infty, & \text{if } n = 1, \\ 1 \leq p < \infty, & \text{if } n > 1. \end{cases}$$

REMARK 6.25.

(1) We will see in the proof that the boundedness of U is not necessary for the case $n = 1$. The same argument holds for $U = \mathbb{R}^n$ with constant equal to 1.

(2) If $U \subseteq \mathbb{R}^n$ bounded, then the constant C depends on U and n and in the case of $n > 1$ additionally on some arbitrarily chosen parameter $q \in [\max(n,p), \infty)$.

PROOF. 1. Let $n = 1$ and $u \in C_c^1(U)$. Let $x \in U$. Then

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(y)|dy.$$

Therefore,

$$\|u\|_\infty \leq \|Du\|_{L^1}.$$

By the same argument as in the proof of Theorem 6.3 we obtain that for $u \in W^{1,n}(U)$ the following equation holds

$$\|u\|_{L^\infty(U)} \leq C\|u\|_{W^{1,n}(U)}, \tag{6.74}$$

where C depends on n and U.

2. Let $n \geq 2$ and choose $n \leq q < \infty$. Set $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. Then $1 \leq s < n$ and $q = \frac{ns}{n-s}$. Note that since $|U| < \infty$ the following estimate holds

$$\|u\|_{W^{1,s}(U)} \leq n^{\frac{1}{s}-\frac{1}{n}}|U|^{1-\frac{s}{n}}\|u\|_{W^{1,n}(U)},$$

Theorem 6.3 yields

$$\|u\|_{L^{s*}} \leq C(n,s,U)\|u\|_{W^{1,s}(U)}.$$

Note that $s^* = q$. Therefore,

$$\|u\|_{L^q(U)} \leq C(n,q,U)\|u\|_{W^{1,n}(U)}.$$

Again, since $|U| < \infty$ the following estimate holds

$$\|u\|_{L^p(U)} \leq C(n,q,U)\|u\|_{W^{1,n}(U)}.$$

for all $1 \leq p \leq q$. Since $q < \infty$ was arbitrarily chosen, we have that for all $1 \leq p < \infty$ there exists a $q \in [n, \infty)$ with $p \leq q$ such that

$$\|u\|_{L^p(U)} \leq C(n,q,U)\|u\|_{W^{1,n}(U)}.$$

Note that the constant does depend on the choice of q. $\square$

REMARK 6.26.
(1) The space $W^{1,n}(\mathbb{R}^n)$ embeds into the space BMO (the space of bounded mean oscillation), cf. Remark 8.4.
(2) Trudinger's inequality (Theorem 6.28) gives the embedding of $W_0^{1,n}(U)$ in another Sobolev space.

THEOREM 6.27 (General case $k = \frac{n}{p}$). *Let $U \subseteq \mathbb{R}^n$ be open and bounded with a C^1 boundary. Assume $u \in W^{k,p}(U)$, $1 \leq p < \infty$. If $k = \frac{n}{p}$, then $u \in L^q(U)$ for all $p \leq q < \infty$ and*

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}} \tag{6.75}$$

where the constant C depends on k, n, p, q.

6.9. Trudinger inequality

THEOREM 6.28 (Trudinger Inequality). *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $u \in W_0^{1,n}(U)$. Then*

$$\int_U \exp\left\{\left(\frac{|u|}{c\,\|Du\|_n}\right)^{\frac{n}{n-1}}\right\} \leq C|U|, \tag{6.76}$$

where $c > 0$ and $C \geq 1$ are constants which depend only on n.

REMARK 6.29. This theorem yields that the Sobolev space $W_0^{1,n}(U)$, U bounded, is embedded in the Orlicz space $L^\varphi(U)$ with $\varphi(t) = \exp\left(|t|^{\frac{n}{n-1}}\right) - 1$. $L^\varphi(U)$ is the space of all measurable functions $u : U \to \mathbb{R}$ such that

$$\|u\|_{L^\varphi(U)} = \inf\{c > 0 : \int_U \varphi\left(\frac{|u(x)|}{c}\right)\, dx \leq 1\} < \infty.$$

See [**1**, Theorem 8.25]

Proof of Trudinger's inequality. Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $f \in L^1_{\text{loc}}(U)$. Recall (Definition 6.9) that the Riesz potential of f of order 1 is given by

$$I_1(f)(x) = (|\cdot|^{1-n} * f)(x) = \int_U \frac{f(y)}{|x-y|^{n-1}} dy. \tag{6.77}$$

PROPOSITION 6.30. *Let $U \subseteq \mathbb{R}^n$ be open and bounded, $n \leq q < \infty$ and $f \in L^n(U)$. Then*

$$\|I_1(f)\|_q \leq q^{1-\frac{1}{n}+\frac{1}{q}} |B(0,1)|^{1-\frac{1}{n}} |U|^{\frac{1}{q}} \|f\|_n. \tag{6.78}$$

REMARK 6.31. Proposition 6.30 is the special case $p = n$ of the following statement: Let $U \subseteq \mathbb{R}^n$ be open and bounded, $1 \leq p \leq n$ and

$$p \leq q < \frac{np}{n-p}.$$

Then, for $f \in L^p(U)$ and $\delta = \frac{1}{p} - \frac{1}{q} \geq 0$,

$$\|I_1(f)\|_q \leq \left(\frac{1-\delta}{\frac{1}{n}-\delta}\right)^{1-\delta} |B(0,1)|^{1-\frac{1}{n}} |U|^{\frac{1}{n}-\delta} \|f\|_p. \tag{6.79}$$

Note that Proposition 6.10 states that for $1 \leq p < n$

$$\|I_1(|f|)\|_{\frac{np}{n-p}} \leq C_p(n) \|f\|_p .$$

and for $p > n$

$$\|I_1(f)\|_\infty \leq C(n, p, |U|)\|f\|_p.$$

Proof. Let

$$\frac{1}{s} = 1 + \frac{1}{q} - \frac{1}{n}.$$

Since $n \leq q < \infty$ we have

$$1 \leq s < \frac{n}{n-1}.$$

We show that the function $h(y) = |y|^{1-n}$ is in $L^s(U)$. Let $R > 0$, so that $|U| = |B(0, R)| = |B(0, 1)|R^n$. Then

$$\int_U \frac{1}{|y|^{s(n-1)}} dy \leq \int_{B(0,R)} \frac{1}{|y|^{s(n-1)}} dy. \tag{6.80}$$

In order to prove equation (6.80) we have to consider two cases. Case 1: $B(0, R) \cap U \neq \emptyset$. Then

$$\begin{aligned}
\int_U |y|^{s(1-n)} dy &= \int_{U \setminus B} |y|^{s(1-n)} dy + \int_{U \cap B} |y|^{s(1-n)} dy \\
&\leq \int_{U \setminus B} R^{s(1-n)} dy + \int_{U \cap B} |y|^{s(1-n)} dy \\
&= |U \setminus B| R^{s(1-n)} + \int_{U \cap B} |y|^{s(1-n)} dy \\
&= |B \setminus U| R^{s(1-n)} + \int_{U \cap B} |y|^{s(1-n)} dy \\
&\leq \int_{B \setminus U} |y|^{s(1-n)} dy + \int_{U \cap B} |y|^{s(1-n)} dy \\
&= \int_B |y|^{s(1-n)} dy
\end{aligned}$$

Case 2: $B(0, R) \cap U = \emptyset$. Then

$$\int_U |y|^{s(1-n)} dy \leq \int_U R^{s(1-n)} dy = |U| R^{s(1-n)} = |B(0, R)| R^{s(1-n)} \leq \int_B |y|^{s(1-n)} dy.$$

Applying integration in polar coordinates (Theorem 11.16) to equation (6.80) yields

$$\begin{aligned}\int_{B(0,R)} |y|^{s(1-n)} dy &= \int_0^R \int_{S^{n-1}} r^{s(1-n)+n-1} dS(\omega) dr \\ &= |\partial B(0,1)| \frac{R^{s(1-n)+n}}{s(1-n)+n} \\ &= \frac{n}{s(1-n)+n} |B(0,1)| R^{s(1-n)+n} \\ &= \frac{1}{s} \frac{1}{1/n - 1 + 1/s} |B(0,1)| R^{sn(1/n-1+1/s)} \\ &= \frac{q}{s} |B(0,1)| R^{\frac{sn}{q}},\end{aligned}$$

where we used $\frac{1}{s} = 1 + \frac{1}{q} - \frac{1}{n}$. Since $|U| = |B(0,1)|R^n$, $s \geq 1$ and $\frac{1}{s} = 1 + \frac{1}{q} - \frac{1}{n}$ the following estimates hold

$$\begin{aligned}\|h\|_s &\leq \left(\frac{q}{s}\right)^{\frac{1}{s}} |B(0,1)|^{\frac{1}{s}} R^{\frac{n}{q}} \\ &\leq q^{\frac{1}{s}} |B(0,1)|^{\frac{1}{s} - \frac{1}{q}} |U|^{\frac{1}{q}} \\ &= q^{1 + \frac{1}{q} - \frac{1}{n}} |B(0,1)|^{1 - \frac{1}{n}} |U|^{\frac{1}{q}}.\end{aligned}$$

We apply Young's theorem (Theorem 11.11) with $\frac{1}{s} + \frac{1}{n} = 1 + \frac{1}{q}$ and obtain by the above estimate

$$\begin{aligned}\|I_1(f)\|_q &= \|h * f\|_q \leq \|h\|_s \|f\|_n \\ &\leq q^{1 + \frac{1}{q} - \frac{1}{n}} |B(0,1)|^{1 - \frac{1}{n}} |U|^{\frac{1}{q}} \|f\|_n.\end{aligned}$$

□

Proof of Theorem 6.28. Let $U \subseteq \mathbb{R}^n$ and $f \in L^n(U)$. Let $q \geq n$, then $\frac{qn}{n-1} \geq n$ and therefore, Proposition 6.30 asserts

$$\|I_1(f)\|_{qn'} \leq (qn')^{1 - \frac{1}{n} + \frac{1}{qn'}} |B(0,1)|^{1 - \frac{1}{n}} |U|^{\frac{1}{qn'}} \|f\|_n, \tag{6.81}$$

where $n' = \frac{n}{n-1}$. Since $qn'(1 - \frac{1}{n}) = q$, we get

$$\begin{aligned}\int_U |I_1(f)(x)|^{qn'} dx &\leq (qn')^{qn' - \frac{qn'}{n} + 1} |B(0,1)|^{qn' - \frac{qn'}{n}} |U| \|f\|_n^{qn'} \\ &= (qn')^{q+1} |B(0,1)|^q |U| \|f\|_n^{qn'}.\end{aligned} \tag{6.82}$$

Hence,

$$\int_U \left(\frac{|I_1(f)(x)|}{c_1 \|f\|_n}\right)^{qn'} dx \leq qn' \left(\frac{qn' |B(0,1)|}{c_1^{n'}}\right)^q |U|. \tag{6.83}$$

We show the following estimate by applying the equation (6.83)

$$\int_U \sum_{k=0}^N \frac{1}{k!} \left(\frac{|I_1(f)(x)|}{c_1 \|f\|_n}\right)^{kn'} dx \leq C_2 |U|. \tag{6.84}$$

We divide the sum on the left-hand side into two parts

$$\begin{aligned}\int_U \sum_{k=0}^{N} \frac{1}{k!}\left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx &= \sum_{k=0}^{N} \frac{1}{k!}\int_U \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx \\ &= \sum_{k=0}^{n-1} \frac{1}{k!}\int_U \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx + \sum_{k=n}^{N} \frac{1}{k!}\int_U \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx \\ &=: I + II.\end{aligned} \tag{6.85}$$

Applying (6.83) yields

$$\begin{aligned} II = \sum_{k=n}^{N} \frac{1}{k!}\int_U \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx &\le |U|n' \sum_{k=1}^{\infty} \frac{k^{k+1}}{k!}\left(\frac{n'|B(0,1)|}{c_1^{n'}}\right)^k \\ = |U|n' \sum_{k=1}^{\infty} \frac{k^k}{(k-1)!}\left(\frac{n'|B(0,1)|}{c_1^{n'}}\right)^k . \end{aligned} \tag{6.86}$$

Stirling's formula yields

$$\frac{k^k}{(k-1)!} \le \frac{k}{\sqrt{2\pi k}} e^k \le e^{2k}, \quad k \ge 1.$$

Summarizing we obtain

$$II \le |U|n' \sum_{k=1}^{\infty} \left(\frac{e^2 n'|B(0,1)|}{c_1^{n'}}\right)^k . \tag{6.87}$$

We choose $c_1(n)$ so that $e^2 n'|B(0,1)| < c_1^{n'}$. Then the sum on the right-hand side converges and there exists a constant $C(n)$ such that

$$II \le C(n)|U|. \tag{6.88}$$

In order to get an estimate for I we set $g(x) = \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{n'}$. Then, by Hölder's inequality, we obtain for $1 \le k < n$:

$$\begin{aligned}\int_U \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx &= \int_U |g(x)|^k dx \\ &\le \left(\int_U |g(x)|^n dx\right)^{\frac{k}{n}} |U|^{\frac{1}{k}-\frac{1}{n}} \\ &= |U|^{\frac{1}{k}-\frac{1}{n}} \left(\int_U \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{nn'} dx\right)^{\frac{k}{n}} .\end{aligned}$$

We apply equation (6.83) to the integral on the right-hand side and obtain

$$
\begin{aligned}
\int_U \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx &\le |U|^{1-\frac{k}{n}} \left(nn' \left(\frac{nn'|B(0,1)|}{c_1^{n'}}\right)^n |U|\right)^{\frac{k}{n}} \\
&= |U|(nn')^{k+\frac{k}{n}} \left(\frac{|B(0,1)|}{c_1^{n'}}\right)^k \\
&\le |U|(nn')^{k+\frac{k}{n}} \left(\frac{|B(0,1)|}{e^2 n'|B(0,1)|}\right)^k \\
&= c(n)^k e^{-2k}|U|.
\end{aligned}
$$

Summarizing we obtain

$$
\begin{aligned}
I &= |U| + \sum_{k=1}^{n-1} \frac{1}{k!}\int_U \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx \\
&\le |U| + |U| \sum_{k=1}^{n-1} \frac{c(n)^k e^{-2k}}{k!} \\
&= |U|C(n),
\end{aligned}
\tag{6.89}
$$

where $C(n) \ge 1$. Collectin the equations (6.85), (6.88) and (6.89) we obtain that there exist constants $c_1(n)$ and $C_2(n)$ such that

$$
\int_U \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{kn'} dx \le C_2|U|. \tag{6.90}
$$

The monotone convergence theorem yields

$$
\int_U \exp\left\{\left(\frac{|I_1(f)(x)|}{c_1\|f\|_n}\right)^{\frac{n}{n-1}}\right\} dx \le C_2|U|. \tag{6.91}
$$

Let $u \in C_c^\infty(U)$. Then equation (6.41) yields

$$
|u(x)| \le C(n)|I_1(Du)(x)|
$$

and by (6.91) we obtain

$$
\int_U \exp\left\{\left(\frac{|u|}{c_1\,\|Du\|_n}\right)^{\frac{n}{n-1}}\right\} \le \int_U \exp\left\{\left(\frac{I_1(|Du|)}{c_2\|Du\|_n}\right)^{\frac{n}{n-1}}\right\} dx \le C_2(n)|U|, \tag{6.92}
$$

where $C_2(n) \ge 1$. $\square$

CHAPTER 7

Compact embeddings

The Gagliardo-Nirenberg-Sobolev inequality shows that $W^{1,p}(U)$ is continuously embedded into $L^{p^*}(U)$, if $1 \leq p < n$. Now we show that $W^{1,p}(U)$ is in fact compactly embedded into some $L^q(U)$ space.

Definition 7.1 (compactly embedded). Let X and Y be Banach spaces, $X \subset Y$. We say X is compactly embedded in Y if and only if the operator

$$\mathrm{Id}\colon X \to Y,\, x \mapsto x$$

is continuous and compact, i.e.

(1) $\exists\, C\, \forall x \in X: \|x\|_Y \leq C\|x\|_X$,
(2) for all sequences $(x_n)_{n=1}^{\infty}$ in X with $\sup_n \|x_n\|_X \leq \infty$ there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ and $y \in Y$ such that $\|I(x_{n_i}) - y\|_Y \xrightarrow{i\to\infty} 0$.

Theorem 7.2 (Rellich-Kondrachov Compactness Theorem). *Let $U \subseteq \mathbb{R}^n$ open and bounded and let ∂U be C^1. Let $1 \leq p < n$. Then*

$$W^{1,p}(U) \subset\subset L^q(U),$$

for all $1 \leq q < p^*$.

Proof. We fix $q \in [1, p^*)$. Let $u \in W^{1,p}(U)$. Theorem 6.3 yields

$$\|u\|_{L^q} \leq C\|u\|_{W^{1,p}(U)}.$$

Hence, the operator $\mathrm{Id}\colon W^{1,p} \to L^q$ is continuous.

We have to show compactness. Let $(\hat{u}_m)_{m=1}^{\infty} \in W^{1,p}(U)$ and $\sup_m \|\hat{u}_m\|_{W^{1,p}(U)} \leq A$. We show that there exists a subsequence $(\hat{u}_{m_k})_{k=1}^{\infty}$ of the bounded sequence $(\hat{u}_m)_{m=1}^{\infty}$ and a $u \in L^q(U)$ so that $\|\hat{u}_{m_k} - u\|_{L^q(U)} \xrightarrow{k\to\infty} 0$. By the extension theorem we may assume that

(1) $(u_m)_{m=1}^{\infty}$ is in $W^{1,p}(\mathbb{R}^n)$ with $u_m\big|_U = \hat{u}_m$,
(2) for all $m \in \mathbb{N}$ there exists V with $U \subset\subset V$ such that $\operatorname{supp} u_m \subset V$,
(3) $\sup_m \|u_m\|_{W^{1,p}(\mathbb{R}^n)} < \infty$.

We first consider the smooth functions

$$u_m^{\varepsilon} = \eta_{\varepsilon} * u_m \in C_c^{\infty}(\mathbb{R}^n) \quad (\varepsilon > 0,\;\; m \in \mathbb{N}).$$

We may assume that for all $m \in \mathbb{N}$ the support of u_m^{ε} is in V.

<u>*Statement 1*</u>:

$$\lim_{\varepsilon\to 0} \sup_{m\in\mathbb{N}} \|u_m^{\varepsilon} - u_m\|_{L^q(V)} \xrightarrow{\varepsilon\to 0} 0 \;\; .$$

Verification: If u_m is smooth, then

$$\begin{aligned} u_m^\varepsilon(x) - u_m(x) &= \int_{B(0,1)} \eta(y)(u_m(x-\varepsilon y) - u_m(x))dy \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} u_m(x - \varepsilon t y) dt\, dy \\ &= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x - \varepsilon t) \cdot y\, dt\, dy. \end{aligned}$$

Thus,

$$\begin{aligned} \int_V |u_m^\varepsilon(x) - u_m(x)| dx &\le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x - \varepsilon t y)|\, dx\, dt\, dy \\ &\le \varepsilon \int_V |Du_m(z)| dz. \end{aligned}$$

Summarizing we have for $u_m \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} u_m^\varepsilon \in V$ the estimate

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \le \varepsilon \|Du_m\|_{L^1(V)}. \tag{7.1}$$

By approximation (Theorem 3.17) this estimate holds for $u_m \in W^{1,p}(V)$. Since V is open and bounded, we obtain

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \le \varepsilon \|Du_m\|_{L^1(V)} \le \varepsilon C \|Du_m\|_{L^p(V)}.$$

By assumption we have that $\sup_m \|u_m\|_{W^{1,p}(V)} < \infty$. Therefore,

$$\lim_{\varepsilon \to 0} \sup_{m \in \mathbb{N}} \|u_m^\varepsilon - u_m\|_{L^1(V)} = 0. \tag{7.2}$$

Note that $1 \le q < p^*$. Let $0 \le \theta \le 1$ such that

$$\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{p^*}.$$

We apply the interpolation theorem for L^p-norms (Theorem 11.10) to obtain

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le \|u_m^\varepsilon - u_m\|_{L^1(V)}^{1-\theta} \|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{\theta}.$$

Theorem 6.3 yields

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le \|u_m^\varepsilon - u_m\|_{L^1(V)}^{1-\theta} \|u_m^\varepsilon - u_m\|_{W^{1,p}(V)}^{\theta}$$

and by equation (7.2)

$$\lim_{\varepsilon \to 0} \sup_{m \in \mathbb{N}} \|u_m^\varepsilon - u_m\|_{L^q(V)} = 0. \tag{7.3}$$

Statement 2: Let $\varepsilon > 0$ be fixed. The sequence $(u_m^\varepsilon)_{m=1}^\infty$ in $C_c^\infty(\mathbb{R}^n)$ is uniformly bounded and uniformly equicontinuous, i.e.

(1) $\sup_m \|u_m^\varepsilon\|_\infty < \infty$
(2) $\forall \eta > 0\, \exists \delta > 0\, \forall m \in \mathbb{N}\, \forall x, y \in \mathbb{R}^n : |x - y| < \delta \Rightarrow |u_m^\varepsilon(x) - u_m^\varepsilon(y)| < \eta$.

Verification: Let $x \in \mathbb{R}^n$.

$$|u_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} \eta^\varepsilon(x-y)|u_m(y)|dy \le \sup_{x\in\mathbb{R}^n} |\eta^\varepsilon(x)| \int_V |u_m(y)|dy$$
$$\le \frac{1}{\varepsilon^n} \|u_m\|_{L^1(V)} \le \frac{1}{\varepsilon^n} |V|^{\frac{1}{p}} \|u_m\|_{L^p(V)} = \frac{C}{\varepsilon^n}.$$

Hence,

$$\sup_{m\in\mathbb{N}} \|u_m^\varepsilon\|_\infty \le \frac{C}{\varepsilon^n}. \tag{7.4}$$

By Lemma 3.7 and the Minkowski inequality for integrals (Theorem 11.4) we have

$$|Du_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} |D\eta^\varepsilon(x-y)|\,|u_m(y)|dy \le \frac{1}{\varepsilon^{n+1}} |V|^{\frac{1}{p}} \|u_m\|_{L^p(V)} = \frac{C}{\varepsilon^{n+1}}. \tag{7.5}$$

Hence,

$$\sup_{m\in\mathbb{N}} \|Du_m^\varepsilon\|_\infty \le \frac{C}{\varepsilon^{n+1}}. \tag{7.6}$$

Equation (7.6) yields

$$\forall \eta > 0\, \exists \delta > 0\, \forall m \in \mathbb{N}\, \forall x, y \in \mathbb{R}^n : \, |x-y| < \delta \Rightarrow |u_m^\varepsilon(x) - u_m^\varepsilon(y)| < \eta.$$

The sequence $(u_m^\varepsilon)_{m=1}^\infty$ satisfies the requirements of the Arzela-Ascoli compactness criterion (Theorem 11.18), which asserts that for the uniformly bounded and uniformly equicontinuous family of functions (u_m^ε) there exists a subsequence that converges uniformly to a continuous function on compact subsets of $\mathbb{R}^n$, i.e.

$$\forall \varepsilon > 0\, \exists N_\varepsilon \subseteq \mathbb{N}, \#N_\varepsilon = \infty : \, (u_j^\varepsilon)_{j\in N_\varepsilon} \text{converges uniformly on V.} \tag{7.7}$$

This implies that $(u_j^\varepsilon)_{j\in N^\varepsilon}$ converges in $L^q(V)$ $(1 \le q \le \infty)$. Summarizing we have

$$\forall \ell \in \mathbb{N}\, \exists \varepsilon_\ell > 0\, \forall \varepsilon < \varepsilon_\ell\, \forall m \in \mathbb{N} : \, \|u_m^\varepsilon - u_m\|_{L^q(V)} \le \frac{1}{\ell} \tag{7.8}$$

$$\forall \varepsilon > 0\, \exists N_\varepsilon \subseteq \mathbb{N}\, \forall \ell \in \mathbb{N}\, \exists N_\ell \subseteq N_\varepsilon\, \forall i, j \in N_\ell : \, \left\|u_j^\varepsilon - u_i^\varepsilon\right\|_{L^q(V)} \le \frac{1}{\ell}. \tag{7.9}$$

We combine equation (7.8) and (7.9) to obtain

$$\forall \ell \in \mathbb{N}\, \exists N_\ell \subseteq \mathbb{N}\, \forall i, j \in N_\ell : \, \|u_i - u_j\|_{L^q(V)} \le \frac{3}{\ell}. \tag{7.10}$$

We apply Cantor's diagonal argument. Note that by definition

$$N_1 \supset N_2 \supset N_3 \supset \cdots \supset N_\ell \supset \ldots.$$

Let $d_i = \min N_i$. Then $d_j \in N_i$ for all $j \ge i$ and by equation (7.10) we have for all $i, j \in \mathbb{N}$:

$$\left\|u_{d_i} - u_{d_j}\right\|_{L^q(V)} \le \max\{\frac{1}{i}, \frac{1}{j}\}.$$

Then we have

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall i, j \ge N : \, \left\|u_{d_i} - u_{d_j}\right\|_{L^q(V)} < \varepsilon.$$

Hence, $(u_{d_i})_{i\in\mathbb{N}}$ is Cauchy sequence in $L^q(V)$ with limit $u \in L^q(V)$, i.e. there exists $u \in L^q(V)$ such that

$$\|u_{d_i} - u\|_{L^q(V)} \to 0, \qquad \text{if } i \to \infty.$$

Since $U \subseteq V$ we have

$$\|u_{d_i} - u\|_{L^q(U)} \to 0, \qquad \text{if } i \to \infty. \tag{7.11}$$

□

The Arzela-Ascoli theorem (Theorem 11.18) gives the compact embedding of $W^{1,p}(U)$, $n < p \leq \infty$ in L^q, $1 \leq q \leq \infty$.

THEOREM 7.3. *Let $U \subseteq \mathbb{R}^n$ open and bounded and let ∂U be C^1. Let $n < p \leq \infty$. Then*

$$W^{1,p}(U) \subset\subset L^q(U),$$

for all $1 \leq q \leq \infty$.

SKETCH OF THE PROOF. By the Arzela-Ascoli theorem (Theorem 11.18) we obtain that

$$C^{0,\gamma}(\overline{U}) \subset\subset C(\overline{U})$$

for all $0 < \gamma \leq 1$. Then use Morrey's inequality (Theorem 6.20) to obtain the statement. □

Lemma 6.24 and Theorem 7.2 give the statement for the borderline case $p = n$:

THEOREM 7.4. *Let $U \subseteq \mathbb{R}^n$ open and bounded and let ∂U be C^1. Then*

$$W^{1,n}(U) \subset\subset L^q(U),$$

for all $1 \leq q \leq n$.

PROOF. Lemma 6.24 yields the continuous embedding. Choose a bounded sequence $(u_m)_{m=1}^{\infty}$ in $W^{1,n}(U)$. Then, since U is bounded, we have that for every $p < n$ the sequence $(u_m)_{m=1}^{\infty}$ is bounded in $W^{1,p}(U)$. Theorem 7.2 asserts that there exists a limit $u \in L^{p^*}(U)$ such that

$$\|u_m - u\|_{L^{p^*}(U)} \to 0.$$

If we choose $\frac{n}{2} < p < n$, then $n < p^*$ and we have

$$\|u_m - u\|_{L^n(U)} \to 0.$$

□

REMARK 7.5. Summarizing we have by Theorem 7.2, Theorem 7.3 and Theorem 7.4 the following statement:

$$W^{1,p}(U) \subset\subset L^p, \tag{7.12}$$

for all $1 \leq p \leq \infty$.

Note also that

$$W_0^{1,p}(U) \subset\subset L^p, \tag{7.13}$$

for all $1 \leq p \leq \infty$, even if we do not assume ∂U to be C^1.

CHAPTER 8

Poincaré's inequality

The inequality

$$\|u\|_{L^{p*}} \leq C\|Du\|_{L^p} \tag{8.1}$$

does not hold for every $u \in W^{1,p}(U)$, where $U \subseteq \mathbb{R}^n$ is open and bounded. The Gagliardo-Nirenberg inequality (Theorem 6.2) states that it holds for $u \in C_c^1(\mathbb{R}^n)$ and hence for $u \in W^{1,p}(\mathbb{R}^n)$. Theorem 6.13 states that it holds for $u \in W_0^{1,p}(U)$, $U \subseteq \mathbb{R}^n$ open. If we consider an arbitrary smooth function that is constant and nonzero on the unit ball $B(0,1)$ and zero outside, we have

$$\|Du\|_{L^p} = 0 \text{ and } \|u\|_{L^{p^*}} = |B(0,1)|^{\frac{1}{p^*}}.$$

However, if we replace the integrand on the left-hand side of (8.1) by $\|u - (u)_U\|_{L^p}$, where

$$(u)_U = \int_U u(x)\frac{dx}{|U|} \quad \text{(mean value of u in U)}$$

we obtain an inequality that holds for all $u \in W^{1,p}(U)$.

8.1. General formulation and proof by contradiction

Theorem 8.1 (Poincaré's Inequality). *Let $U \subseteq \mathbb{R}^n$ be open, bounded and connected with a C^1-boundary ∂U. Let $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p and U, such that*

$$\|u - (u)_U\|_{L^p(U)} \leq C\|Du\|_{L^p(U)} \tag{8.2}$$

for all $u \in W^{1,p}(U)$.

Proof. By contradiction. We assume that the statement is not true, i.e.

$$\forall k \in \mathbb{N}\, \exists u_k \in W^{1,p}(U): \; \|u_k - (u_k)_U\|_{L^p(U)} > k\|Du_k\|_{L^p(U)}. \tag{8.3}$$

We define

$$v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}.$$

Then $\|v_k\|_{L^p(U)} = 1$ and $(v_k)_U = 0$. The gradient of v_k

$$Dv_k = \frac{Du_k}{\|u_k - (u_k)_U\|_{L^p(U)}},$$

satisfies by assumption (8.3)

$$\|Dv_k\|_{L^p(U)} = \frac{\|Du_k\|_{L^p(U)}}{\|u_k - (u_k)_U\|_{L^p(U)}} < \frac{1}{k}. \tag{8.4}$$

Hence,

$$\|v_k\|_{W^{1,p}(U)} \le C(n,p)\left(\|Dv_k\|_{L^p(U)} + \|v_k\|_{L^p(U)}\right) \le C(n,p)\left(1+\frac{1}{k}\right)$$

and

$$\sup_{k\in\mathbb{N}} \|v_k\|_{W^{1,p}(U)} \le 2C(n,p).$$

By Remark 7.5 we have that

$$W^{1,p}(U) \subset\subset L^p(U), \quad 1 \le p \le \infty.$$

By definition there exists a subsequence $(v_{k_j})_{j=1}^{\infty}$ and a $v \in L^p(U)$ with $\|v\|_{L^p(U)} = 1$ and $(v)_U = 0$ such that

$$\lim_{j\to\infty} \left\|v_{k_j} - v\right\|_{L^p(U)} = 0.$$

Let $\phi \in C_c^\infty(U)$. Then, using Lebesgue's Theorem and the definition of the weak derivative, we have

$$\int v\,\phi_{x_i} dx = \lim_{j\to\infty} \int v_{k_j}\,\phi_{x_i} dx = -\lim_{j\to\infty} \int (v_{k_j})_{x_i}\,\phi\, dx = 0,$$

where the last equality follows from $\lim_{j\to\infty} \|Dv_{k_j}\|_{L^p(U)} = 0$. Hence, $Dv = 0$. Since U is connected, Proposition 8.2 implies that v is constant a.e on U. As $(v)_U = 0$ we have $v = 0$ a.e. on U, which is a contradiction to $\|v\|_{L^p(U)} = 1$. $\square$

Proposition 8.2. *Let $U \subseteq \mathbb{R}^n$ open and bounded and connected. Let $u \in W^{1,p}(U)$ and $Du = 0$ a.e. in U. Then u is constant a.e. on U.*

Proof. **Step 1:** Let $\varepsilon > 0$. We consider the smooth functions

$$u^\varepsilon = \eta_\varepsilon * u \ \in C^\infty(U_\varepsilon),$$

where $U_\varepsilon = \{x \in U : d(x,\partial U) > \varepsilon\}$. Corollary 3.8 yields

$$D_{x_i}(u_\varepsilon) = \eta_\varepsilon * D_{x_i} u.$$

Hence, by assumption $D_{x_i} u_\varepsilon = 0$ a.e. on U_ε. Consequently, u_ε is constant on each connected subset of U_ε.

Step 2: Choose $x, y \in U$. Since U is connected there exists a polygonal path $\Gamma \subseteq U$ that connects x and y. Let $\delta = \min_{z\in\Gamma} d(z,\partial U)$ and $\varepsilon < \delta$. Then $\Gamma \subseteq U_\varepsilon$ and x, y lie in the same connected subset of U_ε. Hence, $u^\varepsilon(x) = u^\varepsilon(y) = \text{const}$.

Step 3: $u \in W^{1,p}(U)$. Theorem 3.9 yields that

$$u^\varepsilon \xrightarrow{\varepsilon\to 0} u \text{ a.e. on } U.$$

Hence, u is constant a.e. on U. $\square$

8.2. Poincaré's inequality for a ball

THEOREM 8.3 (Poincaré's inequality for a ball). *Let $1 \leq p \leq \infty$. Then there exists a constant C, that depends only on n and p, such that*

$$\|u - (u)_{B(x,r)}\|_{L^p(B(x,r))} \leq C\, r\, \|Du\|_{L^p(B(x,r))}, \tag{8.5}$$

for each ball $B(x,r) \subseteq \mathbb{R}^n$ and each function $u \in W^{1,p}(B(x,r))$.

PROOF. Let $U = B(0,1)$ and $u \in W^{1,p}(U)$. Theorem 8.1 yields the estimate

$$\|u - (u)_{B(0,1)}\|_{L^p(B(0,1))} \leq C\|Du\|_{L^p(B(0,1))}. \tag{8.6}$$

Let now $u \in W^{1,p}(B(x,r))$. We define

$$v(y) = u(x + ry), \quad y \in B(0,1).$$

Then $v \in W^{1,p}(B(0,1))$ and by equation (8.6) we have

$$\|v - (v)_{B(0,1)}\|_{L^p(B(0,1))} \leq C\|Dv\|_{L^p(B(0,1))}. \tag{8.7}$$

Changing variables, we recover equation (8.5). □

REMARK 8.4. Let $u \in W^{1,n}(\mathbb{R}^n)$ and $B(x,r) \subseteq \mathbb{R}^n$. Then Theorem 8.3 yields

$$\begin{aligned}\left(\int_{B(x,r)} \left|u(y) - (u)_{B(x,r)}\right|^n \frac{dy}{|B(x,r)|}\right)^{\frac{1}{n}} &\leq Cr\left(\int_{B(x,r)} |Du(y)|^n dy\right)^{\frac{1}{n}} \\ &\leq \frac{Cr}{|B(x,r)|^{\frac{1}{n}}}\|Du\|_{L^n(\mathbb{R}^n)} \\ &= \frac{C}{|B(0,1)|^{\frac{1}{n}}}\|Du\|_{L^n(\mathbb{R}^n)}.\end{aligned}$$

By Hölder's inequality we obtain for the left-hand side

$$\int_{B(x,r)} \left|u(y) - (u)_{B(x,r)}\right| \frac{dy}{|B(x,r)|} \leq \left(\int_{B(x,r)} \left|u(y) - (u)_{B(x,r)}\right|^n \frac{dy}{|B(x,r)|}\right)^{\frac{1}{n}}.$$

Hence,

$$\int_{B(x,r)} \left|u(y) - (u)_{B(x,r)}\right| \frac{dy}{|B(x,r)|} \leq C\|Du\|_{L^n(\mathbb{R}^n)}, \tag{8.8}$$

where C depends only on n.

Space of bounded mean oscillation. A function $f \in L^1_{loc}(\mathbb{R}^n)$ is called of bounded mean oscillation if

$$\sup_{B(x,r)\subseteq\mathbb{R}^n} \int_{B(x,r)} \left|f(y) - (f)_{B(x,r)}\right| \frac{dy}{|B(x,r)|} < \infty. \tag{8.9}$$

The space of all such functions is called the space of functions of bounded mean oscillation ($\mathrm{BMO}(\mathbb{R}^n)$) and the left-hand side of equation (8.9) defines a norm $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$ on this space.

Therefore, we have that $W^{1,n}(\mathbb{R}^n)$ is continuously embedded into $\mathrm{BMO}(\mathbb{R}^n)$ with

$$\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq C\|Du\|_{L^n(\mathbb{R}^n)} \leq C\|u\|_{W^{1,n}(\mathbb{R}^n)}. \tag{8.10}$$

8.3. Poincaré's inequality - an alternative proof

PROOF II OF THEOREM 8.3. Let $B \subseteq \mathbb{R}^n$ be a ball with radius R. We show that for $u \in C^\infty(\overline{B})$ the following estimate holds

$$|u(x) - (u)_B| \leq C(n) \int_B \frac{|Du(y)|}{|x-y|^{n-1}} dy, \quad x \in B. \tag{8.11}$$

Let $x, y \in B$ and $\omega = \frac{y-x}{|x-y|}$. Then

$$\begin{aligned} u(x) - u(y) &= -\int_0^{|x-y|} \frac{d}{dr} u(x + r\omega) dr \\ &= -\int_0^{|x-y|} Du(x + r\omega) \cdot \omega \, dr. \end{aligned}$$

We integrate over B with respect to y and obtain

$$|B|(u(x) - (u)_B) = -\int_B \int_0^{|x-y|} Du(x + r\omega) \cdot \omega \, dr dy.$$

By integration in polar coordinates (Theorem 11.16) on the right-hand side we obtain equation (8.11). Using Definition 6.9 we get the potential estimate

$$|u(x) - u_B| \leq C(n) I_1(|\nabla u|)(x). \tag{8.12}$$

Case: $n = 1$**:** Equation (8.11) yields

$$|u(x) - (u)_B| \leq C(n) \int_B |Du(y)| dy. \tag{8.13}$$

Hence,

$$\left(\int_B |u(x) - (u)_B|^p \right)^{\frac{1}{p}} \leq C(n) |B|^{\frac{1}{p}} \|Du\|_{L^1}.$$

We apply Hölder's inequality to the right-hand side with exponents $\frac{1}{p} + 1 - \frac{1}{p} = 1$ and obtain

$$\begin{aligned} \left(\int_B |u(x) - (u)_B|^p \right)^{\frac{1}{p}} &\leq C(n) |B|^{\frac{1}{p}} |B|^{1-\frac{1}{p}} \|Du\|_{L^p} \\ &= C(n) |B| \|Du\|_{L^p} = 2C(n)\, R \, \|Du\|_{L^p}. \end{aligned}$$

Case $n > 1$**:**

1. Let $1 < p < n$. We apply Proposition 6.10 to the potential estimate (8.12) and obtain

$$\left(\int_B |u(x) - (u)_B|^{p^*} \right)^{\frac{1}{p^*}} \leq C(n) \|I_1(|Du|)\|_{L^{p^*}(B)} \leq C(n) \|Du\|_{L^p(B)}.$$

Hölder's inequality applied to the left-hand side with exponents $\frac{p}{p^*} + 1 - \frac{p}{p^*} = 1$ yields

$$\begin{aligned}\left(\int_B |u(x) - (u)_B|^p\right)^{\frac{1}{p}} &\leq |B|^{\frac{1}{p}-\frac{1}{p^*}} \left(\int_B |u(x) - (u)_B|^{p^*}\right)^{\frac{1}{p^*}} \\ &\leq |B|^{\frac{1}{n}} C(n) \|Du\|_{L^p(B)} \\ &= C(n)\, R\, \|Du\|_{L^p(B)}.\end{aligned}$$

2. Let $n \leq p < \infty$. Choose $1 < q < n$ so that $q^* \geq p$. (This is always possible!!) Then, by the above computations

$$\left(\int_B |u(x) - (u)_B|^{q^*}\right)^{\frac{1}{p^*}} \leq C(n) \|Du\|_{L^q(B)}.$$

Applying Hölder's inequality to the left-hand side with exponents $\frac{p}{q^*} + 1 - \frac{p}{q^*} = 1$ yields

$$\begin{aligned}\left(\int_B |u(x) - (u)_B|^p\right)^{\frac{1}{p}} &\leq |B|^{\frac{1}{p}-\frac{1}{q^*}} \left(\int_B |u(x) - (u)_B|^{q^*}\right)^{\frac{1}{q^*}} \\ &\leq |B|^{\frac{1}{p}-\frac{1}{q^*}} C(n) \|Du\|_{L^q(B)}.\end{aligned}$$

Now applying Hölder's inequality to the right-hand side with exponents $\frac{q}{p} + 1 - \frac{q}{p} = 1$ yields

$$\begin{aligned}\left(\int_B |u(x) - (u)_B|^p\right)^{\frac{1}{p}} &\leq |B|^{\frac{1}{p}-\frac{1}{q^*}} |B|^{\frac{1}{q}-\frac{1}{p}} C(n) \|Du\|_{L^p(B)} \\ &= |B|^{\frac{1}{q}-\frac{1}{q^*}} C(n) \|Du\|_{L^p(B)} \\ &= |B|^{\frac{1}{n}} C(n) \|Du\|_{L^p(B)} \\ &= C(n)\, R\, \|Du\|_{L^p(B)}.\end{aligned}$$

3. Let $p = \infty$. Then equation (8.11) yields

$$|u(x) - (u)_B| \leq C(n) \|Du\|_\infty \int_B \frac{1}{|x-y|^{n-1}} dy$$

Integration in polar coordinates (Theorem 11.16) gives

$$|u(x) - (u)_B| \leq C(n) R \|Du\|_\infty.$$

Hence,

$$\|u - (u)_B\|_\infty \leq C(n) R \|Du\|_\infty.$$

4. Let $p = 1$. We apply the same procedure as in the case $p = 1$ in the alternative proof of the Gagliardo-Nirenberg inequality (Proof II of Theorem 6.2 in Section 6.3):

Let $h(x) = u(x) - (u)_B$, $x \in \overline{B}$. We set

$$h^+(x) = \max\{h(x), 0\} \quad \text{ and } h^-(x) = \max\{-h(x), 0\}.$$

In the following let $h = h^+$ or $h = h^-$. The support of h can be written as union of the sets

$$A_j := \left\{x \in \mathbb{R}^n : 2^j < h(x) \leq 2^{j+1}\right\} \quad , j \in \mathbb{Z}.$$

We consider the function

$$v_j(x) = \begin{cases} 0, & \text{if } h(x) \le 2^j \\ h(x) - 2^j, & \text{if } 2^j < h(x) \le 2^{j+1} \\ 2^j, & \text{if } 2^{j+1} < h(x). \end{cases} \tag{8.14}$$

Since $v_j(x) > 2^{j-1}$ if and only if $h(x) - 2^j > 2^{j-1}$, we obtain

$$\begin{aligned} |A_{j+1}| &= \left| \left\{ 2^{j+1} < h \le 2^{j+2} \right\} \right| \le \left| \left\{ h > 2^{j+1} \right\} \right| = \left| \left\{ h > 4 \cdot 2^{j-1} \right\} \right| \\ &\le \left| \left\{ h > 3 \cdot 2^{j-1} \right\} \right| = \left| \left\{ v_j > 2^{j-1} \right\} \right|. \end{aligned} \tag{8.15}$$

The function v_j is continuous on $\overline{B}$. Hence, by smoothing by convolution we can construct a sequence of smooth functions which converges by Theorem 3.9 uniformly to v_j on B. This approximation argument allows us to apply the potential estimate (8.12) to v_j and obtain

$$|v_j(x) - (v_j)_B| \le C(n) I_1(|Dv_j|)(x). \tag{8.16}$$

Equation (8.16) and (8.15) yield

$$\begin{aligned} |A_{j+1}| &\le \left| \left\{ v_j > 2^{j-1} \right\} \right| \\ &\le \left| \left\{ v_j - (v_j)_B > 2^{j-1} \right\} \right| \\ &\le \left| \left\{ |v_j - (v_j)_B| > 2^{j-1} \right\} \right| \\ &\le \left| \left\{ I_1(|Dv_j|) > C(n)^{-1} 2^{j-1} \right\} \right|. \end{aligned}$$

Using the weak estimate (6.23) in Proposition 6.10 for $\lambda = C(n)^{-1} 2^{j-1}$ we get

$$|A_{j+1}| \le C_1(n) \left(C(n) 2^{-j+1} \int_{\mathbb{R}^n} |Dv_j| dx \right)^{\frac{n}{n-1}}.$$

The definition of v_j yields that the support of Dv_j is contained in A_j and $Dv_j = Dh$ on A_j. Hence,

$$|A_{j+1}| \le C(n) \left(2^{-j} \int_{A_j} |Dh| dx \right)^{\frac{n}{n-1}}. \tag{8.17}$$

By the definition of A_j we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |h(x)|^{\frac{n}{n-1}} dx &= \sum_{j \in \mathbb{Z}} \int_{A_j} |h(x)|^{\frac{n}{n-1}} dx \\ &\le \sum_{j \in \mathbb{Z}} \left(2^{j+1} \right)^{\frac{n}{n-1}} |A_j| \\ &= 2^{\frac{n}{n-1}} \sum_{j \in \mathbb{Z}} \left(2^{j+1} \right)^{\frac{n}{n-1}} |A_{j+1}|. \end{aligned} \tag{8.18}$$

Equation (8.17) yields

$$2^{\frac{n}{n-1}} \sum_{j\in\mathbb{Z}} \left(2^{j+1}\right)^{\frac{n}{n-1}} |A_{j+1}| \le C(n) \sum_{j\in\mathbb{Z}} \left(\int_{A_j} |Dh(x)|dx\right)^{\frac{n}{n-1}}$$
$$\le C(n) \left(\sum_{j\in\mathbb{Z}} \int_{A_j} |Dh(x)|dx\right)^{\frac{n}{n-1}} \tag{8.19}$$
$$= C(n) \left(\int_{\mathbb{R}^n} |Dh(x)|dx\right)^{\frac{n}{n-1}}.$$

Equation (8.18) and (8.19) give the estimate

$$\left\|h^+\right\|_{L^{\frac{n}{n-1}}} \le C(n) \left\|Dh^+\right\|_{L^1}.$$

The same argument holds for h^-. Hence, we have

$$\begin{aligned}\|h\|_{L^{\frac{n}{n-1}}} &= \left\|h^+ - h^-\right\|_{L^{\frac{n}{n-1}}} \le \left\|h^+\right\|_{L^{\frac{n}{n-1}}} + \left\|h^-\right\|_{L^{\frac{n}{n-1}}} \\ &\le C(n) \int_B \left|Dh^+(x)\right| + \left|Dh^-(x)\right|dx \\ &= C(n) \int_B |Dh(x)|dx \\ &= C(n)\|Dh\|_{L^1}.\end{aligned}$$

Since $h(x) = u(x) - (u)_B$, we have

$$\left(\int_B |u(x) - (u)_B|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le C(n)\|Du\|_{L^1(B)}.$$

Applying Hölder's inequality with $\frac{n-1}{n} + 1 - \frac{n-1}{n} = 1$ yields

$$\begin{aligned}\int_B |u(x) - (u)_B| &\le |B|^{1-\frac{n-1}{n}} \left(\int_B |u(x) - (u)_B|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \\ &\le |B|^{\frac{1}{n}} C(n)\|Du\|_{L^1(B)} \\ &= C(n)\, R\, \|Du\|_{L^1(B)}.\end{aligned}$$

□

CHAPTER 9

Fourier transform

Definition 9.1 (L^1-Fourier transform). For $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx \tag{9.1}$$

and its inverse Fourier transform

$$\check{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(x)dx. \tag{9.2}$$

Remark 9.2. Since $\left|e^{\pm 2\pi i x\xi}\right| = 1$ and $u \in L^1(\mathbb{R}^n)$, these integrals converge absolutely for all $\xi \in \mathbb{R}^n$ and define bounded functions:

$$|\int f(x)e^{\pm 2\pi i x\xi}dx| \leq \int |e^{\pm 2\pi i x\xi}||f(x)|dx = \|f\|_1 .$$

Theorem 9.3 (Fourier Inversion). *Let $f, \widehat{f} \in L^1(\mathbb{R}^n)$. Then*

$$f(x) = \int \widehat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi, \quad \textit{for a.e. } x \in \mathbb{R}^n. \tag{9.3}$$

Proof. See[**5**]. $\square$

Remark 9.4. Let $f, \widehat{f}, \check{f} \in L^1(\mathbb{R}^n)$. Then we have

$$\left(\widehat{f}\right)^{\vee} = f = \left(\check{f}\right)^{\wedge} \text{ a.e. in } \mathbb{R}^n. \tag{9.4}$$

The Fourier transform on L^2. We intend now to extend the definition of the Fourier transform and its inverse to $L^2(\mathbb{R}^n)$.

Theorem 9.5 (Plancherel). *Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{f}, \check{f} \in L^2(\mathbb{R}^n)$ and*

$$\left\|\widehat{f}\right\|_{L^2(\mathbb{R}^n)} = \left\|\check{f}\right\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}. \tag{9.5}$$

Proof. See [**4**]. $\square$

In view of the equality (9.5) we can define the Fourier transform of $f \in L^2(\mathbb{R}^n)$ as follows. Choose a sequence $(f_k) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $f_k \to f$ in $L^2(\mathbb{R}^n)$. Note that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. According to (9.5) we have

$$\left\|\widehat{f_k} - \widehat{f_j}\right\|_{L^2} = \left\|\widehat{f_k - f_j}\right\|_{L^2} = \|f_k - f_j\|_{L^2}.$$

Hence, $(\widehat{f_k})$ is Cauchy sequence in $L^2(\mathbb{R}^n)$ with limit in L^2. We define

$$\widehat{f} = \lim_{k\to\infty} \widehat{f_k}.$$

Note that the limit does not depend on the particular choice of the sequence (f_k). We similarly define $\check{f}$. This gives the following theorem.

THEOREM 9.6. *There exists a unique linear and bounded operator $\mathcal{F} : L^2 \to L^2$, such that*

(1) $\|\mathcal{F}f\|_2 = \|f\|_2$ *for* $f \in L^2$,
(2) $\mathcal{F}f = \widehat{f}$ *for* $f \in L^1 \cap L^2$.

THEOREM 9.7 (Properties). *Let $f \in L^2(\mathbb{R}^n)$ and $\tau \in \mathbb{R}^n$.*

(1) *Let $f_\tau(x) = f(x - \tau)$. Then*

$$\widehat{f_\tau}(\xi) = e^{-2\pi i\tau\,\xi}\widehat{f}(\xi). \tag{9.6}$$

(2) *Let $e_\tau(x) = e^{2\pi i x\,\tau}$. Then*

$$\widehat{e_\tau f}(\xi) = \widehat{f}(\xi - \tau). \tag{9.7}$$

(3) *Let $f^\varepsilon(x) = \varepsilon^{-n} f(\frac{1}{\varepsilon}x)$. Then*

$$\widehat{f^\varepsilon}(\xi) = \widehat{f}(\varepsilon\xi) \tag{9.8}$$

(4) *Let $D^\alpha f \in L^2(\mathbb{R}^n)$ for some multiindex α. Then*

$$\widehat{D^\alpha f}(\xi) = (2\pi i)^{|\alpha|}\xi^\alpha \widehat{f}(\xi). \tag{9.9}$$

(5) *Let $g \in L^2(\mathbb{R}^n)$. Then*

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi). \tag{9.10}$$

(6) *Let $g \in L^2(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} f(x)\overline{g}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(x)\overline{\widehat{g}}(x)dx.$$

PROOF. We prove (4) and (6) only. Let $f \in C_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
\widehat{D^\alpha u}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} D^\alpha f(x)dx \\
&= (-1)^{|\alpha|}\int_{\mathbb{R}^n} D^\alpha e^{-2\pi i x\cdot\xi} f(x)dx \\
&= (-1)^{|\alpha|}(-2\pi i\xi)^\alpha \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx \\
&= (2\pi i)^{|\alpha|}\xi^\alpha \widehat{f}(\xi).
\end{aligned}$$

Let now $f \in L^2(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, there exists a sequence $(f_k)_{k=1}^\infty \subseteq C_c^\infty(\mathbb{R}^n)$ that converges to f in L^2. Let f^α be the L^2-limit of the sequence $(D^\alpha f_k)_{k=1}^\infty \subseteq C_c^\infty$.

Then, for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n} f^\alpha(x)\varphi(x)dx &= \lim_{k\to\infty}\int_{\mathbb{R}^n} D^\alpha f_k(x)\varphi(x)dx\\ &= (-1)^{|\alpha|}\lim_{k\to\infty}\int_{\mathbb{R}^n} f_k(x)D^\alpha\varphi(x)dx\\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(x)D^\alpha\varphi(x)dx\\ &= \int_{\mathbb{R}^n} D^\alpha f(x)\varphi(x)dx.\end{aligned}$$

Hence, $D^\alpha f = f^\alpha$. Note that by Theorem 9.5 (Plancherel) we have that $\widehat{f_k}$ converges to $\widehat{f}$ in L^2. By the above we obtain

$$\widehat{D^\alpha f} = \lim_{k\to\infty}\widehat{D^\alpha f_k} = \lim_{k\to\infty}(2\pi i)^{|\alpha|}\xi^\alpha\widehat{f_k} = (2\pi i)^{|\alpha|}\xi^\alpha\widehat{f}.$$

This gives (4).

Let $\alpha \in \mathbb{C}$. Then, by Theorem 9.5

$$\|f+\alpha g\|_{L^2}^2 = \left\|\widehat{f}+\widehat{\alpha g}\right\|_{L^2}^2. \tag{9.11}$$

Expanding we deduce

$$\begin{aligned}&\int_{\mathbb{R}^n} |f(x)|^2 + \alpha g(x)\overline{f}(x) + \overline{\alpha g}(x)f(x) + |\alpha g(x)|^2 dx\\ &= \int_{\mathbb{R}^n}\left|\widehat{f}(x)\right|^2 + \widehat{\alpha g}(x)\overline{\widehat{f}}(x) + \overline{\widehat{\alpha g}}(x)\widehat{f}(x) + |\widehat{\alpha g}(x)|^2 dx.\end{aligned}$$

Again, by Theorem 9.5 we obtain

$$\int_{\mathbb{R}^n} \alpha g(x)\overline{f}(x) + \overline{\alpha g}(x)f(x)dx = \int_{\mathbb{R}^n}\widehat{\alpha g}(x)\overline{\widehat{f}}(x) + \overline{\widehat{\alpha g}}(x)\widehat{f}(x)dx.$$

If we take $\alpha = 1$ we obtain

$$\int_{\mathbb{R}^n} g(x)\overline{f}(x) + \overline{g}(x)f(x)dx = \int_{\mathbb{R}^n}\widehat{g}(x)\overline{\widehat{f}}(x) + \overline{\widehat{g}}(x)\widehat{f}(x)dx. \tag{9.12}$$

If we take $\alpha = i$ we obtain

$$\int_{\mathbb{R}^n} ig(x)\overline{f}(x) - i\overline{g}(x)f(x)dx = \int_{\mathbb{R}^n} i\,\widehat{g}(x)\overline{\widehat{f}}(x) - i\,\overline{\widehat{g}}(x)\widehat{f}(x)dx. \tag{9.13}$$

We multiply equation (9.13) with i and obtain

$$\int_{\mathbb{R}^n} -g(x)\overline{f}(x) + \overline{g}(x)f(x)dx = \int_{\mathbb{R}^n} -\widehat{g}(x)\overline{\widehat{f}}(x) + \overline{\widehat{g}}(x)\widehat{f}(x)dx. \tag{9.14}$$

Combining equation (9.12) and (9.14) yields the statement.

$\square$

THEOREM 9.8 (Characterization of H^k by the Fourier Transform). *Let $k \in \mathbb{N}$. A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1+|\cdot|^k)\widehat{u} \in L^2(\mathbb{R}^n). \tag{9.15}$$

In addition, there exists a positive constant C such that

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leq \left\|1 + |\cdot|^k \widehat{u}\right\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}. \tag{9.16}$$

PROOF. **1.** Assume first that $u \in H^k(\mathbb{R}^n)$. Then for each multiindex $|\alpha| \leq k$, we have $D^\alpha u \in L^2(\mathbb{R}^n)$. Theorem 9.7 (4) asserts that

$$\widehat{D^\alpha u}(\xi) = (2\pi i)^{|\alpha|}\xi^\alpha \widehat{u}(\xi) \quad \text{ for a.e. } \xi \in \mathbb{R}^n. \tag{9.17}$$

Thus, by Theorem 9.5, we have that $(2\pi i \xi)^{|\alpha|}\widehat{u}(\xi) \in L^2(\mathbb{R}^n)$ for each $|\alpha| \leq k$.

Let $1 \leq j \leq n$ and $\alpha_j = (0, \dots, k, \dots, 0)$, where k is at the j^{th} position of the multiindex α_j. Then we have

$$\widehat{D^{\alpha_j} u}(\xi) = (2\pi i)^k \xi_j^k \widehat{u}(\xi).$$

Hence, by Theorem 9.5,

$$\begin{aligned}
\sum_{1 \leq |\alpha| \leq k} \|D^\alpha u\|_{L^2}^2 &= \sum_{1 \leq |\alpha| \leq k} \left\|\widehat{D^\alpha u}\right\|_{L^2}^2 \\
&\geq \sum_{j=1}^n \left\|\widehat{D^{\alpha_j} u}\right\|_{L^2}^2 \\
&= \sum_{j=1}^n \int_{\mathbb{R}^n} \left|(2\pi i)^k y_j^k \widehat{u}(y)\right|^2 dy \\
&= (2\pi)^{2k} \int_{\mathbb{R}^n} |\widehat{u}(y)|^2 \sum_{j=1}^n |y_j|^{2k} dy \\
&\geq c(n,k)(2\pi)^{2k} \int_{\mathbb{R}^n} |\widehat{u}(y)|^2 |y|^{2k} dy.
\end{aligned}$$

Summarizing we have

$$\int_{\mathbb{R}^n} |\widehat{u}(y)|^2 |y|^{2k} dy \leq C(n,k) \sum_{1 \leq |\alpha| \leq k} \|D^\alpha u\|_{L^2}^2. \tag{9.18}$$

Since $|y|^k \leq 1$, if $|y| \leq 1$ and $|y|^k \leq |y|^{2k}$, if $|y| \geq 1$, the following estimate holds:

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\widehat{u}(y)(1 + |y|^k)\right|^2 dy &= \int_{\mathbb{R}^n} |\widehat{u}(y)|^2 (1 + |y|^k)^2 dy \\
&= \int_{\mathbb{R}^n} |\widehat{u}(y)|^2 dy + \int_{\mathbb{R}^n} |\widehat{u}(y)|^2 |y|^{2k} dy + 2\int_{\mathbb{R}^n} |\widehat{u}(y)|^2 |y|^k dy \\
&\leq 3\left(\int_{\mathbb{R}^n} |\widehat{u}(y)|^2 dy + \int_{\mathbb{R}^n} |\widehat{u}(y)|^2 |y|^{2k} dy\right).
\end{aligned}$$

We apply Theorem 9.5 and equation (9.18) to the right-hand side and obtain

$$\int_{\mathbb{R}^n} \left|\widehat{u}(y)(1 + |y|^k)\right|^2 dy \leq C(n,k)\left(\|u\|_{L^2}^2 + \sum_{1 \leq |\alpha| \leq k} \|D^\alpha u\|_{L^2}^2\right) = C(n,k)\|u\|_{H^k}^2.$$

2. Assume $(1+|\cdot|^k)\widehat{u} \in L^2(\mathbb{R}^n)$. We show that $u \in H^k(\mathbb{R}^n)$. Let $|\alpha| \leq k$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|(2\pi i)^{|\alpha|} y^\alpha \widehat{u}(y)\right|^2 dy &= (2\pi)^{2|\alpha|} \int_{\mathbb{R}^n} |y^\alpha|^2 |\widehat{u}(y)|^2 dy \\
&\leq (2\pi)^{2k} \int_{\mathbb{R}^n} |y|^{2k} |\widehat{u}(y)|^2 dy \\
&\leq (2\pi)^{2k} \int_{\mathbb{R}^n} (1+|y|^k)^2 |\widehat{u}(y)|^2 dy \\
&= (2\pi)^{2k} \left\|(1+|\cdot|^k)\widehat{u}\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{9.19}$$

Hence, $(2\pi i)^{|\alpha|} y^\alpha \widehat{u}(y) \in L^2(\mathbb{R}^n)$. Let

$$u_\alpha = \left((2\pi i)^{|\alpha|} y^\alpha \widehat{u}(y)\right)\check{}\,.$$

We show that u_α is the weak derivative of u. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. We use Theorem 9.7 to obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} D^\alpha \varphi(x) \overline{u}(x) dx &= \int_{\mathbb{R}^n} \widehat{D^\alpha \varphi}(x) \widehat{\overline{u}}(x) dx \\
&= \int_{\mathbb{R}^n} (2\pi i)^{|\alpha|} x^\alpha \widehat{\varphi}(x) \widehat{\overline{u}}(x) dx \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \overline{(2\pi i)^{|\alpha|} x^\alpha \widehat{u}(x)} \widehat{\varphi}(x) dx \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \overline{u_\alpha}(x) \varphi(x) dx,
\end{aligned}$$

where $\overline{u}$ denotes the complex conjugate of the function $u : \mathbb{R}^n \to \mathbb{C}$. Hence,

$$\int_{\mathbb{R}^n} D^\alpha \varphi(x) \overline{u}(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \overline{u_\alpha}(x) \varphi(x) dx, \qquad \text{for all } \varphi \in C_c^\infty.$$

If we take the complex conjugate on both sides of the equation we obtain

$$\int_{\mathbb{R}^n} D^\alpha \overline{\varphi(x)} u(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} u_\alpha(x) \overline{\varphi}(x) dx, \qquad \text{for all } \varphi \in C_c^\infty.$$

Hence, u_α is the weak derivative of u and by equation (9.19) the weak derivative is in $L^2(\mathbb{R}^n)$. It remains to show the left-hand side inequality of (9.16). By Theorem 9.5 and equation (9.19) we have that for every $|\alpha| \leq k$ the following estimate holds

$$\|D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \leq (2\pi)^{2k} \int_{\mathbb{R}^n} |y|^{2k} |\widehat{u}(y)|^2 dy.$$

Hence,

$$\sum_{1 \leq |\alpha| \leq k} \|D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \leq C(n,k) \int_{\mathbb{R}^n} |y|^{2k} |\widehat{u}(y)|^2 dy. \tag{9.20}$$

Theorem 9.5 yields

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{u}(y)|^2 dy. \tag{9.21}$$

By combining equation (9.20) and (9.21) we obtain

$$\begin{aligned}\|u\|_{H^k}^2 &\leq C(n,k)\int_{\mathbb{R}^n}(1+|y|^{2k})|\widehat{u}(y)|^2dy\\ &\leq C(n,k)\int_{\mathbb{R}^n}(1+|y|^k)^2|\widehat{u}(y)|^2dy\\ &= C(n,k)\Big\|(1+|\cdot|^k)\widehat{u}\Big\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

□

We can define the fractional Sobolev spaces.

DEFINITION 9.9. Let $0 < s < \infty$ and $u \in L^2(\mathbb{R}^n)$. We say $u \in H^s(\mathbb{R}^n)$ if $(1+|\cdot|^s)\widehat{u} \in L^2(\mathbb{R}^n)$ and define for $s \notin \mathbb{N}$ the norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|(1+|\cdot|^s)\widehat{u}\|_{L^2(\mathbb{R}^n)}.$$

REMARK 9.10. Note that

$$(1+|y|^k) \approx (1+|y|^2)^{\frac{k}{2}}, \quad \text{for all } y \in \mathbb{R}^n.$$

Hence, we have the following equivalent characterization: a function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if

$$(1+|\cdot|^2)^{\frac{k}{2}}\widehat{u} \in L^2(\mathbb{R}^n). \tag{9.22}$$

Alternatively, the following definition is common as well

DEFINITION 9.11. Let $0 < s < \infty$ and $u \in L^2(\mathbb{R}^n)$. We say $u \in H^s(\mathbb{R}^n)$ if $(1+|\cdot|^2)^{\frac{s}{2}}\widehat{u} \in L^2(\mathbb{R}^n)$ and define for $s \notin \mathbb{N}$ the norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \Big\|(1+|\cdot|^2)^{\frac{s}{2}}\widehat{u}\Big\|_{L^2(\mathbb{R}^n)}.$$

CHAPTER 10

Exercises

Chapter 2

Exercise 1. Let $u \in L^1_{\text{loc}}(\mathbb{R})$ and

$$T_u : C_c^\infty(\mathbb{R}) \to \mathbb{R},\ T_u(\phi) = \int_{\mathbb{R}} u(x)\phi(x)dx.$$

Show that the integral exists and that T_u is linear and continuous.

Remark: Note that a sequence $\varphi_n \in C_c^\infty(\mathbb{R})$ converges to φ in $C_c^\infty(\mathbb{R})$, if

(1) there exists a compact interval $[a,b]$ such that $\operatorname{supp}\varphi_n \subseteq [a,b]$ for all $n \in \mathbb{N}$.
(2) $\forall \epsilon > 0\, \forall l \in \mathbb{N}_0\, \exists\, N_l\, \forall n \geq N_l :\ \sup_x \left|\varphi_n^{(l)}(x) - \varphi^{(l)}(x)\right| < \epsilon.$

Exercise 2. Consider the function $u : \mathbb{R} \to \mathbb{R}$, given by

$$u(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } x > 0. \end{cases}$$

Determine the distributional and the weak derivative (if it exists) of u.

Exercise 3. The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Determine the distributional derivative of H. Prove or disprove that H does not have a weak derivative.

Exercise 4. Prove Lemma 2.6.

Exercise 5. Prove Lemma 2.7.

Exercise 6. Show that $W^{k,p}(U)$ is a normed vector space.

Exercise 7. Show that

$$\langle u, v\rangle = \sum_{|\alpha|\leq k} \int_U D^\alpha u(x) D^\alpha v(x) dx$$

defines an inner product on H^k and $\sqrt{\langle u,u\rangle} = \|\cdot\|_{H^k}$.

Exercise 8. Prove Lemma 2.16

Chapter 3

Exercise 9. Let U_ε and f^ε be as in Definition 3.4. Show that for $x \in U_\varepsilon$ and $1 \le i \le n$

$$\frac{\partial}{\partial_{x_i}} f^\varepsilon(x) := \lim_{h\to 0} \frac{f^\varepsilon(x+e_i h) - f^\varepsilon(x)}{h} = \int_U \partial_{x_i} \eta^\varepsilon(x-y) f(y) dy \ .$$

Exercise 10. Let u be absolutely continuous (cf. Section 11.5) on an interval $U = (a,b) \subseteq \mathbb{R}$. Show that u admits a weak derivative $v \in L^1(U)$ and that the weak derivative coincides with the classical derivative almost everywhere.

Exercise 11. Let $U = (a,b) \subseteq \mathbb{R}$ be an open interval. Assume that $u \in L^1_{\mathrm{loc}}(U)$ admits a weak derivative $v \in L^1(U)$. Show that there exists an absolutely continuous function $\tilde{u}$ such that

$$\tilde{u}(x) = u(x), \qquad \text{for a.e. } x \in U, \tag{10.1}$$

$$v(x) = \lim_{h\to 0} \frac{\tilde{u}(x+h) - \tilde{u}(x)}{h}, \qquad \text{for a.e. } x \in U. \tag{10.2}$$

Hint: Let $x_0 \in U$ be a Lebesgue point of u, i.e.

$$\lim_{r\to 0^+} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |u(y) - u(x_0)| \, \mathrm{d}y = 0.$$

Define $\tilde{u}(x) = u(x_0) + \int_{x_0}^x v(y) dy$ and use Theorem 3.9.

Exercise 12. Let $U = (a,b) \subseteq \mathbb{R}$. Show that $u \in W^{1,p}(U)$ if and only if u coincides a.e. with an absolutely continuous function $\tilde{u} : U \to \mathbb{R}$ with $\tilde{u} \in L^p(U)$ and its derivative $\tilde{u}' \in L^p(U)$.

Hint for exercises 10-12: The following facts for absolutely continuous functions follow from the fundamental theorem (Theorem 11.23)

(1) $u \colon I = (a,b) \to \mathbb{R}$ is absolutely continuous if and only if there exists a function $v \in L^1(I)$ such that

$$u(x) = u(a) + \int_a^x v(t) dt, \quad x \in I.$$

(2) $u \colon I \to \mathbb{R}$ is absolutely continuous if and only if its classical derivative u' exists a.e. in I and belongs to $L^1(I)$.

Exercise 13. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Let $f_h(t) = f(t-h)$, $h \in \mathbb{R}^n$. Prove

$$\|f_h - f\|_{L^p} \to 0.$$

Hint. Use that $C_c(\mathbb{R}^n)$ is dense in L^p, $1 \le p < \infty$.

Exercise 14. Give an example to show that the result in Exercise 13 is not true when $p = \infty$.

Exercise 15. Reconsider the estimate in Remark 3.18

Chapter 4

Exercise 16. Show that

$$\|\overline{u}\|_{W^{1,p}(B)} \leq C_p \|u\|_{W^{1,p}(B^+)},$$

where B , B^+, u and $\overline{u}$ are as in the proof of Theorem 4.1 (Extension Theorem).

Exercise 17. Let $u \in W^{2,p}(U) \cap C^2(\overline{U})$. Show that $\overline{u} \notin C^2(B)$, but

$$\|\overline{u}\|_{W^{2,p}(B)} \leq C_p \|u\|_{W^{2,p}(B^+)},$$

where B , B^+ and $\overline{u}$ are as in the proof of Theorem 4.1 (Extension Theorem).

Exercise 18. Let $u \in W^{1,\infty}(U)$. Let

$$\overline{u}(x) = \begin{cases} u(x), & \text{if } x \in B^+ \\ u(x_1, \ldots, x_{n-1}, -x_n), & \text{if } x \in B^-. \end{cases}$$

Show that the weak derivatives of $\overline{u}$ are given by

$$\frac{\partial \overline{u}}{\partial x_i} = \begin{cases} u_{x_i}, & \text{on } B^+ \\ u_{x_i}(x_1, \ldots, x_{n-1}, -x_n), & \text{on } B^- \end{cases}$$

if $1 \leq i < n$ and

$$\frac{\partial \overline{u}}{\partial x_n} = \begin{cases} u_{x_n}, & \text{on } B^+ \\ -u_{x_n}(x_1, \ldots, x_{n-1}, -x_n), & \text{on } B^-. \end{cases}$$

Exercise 19. Show that the operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ defined by

$$Eu = \lim_{m\to\infty} Eu_m$$

does not depend on the particular choice of the sequence $(u_m)_{m=1}^{\infty} \subseteq C^1(\overline{U})$ and satisfies the properties of Theorem 4.1.

Chapter 5

Exercise 20. Verify Step 2 in the proof of Theorem 5.1 analogously to Step 4 in the proof of Theorem 4.1.

Exercise 21. Let $U = (a, b) \subseteq \mathbb{R}$ and $u \in W^{1,p}(U)$, $1 \leq p < \infty$. Then

$$Tu \equiv 0 \Leftrightarrow \tilde{u}(a) = \tilde{u}(b) = 0,$$

where $T : W^{1,p}(U) \to L^p(\partial U)$ is the trace operator and $\tilde{u}$ is the absolutely continuous representation of u, cf. Exercise 12.

Exercise 22. Let $u \in W^{k,p}(U)$ and $x_0 \in U$. Show that

$$x_0 \in \operatorname{supp} u \Leftrightarrow \forall\, V \subseteq U \text{ open with } x_0 \in V\, \exists\, \varphi \in C_c^\infty(V) : \int_V u(x)\varphi(x) \neq 0,$$

where $\operatorname{supp} u = U \setminus \bigcup\{V \subseteq U \text{ open} : u = 0 \text{ a.e. on } V\}$.

EXERCISE 23. Let $u \in C(U)$. Let

$$\operatorname{supp}(u) = \overline{\{x \in U : u(x) \neq 0\}}$$

and

$$\operatorname{ess\,supp}(u) = U \setminus \bigcup \{V \subseteq U \text{ open} : u = 0 \text{ a.e. on } V\}.$$

Show that

$$\operatorname{supp} u = \operatorname{ess\,supp} u.$$

EXERCISE 24. Let $U = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Construct a sequence $(u_n)_{n=1}^\infty$ in $C(\overline{U}) \cap W^{1,1}(U)$ such that

$$\|u_n\|_{L^1(U)} \leq \frac{C}{n} \quad \text{and} \quad \|u_n\|_{L^1(\partial U)} = C,$$

where C is some constant independent from n.

EXERCISE 25. Show that the sequence $(u_n)_{n=1}^\infty$ constructed in Exercise 24 satisfies

$$\|u_n\|_{L^1(\partial U)} \leq C \|u_n\|_{W^{1,1}(U)},$$

where C is some constant independent from n.

Chapter 6

EXERCISE 26. Prove the general Hölder inequality (Theorem 11.6): Let $1 \leq p_1, \ldots, p_m \leq \infty$, with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$. Assume $u_k \in L^{p_k}$ for $k = 1 \ldots, m$. Then

$$\int_U |u_1 \cdots u_m| dx \leq \prod_{k=1}^m \|u_i\|_{L^{p_k}(U)} \quad .$$

EXERCISE 27. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta(0) = 1$, $0 \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^n$, and $\operatorname{supp} \eta \subseteq B(0,1)$. Let $f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$. Show that $f_R(x) := f(x)\eta(x/R)$ converges to f in $W^{k,p}(\mathbb{R}^n)$ for $R \to \infty$. As a consequence show that $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$ for all $1 \leq p < \infty$.

EXERCISE 28. Show that the statement of Theorem 6.2 is true for $p = n$, if $n = 1$.

EXERCISE 29. Recalculate the proof of Theorem 6.2 for $p = 1$ and $n = 2$.

EXERCISE 30. Let $U \subseteq \mathbb{R}^n$ open and bounded. Let $1 \leq p < n$. Show that for $u \in W_0^{1,p}$ we have that $\|u\|_{W^{1,p}(U)}$ is equivalent to $\|Du\|_{L^p}(U)$.

EXERCISE 31. Let $U \subseteq \mathbb{R}^n$ be open and bounded and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{2,p}(U)$. Show that

$$\|u\|_{W^{1,q}(U)} \leq C \|u\|_{W^{2,p}(U)}$$

for all $q \in [1, p^*]$.

EXERCISE 32. Show that $f(x) = x^\alpha$, $x \in [0,1]$, is Hölder continuous with exponent α', $0 < \alpha' \leq \alpha < 1$, but not for $\alpha' > \alpha$.

EXERCISE 33. Show that $(C^{k,\gamma}(\overline{U}), \|\cdot\|_{k,\gamma})$ is a Banach space.

Exercise 34. Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$. Let $u(x) = |x|^\alpha$, $\alpha \in (0,1)$. Show that $u \in W^{1,p}(U)$, $p > n$, if and only if $\alpha > 1 - \frac{n}{p}$.

Exercise 35. Let $U \subseteq \mathbb{R}^n$ be open, bounded and suppose ∂U is C^1. Assume $u \in W^{1,p}(U)$, $n < p \leq \infty$. Show that for all $\gamma \in (0, 1 - \frac{n}{p}]$

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|u\|_{W^{1,p}(U)}. \tag{10.3}$$

where u^* is a version of u. Use Exercise 34 to show that equation (10.3) does not hold for $\gamma \in (1 - \frac{n}{p}, 1]$. Then show that

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

does not hold for $\gamma \in (1 - \frac{n}{p}, 1]$.

Exercise 36. Prove Theorem 6.20.

Exercise 37. Let $U \subsetneq \mathbb{R}^n$ open and bounded. Assume $u \in W_0^{1,p}(U)$, $n < p \leq \infty$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ for $\gamma = 1 - \frac{n}{p}$, with the estimate

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|Du\|_{L^p}.$$

Exercise 38. Present the proof of Proposition 6.10. Introduce first all definitions needed.

Exercise 39. Present the alternative proof of Theorem 6.2 (given in Section 6.3) in the case $1 < p < n$.

Exercise 40. Present the alternative proof of Theorem 6.2 (given in Section 6.3) in the case $p = 1$.

Exercise 41. Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$, where $|x|$ is the euclidean norm on $\mathbb{R}^n$, i.e.

$$|x| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}.$$

Let $u(x) = \ln\left(\ln\left(1 + \frac{1}{|x|}\right)\right)$ on $U \setminus \{0\}$. Show that $u \in L^n(U)$.

Hint:

(1) $\ln(1 + x) \leq x$ for all $x \geq 0$.
(2) $\int_0^\infty t^{n-1}e^{-t}dt = \Gamma(n)$.

Exercise 42. Let U and u be as in Exercise 41. Show that the classical derivatives

$$\frac{\partial u}{\partial x_i},$$

which exist on $U \setminus \{0\}$, are the weak derivatives of u on U and

$$\frac{\partial u}{\partial x_i} \in L^n(U).$$

Chapter 7

Exercise 43. Show that the estimate (7.1) in the proof of Theorem 7.2 is valid for $(u_m)_{m=1}^{\infty}$ in $W^{1,p}(V)$.

Exercise 44. Verify equation 7.6, by applying Lemma 3.7 and Minkowski's inequality for integrals (Theorem 11.4).

Exercise 45. Prove Theorem 7.3.

Exercise 46. What do we have to change in the proof of Theorem 7.2 to obtain a proof for Theorem 7.4?

Chapter 8

Exercise 47. Let $f \in L^2([0, 2\pi], \frac{dt}{2\pi})$ be given by its Fourier series

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{int},$$

where

$$a_n = \left\langle f, e^{in\cdot} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt \;\in \mathbb{C}.$$

Let $\sum_{n\in\mathbb{Z}} n^2 |a_n|^2 < \infty$. Show that the weak derivative of f is given by

$$g(t) = \sum_{n\in\mathbb{Z}} ina_n e^{int}$$

and is in $L^2([0, 2\pi], \frac{dt}{2\pi})$.

Remark: The partial sums

$$S_N = \sum_{n=-N}^{N} a_n e^{int}$$

converge to f in $L^2([0, 2\pi], \frac{dt}{2\pi})$.

Exercise 48. Let $f \in L^2([0, 2\pi], \frac{dt}{2\pi})$ be given by its Fourier series

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}$$

with $\sum_{n\in\mathbb{Z}} n^2 |a_n|^2 < \infty$. Show that Poincaré's inequality (Theorem 8.1) holds with constant one.

CHAPTER 11

Appendix

11.1. Notation

11.1.1. Geometric notation.

(1) $\mathbb{R}^n$ is the n-dimensional real euclidean space equipped with the euclidean norm

$$\|x\|_{\mathbb{R}^n} = |x| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}.$$

(2) $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

11.1.2. Notation for functions. Let $U \subseteq \mathbb{R}^n$.

(1) If $u : U \longrightarrow \mathbb{R}$, we write

$$u(x) = u(x_1, \dots x_n), \quad x \in U.$$

We say u is *smooth* if u is infinitely differentiable.

(2) If $\mathbf{u} : U \longrightarrow \mathbb{R}^m$, we write

$$\mathbf{u}(x) = (u^1(x), \dots, u^m(x)), \quad x \in U.$$

The function u^k is the k^{th} component of u, $k = 1, \dots, m$.

11.1.3. Function Spaces. Let $U \subseteq \mathbb{R}^n$.

(1)

$$C^k(U) = \{u : U \to \mathbb{R} \mid u \text{ is continuous}\},$$

$$C(\overline{U}) = \{u \in C(U) \mid u \text{ is uniformly continuous on bounded subsets of U}\},$$

$$C^k(U) = \{u : U \to \mathbb{R} \mid u \text{ is k-times continuously differentiable}\},$$

$$C^k(\overline{U}) = \left\{ u \in C^k(U) \,\middle|\, \begin{matrix} D^\alpha u \text{ is uniformly continuous on bounded subsets of } U \\ \text{for all multiindex } \alpha \text{ with } |\alpha| \le k \end{matrix} \right\}.$$

Thus, if $u \in C^k(\overline{U})$, then $D^\alpha u$ continuously extends to $\overline{U}$ for each multiindex α, $|\alpha| \le k$. On the other hand, if $V \subset U$ is compact, then every continuous function on V is uniformly continuous (Heine-Cantor).

(2)

$$C^\infty(U) = \{u : U \to \mathbb{R} \mid u \text{ is infinitely differentiable }\} = \bigcap_{k=0}^{\infty} C^k(U)$$

$$C^\infty(\overline{U}) = \bigcap_{k=0}^{\infty} C^k(\overline{U})$$

(3) $C_c(U), C_c^k(U)$, etc. denote the functions in $C(U), C^k(U)$, etc. with compact support. $C_c^\infty(U)$, cf. page 7.
(4) Let $1 \leq p < \infty$. The space of p-times integrable functions, denoted by $L^p(U)$, consists of equivalence classes of measurable functions $u : U \to \mathbb{R}^n$ such that

$$\|u\|_{L^p} = \left(\int_U |u|^p\right)^{\frac{1}{p}} < \infty,$$

where two such functions are equivalent if they are equal a.e. (w.r.t. the Lebesgue measure).

The space of essentially bounded functions, denoted by $L^\infty(U)$, consists of equivalence (a.e. equivalence) classes of measurable functions $u : U \to \mathbb{R}^n$ such that

$$\|u\|_\infty = \operatorname{ess\,sup}_{\mathbb{R}^n} |u| < \infty, \tag{11.1}$$

where $\operatorname{ess\,sup}_{\mathbb{R}^n} u = \inf\{a \in \mathbb{R} : |\{x \in \mathbb{R}^n : u(x) > a\}| = 0\}$.
(5) $L^p_{\text{loc}}(U) = \{u : U \to \mathbb{R} \mid u \in L^p(V) \text{ for each } V \subset\subset U\}$, cf. page 7.

Note that $V \subset\subset U$ means $V \subset K \subset U$, where K is compact (compactly contained), cf. page 12.
(6) $W^{k,p}(U)$, cf. page 10,
$H^k(U)$, cf. page 11,
$W_0^{k,p}(U)$, cf. page 11.

11.1.4. Notation for derivatives. Assume $u : U \longrightarrow \mathbb{R}$, $x \in U$.
(1) $\frac{\partial u}{\partial x_i}(x) = \lim_{h\to\infty} \frac{u(x+he_i-u(x)}{h}$, provided the limit exists.
(2) We write u_{x_i} for $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ for $u_{x_i x_j}$, etc.
(3) *Multiindex Notation*
  (a) A vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is called a multiindex of order
  $$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$
  (b) Let $x \in \mathbb{R}^n$, then
  $$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$
  (c) We define for a given multiindex α
  $$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial_{x_1}^{\alpha 1} \cdots \partial_{x_n}^{\alpha_n}} = \partial_{x_1}^{\alpha_n} \cdots \partial_{x_n}^{\alpha_n} u$$
  (d) Let $k \in \mathbb{N}_0$. The set of all partial derivatives of order k is denoted by
  $$D^k u(x) := \{D^\alpha u(x) | |\alpha| = k\}.$$
  (e) Let α, β be multiindices, then
  $$\beta \leq \alpha \Longleftrightarrow \beta_1 \leq \alpha_1, \ldots, \beta_n \leq \alpha_n.$$
  (f) Let α, β be multiindices with $\beta \leq \alpha$. Then
  $$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!},$$
  where $\alpha! = \alpha_1! \cdots \alpha_n!$.

(g) *Leibniz formula.* If $u, v \in \mathbf{C}_c^\infty(U)$, then

$$D^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\alpha u D^{\alpha-\beta} u$$

11.2. Inequalities

THEOREM 11.1. *Let $1 \le p < \infty$. Then*

$$(a+b)^p \le 2^{p-1}(a^p + b^p), \quad a, b > 0. \tag{11.2}$$

PROOF. The function $t \mapsto t^p$ is convex for $t \ge 0$. Therefore,

$$\left(\frac{a+b}{2}\right)^p \le \frac{1}{2}(a^p + b^p).$$

□

COROLLARY 11.2. *Let $1 \le p < \infty$. Then*

$$\|f+g\|_{L^p(U)}^p \le 2^{p-1}\left(\|f\|_{L^p(U)}^p + \|g\|_{L^p(U)}^p\right).$$

PROOF. Use the triangle inequality and apply Theorem 11.1 with $a = |f|$ and $b = |g|$. □

THEOREM 11.3 (Young's inequality). *Let $1 < p,\ q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \qquad a, b > 0.$$

PROOF.

$$ab = e^{\log(ab)} = e^{\log a + \log b} = e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q}.$$

The function $x \mapsto e^x$ is convex for all $x \in \mathbb{R}$. Therefore,

$$e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q} \le \frac{1}{p}e^{\log a^p} + \frac{1}{q}e^{\log b^q} = \frac{a^p}{p} + \frac{b^q}{q}.$$

□

THEOREM 11.4 (Minkowski's inequality for integrals). *Let (Ω_1, dx) and (Ω_2, dy) be measure spaces and $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be measurable. Let $r \ge 1$. Then*

$$\left(\int_{\Omega_1}\left(\int_{\Omega_2}|F(x,y)|dy\right)^r dx\right)^{\frac{1}{r}} \le \int_{\Omega_2}\left(\int_{\Omega_1}|F(x,y)|^r dx\right)^{\frac{1}{r}} dy.$$

THEOREM 11.5 (Hölder's inequality). *Let $1 \le p,\ q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, if $u \in L^p(U)$, $v \in L^q(U)$, we have*

$$\int_U |uv|dx \le \|u\|_{L^p(U)}\|v\|_{L^q(U)}\ .$$

THEOREM 11.6 (General Hölder Inequality). *Let* $1 \le p_1, \dots, p_m \le \infty$, *with* $\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$. *Assume* $u_k \in L^{p_k}(U)$ *for* $k = 1 \dots, m$. *Then*

$$\int_U |u_1 \cdots u_m| dx \le \prod_{k=1}^m \|u_i\|_{L^{p_k}(U)} \quad .$$

PROOF. Induction, and using Hölders inequality.
Let $m = 2$. This is clear from Hölders inequality.

Induction step:

$$\frac{1}{p_1} + \dots + \Big(\frac{1}{p_m} + \frac{1}{p_{m+1}}\Big) = 1$$

$$\frac{1}{p_m} + \frac{1}{p_{m+1}} = \frac{p_m + p_{m+1}}{p_m p_{m+1}}; \quad \alpha := \frac{p_m p_{m+1}}{p_m + p_{m+1}}$$

$\Rightarrow$

$$\int_U |u_1(x) \cdots u_m(x) u_{m+1}(x)| dx \le \|u_1\|_{L^{p_1}} \cdots \|u_{m-1}\|_{L^{p_{m-1}}} . \|u_m u_{m+1}\|_{L^\alpha}$$

It remains to show that $\|u_m u_{m+1}\|_{L^\alpha} \le \|u_m\|_{L^{p_m}} . \|u_{m+1}\|_{L^{p_{m+1}}}$.

Note that $\frac{\alpha}{p_m} + \frac{\alpha}{p_{m+1}} = 1$. Hence,we can use Hölder's inequality to obtain

$$\int_U |u_m^\alpha| . |u_{m+1}^\alpha| \, dx \le \left(\int_U |u_m^\alpha|^{\frac{p_m}{\alpha}} dx \right)^{\frac{\alpha}{p_m}} \left(\int_U |u_{m+1}^\alpha|^{\frac{p_{m+1}}{\alpha}} dx \right)^{\frac{\alpha}{p_{m+1}}} .$$

□

COROLLARY 11.7. *Let* $1 \le p \le q \le \infty$. *Let* $U \subseteq \mathbb{R}^n$ *be bounded. Then*

$$\|f\|_{L^p(U)} \le |U|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(U)}.$$

PROOF. Apply Theorem 11.5 with $u = |f|^p$, $v = 1$ and exponents $\frac{p}{q} + 1 - \frac{p}{q} = 1$. □

Similar proofs establish the following discrete versions of the above inequalities

THEOREM 11.8. *Let* $1 \le p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ *and* $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. *Then*

$$\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}} .$$

COROLLARY 11.9. *Let* $1 \le p \le q \le \infty$. *Let* $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. *Then*

$$\left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \le n^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} . \tag{11.3}$$

PROOF. Assume that $(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} = 1$. Then, $|x_i| \le 1$ for every $1 \le i \le n$. Hence, for all $q \ge p$ the following inequality holds

$$\sum_{i=1}^n |x_i|^q \le \sum_{i=1}^n |x_i|^p = 1 = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{q}{p}} .$$

Therefore,

$$\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}. \tag{11.4}$$

If $(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} \neq 1$. Set

$$z_i = \frac{x_i}{(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}}.$$

Then $(\sum_{i=1}^n |z_i|^p)^{\frac{1}{p}} = 1$. Therefore, by the above we obtain

$$\left(\sum_{i=1}^n |z_i|^q\right)^{\frac{1}{q}} \le \left(\sum_{i=1}^n |z_i|^p\right)^{\frac{1}{p}}. \tag{11.5}$$

This is equivalent to

$$\frac{(\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}}{(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}} \le \frac{(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}}{(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}}. \tag{11.6}$$

Hence,

$$\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}. \tag{11.7}$$

This gives the left-hand side of inequality (11.3)

Applying Theorem 11.8 with $y = (1, \ldots, 1)$ and $\frac{p}{q} + 1 - \frac{p}{q} = 1$ yields the right-hand side of equation (11.3).

□

THEOREM 11.10 (Interpolation inequality for L^p-norms). *Let $1 \le p_0 \le q_0 \le \infty$ and $0 < \theta < 1$. We define*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Let $f \in L^{p_0} \cap L^{p_1}$. Then $f \in L^{p_\theta}$ and

$$\|f\|_{L^{p_\theta}} \le \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta}. \tag{11.8}$$

PROOF. Apply Hölder's inequality with $\frac{\theta p_\theta}{p_1} + \frac{(1-\theta)p_\theta}{p_0} = 1$.

$$\begin{aligned}
\int |f|^{p_\theta} dx &= \int |f|^{p_\theta \theta} |f|^{p_\theta(1-\theta)} dx \\
&\le \left(\int |f|^{p_\theta \theta \frac{p_1}{p_\theta \theta}}\right)^{\frac{p_\theta \theta}{p_1}} \left(\int |f|^{p_\theta(1-\theta)\frac{p_0}{p_\theta(1-\theta)}}\right)^{\frac{p_\theta(1-\theta)}{p_0}} \\
&= \|f\|_{L^{p_1}}^{\theta} \|f\|_{L^{p_0}}^{1-\theta}.
\end{aligned}$$

□

THEOREM 11.11 (Young's inequality). *Let $1 \le p, q, r \le \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then $f * g$ exists a.e. and lies in $L^r(\mathbb{R}^n)$ with*

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

11.3. Calculus Facts

Let $U \subset \mathbb{R}^n$ be open and bounded.

DEFINITION 11.12. We say ∂U is C^k if for each point $x_0 \in \partial U$ there exists $r > 0$ and a C^k-function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that we have

$$U \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, ..., x_{n-1})\}.$$

Note that

$$\partial U \cap B(x_0, r) = \{x \in B(x_0, r) : x_n = \gamma(x_1, ..., x_{n-1})\}.$$

∂U is C^∞ if it is C^k for $k = 1, 2, \dots$.

DEFINITION 11.13.

(1) If ∂U is C^1, then along ∂U is defined the outward pointing unit normal vector field
$$\boldsymbol{\nu} = (\nu^1, \dots, \nu^n).$$
(2) The unit normal at any point $x_0 \in \partial U$ is $\boldsymbol{\nu}(x_0) = \nu = (\nu_1, \dots, \nu_n)$.
(3) Let $u \in C^1(\overline{U})$. We call
$$\frac{\partial u}{\partial \nu} = \boldsymbol{\nu} \cdot Du.$$

THEOREM 11.14 (Gauss-Green Theorem). *Suppose $u \in C^1(\overline{U})$. Then*

$$\int_U u_{x_i} dx = \int_{\partial U} u\nu^i dS, \quad (i = 1, \dots, n). \tag{11.9}$$

THEOREM 11.15 (Integration by parts formula). *Let $u, v \in C^1(\overline{U})$. Then*

$$\int_U u_{x_i} v\, dx = -\int_U u v_{x_i}\, dx + \int_{\partial U} u\nu^i dS, \quad (i = 1, \dots, n). \tag{11.10}$$

Polarcoordinates. For $x \in \mathbb{R}^n \setminus \{0\}$ the polar coordinates are given by

$$r = |x| \quad \text{and} \quad \omega = \frac{x}{|x|} \in S^{n-1} = \{\omega \in \mathbb{R}^n : |\omega| = 1\}.$$

THEOREM 11.16 (Integration in polar coordinates). *Let $f \in L^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \int_{S^{n-1}} f(r\omega)r^{n-1}d\sigma(\omega)dr.$$

where σ is the Borel measure on the unit sphere S^{n-1} in $\mathbb{R}^n$.

COROLLARY 11.17. *If f is a measurable function on $\mathbb{R}^n$, nonnegative or integrable, such that $f(x) = g(|x|)$ for some function g on $(0, \infty)$, then*

$$\int_{\mathbb{R}^n} f(x)dx = \sigma(S^{n-1}) \int_0^\infty g(r)r^{n-1}dr.$$

THEOREM 11.18 (Arzela-Ascoli Compactness Criterion). *Let $\{f_k\}_{k=1}^\infty$ be a sequence of real-valued functions on $\mathbb{R}^n$ such that*

$$|f_k(x)| \le M \quad (k = 1, \dots \quad , x \in \mathbb{R}^n)$$

for some constant M and the $\{f_k\}_{k=1}^\infty$ are uniformly equicontinuous, i.e.

$$\forall \eta > 0\, \exists \delta > 0\, \forall k \in \mathbb{N}\, \forall x, y \in \mathbb{R}^n : \, |x - y| < \delta \Rightarrow |f_k(x) - f_k(y)| < \eta.$$

Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subseteq \{f_k\}_{k=1}^{\infty}$ *and a continuous function* f*, such that* $f_{k_j} \to f$ *uniformly on compact subsets of* $\mathbb{R}^n$*.*

11.4. Convergence theorems for integrals

Definition 11.19 (Average of f over a ball $B(x,r)$). We denote by

$$\fint_{B(x,r)} f\,dy = \frac{1}{|B(x,r)|}\int_{B(x,r)} f\,dy$$

the average of f over the ball $B(x,r)$ and by

$$(f)_U = \fint_U f\,dx = \frac{1}{|\,U\,|}\int_U f\,dx$$

the average of f over $U \subset \mathbb{R}^n$.

Theorem 11.20 (Lebesgue Differentiation Theorem). *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be locally integrable.*

(i) *Then for a.e. point* $x_0 \in \mathbb{R}^n$*,*

$$\fint_{B(x_0,r)} f\,dx \longrightarrow f(x_0), \qquad \text{as } r \to 0$$

(ii) *In fact, for a.e. point* $x_0 \in \mathbb{R}^n$*,*

$$\fint_{B(x_0,r)} |\,f(x) - f(x_0)\,|\,dx \longrightarrow 0, \qquad \text{as } r \to 0. \tag{11.11}$$

A point x_0 *at which (11.11) holds, is called a Lebesgue point* of f.

Remark 11.21. More generally, if $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some $1 \le p < \infty$, then for a.e. point $x_0 \in \mathbb{R}^n$ we have

$$\fint_{B(x_0,r)} |\,f(x) - f(x_0)\,|^p\,dx \longrightarrow 0, \qquad \text{as } r \to 0$$

11.5. Absolutely continuous functions

Definition 11.22. Let I be an interval in $\mathbb{R}$. A function $u : I \to \mathbb{R}$ is absolutely continuous on I if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every finite sequence of pairwise disjoint subintervals $((x_k, y_k))_k$ of I we have that

$$\sum_k (y_k - x_k) < \delta \quad \text{implies} \quad \sum_k |u(y_k) - u(x_k)| < \varepsilon.$$

Theorem 11.23 (Fundamental theorem of calculus).

(1) *Let* $f : I \to \mathbb{R}$ *be Lebesgue integrable. The function*

$$F(x) = \int_{x_0}^{x} f(t)dt, \quad x \in I$$

is for every $x_0 \in I$ *absolutely continuous. In particular* F *is differentiable a.e. in* I *and* $F'(x) = f(x)$ *for a.e.* $x \in I$*.*

(2) *Let* $F : I = (a, b) \to \mathbb{R}$ *be absolutely continuous. Then* F *is differentiable a.e. in* I *and* F' *is Lebesgue integrable with*

$$F(b) - F(a) = \int_a^b F'(t)dt.$$

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