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Geometry, Modelling and Control of Infinite Dimensional Port-Hamiltonian Systems

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Eingereicht von:

Andreas Siuka

Am Auring 14, 4533 Piberbach

Betreuung:

o.Univ.-Prof. Dipl.-Ing. Dr.techn. K. Schlacher

Beurteilung:

o.Univ.-Prof. Dipl.-Ing. Dr.techn. K. Schlacher

Univ.-Prof. Dipl.-Ing. Dr.techn. A. Kugi

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Kurzfassung

Im konzentriert-parametrischen Fall hat sich in den letzten Jahren die Klasse der Tor-basierten Hamiltonschen Systeme besonders darin ausgezeichnet, eine strukturierte mathematische Systemdarstellung zu gewährleisten, welche die Anwendung sogenannter energiebasierter Regelungsentwürfe erlaubt. Diese Arbeit widmet sich nun der Analyse und weiteren Verallgemeinerung dieser Systemklasse hinsichtlich der Modellierung verteilt-parametrischer Systeme und der Übertragung energiebasierter Regelungsmethoden vom konzentriert- auf den verteilt-parametrischen Fall basierend auf dem klassischen evolutionären Zugang. Die vorliegende Arbeit ist in drei Hauptteile gegliedert. Der erste Teil behandelt die Analyse und Weiterentwicklung verteilt-parametrischer Tor-basierter Hamiltonscher Systeme, wobei prinzipiell zwei Systemklassen untersucht werden, welcher in direkter Analogie zur Tor-basierten Hamiltonschen Darstellung konzentriert-parametrischer Systeme stehen. Um möglichst Koordinatensystem unabhängige und vor allem hinsichtlich physikalischer Anwendungen allgemein gültige Systemklassen zu formulieren, werden formale differentialgeometrische Konzepte genutzt, welche ein effektives mathematisches Rahmenwerk für die Untersuchung verteilt-parametrischer Systeme darstellen.

Im zweiten Teil der Arbeit wird die Formulierung von Feldtheorien auf Basis des Tor-basierten Hamiltonschen Ansatzes behandelt. Dabei wird zuerst die Tor-basierte Hamiltonsche Modellierung von Balkenmodellen untersucht, welche auf der bekannten Timoshenko Balkentheorie beruhen. Weiters werden fluiddynamische Anwendungen in Lagrangescher Betrachtungsweise betrachtet, welche beispielsweise bei der Modellierung von Einspritzprozessen auftreten können. Dabei wird zuerst die Tor-basierte Hamiltonsche Darstellung eines bewegten, idealen Fluidkontinuums (keine viskosen Spannungen) untersucht, welche dann als Basis für die Tor-basierte Hamiltonsche Formulierung der bekannten Navier-Stokes Gleichungen (in Lagrangescher Betrachtungsweise) dient. Darauf basierend werden weiters elektrisch leitende Fluide untersucht, um so eine Tor-basierte Hamiltonsche Formulierung der Grundgleichungen der Magnetohydrodynamik in Lagrangescher Betrachtungsweise unter der Voraussetzung quasistationärer elektrodynamischer Beziehungen zu erhalten.

Der dritte Teil der Arbeit widmet sich der direkten Übertragung einer, aus dem konzentriert-parametrischen Fall wohl bekannten, energiebasierten Regelungsmethode – basierend auf sogenannten strukturellen Invarianten – auf die verteilt-parametrische Tor-basierte Hamiltonsche Systemklasse. Dieses Konzept wird zur Regelung des Timoshenkobalkens mittels Randeingriff genutzt.

Abstract

With regard to the lumped-parameter case the Port-Hamiltonian framework has proved itself over the past years concerning a structured mathematical system description which allows the application of so-called energy based control methods. This work focuses on the analysis and further generalisation of this system class with respect to the modelling of distributed-parameter systems and the extension of energy based control concepts from the lumped- to the distributed-parameter case on the basis of the classical evolutionary approach. The instant work is structured in three main parts. The first part is dedicated to the analysis and further development of distributed-parameter Port-Hamiltonian systems. In principle, two system classes will be investigated in detail, where the direct analogies to the Port-Hamiltonian framework in the finite dimensional case will become apparent. In order to formulate a coordinate system independent and mainly general system class – with regard to physical applications – formal differential geometric concepts which represent an effective mathematical framework for the investigation of infinite dimensional systems will be used.

The second part of this work deals with the formulation of field-theories based on the Port-Hamiltonian framework. First of all, the Port-Hamiltonian formulation of beams modelled according to the Timoshenko beam theory is investigated. Furthermore, fluid dynamical applications in a Lagrangian setting are taken into account which may occur for the modelling of injection processes, for instance. First, the Port-Hamiltonian formulation of an ideal fluid continuum (no viscous stresses) in motion which will serve as the basis for the Port-Hamiltonian representation of the well-known Navier-Stokes equations (restricted to the Lagrangian point of view) is analysed. In addition, based on these investigations we also take electrically conducting fluids into account leading to a Port-Hamiltonian formulation of the governing equations of magnetohydrodynamics in a Lagrangian setting on the condition of quasi-stationary electrodynamic relations.

The third part of the thesis aims to directly generalise an energy based control concept based on so-called structural invariants – well-known in the lumped-parameter case – to the infinite dimensional Port-Hamiltonian system class. This approach is applied to the energy based boundary control of the Timoshenko beam.

To Sandra

Preface

This thesis was developed within my employment as a research assistant at the Institute of Automatic Control and Control Systems Technology which is part of the technical faculty of the Johannes Kepler University of Linz, Austria.

First of all, my deep respectfulness and my special gratitude with respect to Professor Kurt Schlacher has to be emphasised since he encouraged me and, in particular, he aroused my interests in control theory as well as in differential geometric issues with regard to control purposes. I appreciated very much the inspiring working atmosphere at the institute and the professional discussions; these and his never ending support contributed a lot to the growth of my work and my understanding of complex mathematical and physical relations.

Special thanks and deep acknowledgements go to Markus Schöberl for his scientific comments and, especially, for his inspirations which have influenced my way of scientific thinking a lot. Furthermore, I want to thank all the colleagues and former staff at the institute, namely Bernhard Ramsebner, Harald Daxberger, Karl Rieger, Richard Stadlmayr, Paul Ludwig, Klaus Weichinger and Phillip Wieser as well as Julia Gabriel and Christian Höfler for the excellent working atmosphere and the (not only scientific) discussions which often helped and encouraged me in continuing my research.

Finally, I want to express my thankfulness to my better half, Sandra, in particular for her understanding for my work and for all her love over the last few years. I also want to thank my parents who have offered me the opportunity of a study, my friends and all the persons who have supported me during the last few years. Last but not least I want to thank Stefan Söllradl for the proofreading of this thesis.

Over the past years my research was focused on the modelling and control of infinite dimensional Port-Hamiltonian systems. During this time-period I considered several concepts and I had worked out some ideas; of course, these led to further interesting problems which could not be completely addressed within my thesis. Finally, I hope that the results presented in the forthcoming chapters are useful and beneficial with respect to further control theoretic issues on distributed-parameter systems.

Contents

1	Introduction	1
2	Geometric Preliminaries	4
2.1	Bundles	4
2.1.1	Tangent, Cotangent and Vertical Bundles	4
2.1.2	Bundle Morphisms and Pull-back Bundles	5
2.2	Jet Bundles	6
2.2.1	First-order Jet Bundles	6
2.2.2	Higher-order Jet Bundles	8
2.2.3	Integration on Manifolds	9
2.3	Poisson Structures	10
3	Port-Hamiltonian Systems	13
3.1	Finite Dimensional Port-Controlled Hamiltonian Systems	14
3.2	Infinite Dimensional Port-Controlled Hamiltonian Systems	17
3.2.1	The Geometry of Distributed-Parameter Systems	17
3.2.2	First-order Hamiltonian Densities	19
3.2.3	Hamiltonian Evolution Equations I	21
3.2.4	Hamiltonian Evolution Equations II	29
3.2.5	Concluding Remarks	33
4	Port-Hamiltonian Formulation of Field Theories	35
4.1	Port-Hamiltonian Modelling of the Timoshenko Beam	36
4.2	Port-Hamiltonian Formulation of Fluid Dynamics	40
4.2.1	The Geometry of Lagrangian Fluid Dynamics	40
4.2.2	Conservation of Mass	43
4.2.3	Stress Forms and Constitutive Relations in Fluid Dynamics	44
4.2.4	The Balance of Linear Momentum	48
4.2.5	Port-Hamiltonian Formulation of the Ideal Fluid	50
4.2.6	Port-Hamiltonian Formulation of the Navier-Stokes Equations	55
4.3	Port-Hamiltonian Formulation of Magnetohydrodynamics	60
4.3.1	Electromagnetic Body Forces	61

4.3.2	Port-Hamiltonian Formulation of inductionless Magnetohydrodynamics	68
5	Control of infinite dimensional Port-Hamiltonian Systems	73
5.1	Control of finite dimensional Port-Hamiltonian Systems based on Structural Invariants	74
5.2	Boundary Control of infinite dimensional Port-Hamiltonian Systems based on Structural Invariants	78
5.3	Boundary Control of the Timoshenko Beam based on Structural Invariants .	85
6	Summary and Outlook	94
A	Proofs and Detailed Computations	96
A.1	The Application of the Horizontal Differential	96
A.2	The Rate of Deformation Tensors	96
A.3	The Stored Energy Relation	98
A.4	Hamiltonian Formulation of the Ideal Fluid	99
A.5	The first Viscous Piola-Kirchhoff Stress Form	101
A.6	The Damping Tensor in iMHD	102
A.7	Control of finite dimensional PCHD Systems	104
A.8	The Equivalent Norm on \mathcal{Z}	105
A.9	The Existence of the Inverse Operator \mathcal{A}^{-1}	108
	Bibliography	111

Introduction

The basis for the analysis and control of complex physical systems provides a mathematical model/description of the system which can be used not only for simulation purposes but also for the stability investigations. In particular, for lumped-parameter systems the Port-Hamiltonian framework enjoys great popularity in the modelling and control community since it provides a structured mathematical system representation, where for many applications the physics behind the governing equations becomes apparent in a remarkable way. Even this structured system description allows the application of so-called energy based control methods, see, e.g., [Gómez-Estern et al., 2001, Ortega et al., 2001, 2002, van der Schaft, 2000]. Over the past years the trend was – due to the physical interpretation offered by the Port-Hamiltonian framework – to extend the Port-Hamiltonian system class to the distributed-parameter case, where the governing equations are represented by partial differential equations – abbreviated as PDEs in the sequel. In this context there exist several approaches (which are known to the author) for a possible generalisation of the (Port-)Hamiltonian framework to the infinite dimensional scenario;

- the polysymplectic approach going back to DeDonder/Weyl, see, e.g., [Giachetta et al., 1997, Kanatchikov, 1998] and references therein,
- a concept based on Stokes-Dirac structures, see [van der Schaft and Maschke, 2002], and references therein, and also the extensions for control purposes, e.g., [Macchelli and Melchiorri, 2004a,b, Macchelli et al., 2004c,d, Macchelli and Melchiorri, 2005, Rodriguez et al., 2001],
- and an approach based on the classical evolutionary approach, see, e.g., [Marsden and Ratiu, 1994, Olver, 1993], and references therein, and also the extensions with regard to control purposes [Ennsbrunner and Schlacher, 2005, Ennsbrunner, 2006, Kugi, 2001, Schlacher, 2007, 2008, Schöberl et al., 2008].

Within this thesis the infinite dimensional Port-Hamiltonian system representation based on the (classical) evolutionary approach is considered, where this work aims to analyse and further generalise this system class on the one hand with respect to the formulation of Hamiltonian field theories and on the other hand with regard to control purposes including

the controller design based on the Port-Hamiltonian machinery. Roughly speaking, it is remarkable that this approach may be seen as a direct adaption of the classical evolutionary approach based on, e.g., [Marsden and Ratiu, 1994, Olver, 1993] and references therein, where the main difference lies in the fact that the adapted Port-Hamiltonian approach is able to consider non-trivial boundary conditions/terms. Therefore, for many applications it is possible to introduce so-called (energy) ports acting – besides the distributed ports – through the system boundary in order that the considered infinite dimensional system is able to interact with its environment. Even this fact is essential for concrete physical and engineering applications with regard to control purposes, where it often is of interest to investigate the coupling of such systems via their (energy) ports; this fact may be advantageously for the modelling of networks as well as for the design of controller systems which act through the (energy) ports with the considered plant(s) – the well-known control by interconnection methodology.

Particularly, with regard to the introduction, analysis and further development of the infinite dimensional Port-Hamiltonian framework it is obvious that appropriate and effective mathematical tools are necessary for the purpose of system formulations which should be general enough in order that – besides the covering of a wide range of physical applications – the system descriptions do not depend on the used coordinate system. In fact, we are interested in (a kind of) coordinate free introduction and formulation of the system classes; even this fact makes it, in principle, possible to specify and analyse the structural properties which are offered by the Port-Hamiltonian machinery in an intrinsic manner.

The thesis is organised as follows; In **Chapter 2** the mathematical tools which are necessary for a coordinate system independent treatment are briefly introduced and summarised, namely we will use formal differential geometric concepts. In fact, this chapter presents only a brief survey of the geometric objects and basic concepts which are used throughout this work. For detailed proofs and more profound discussions concerning the geometric machinery the interested reader is referred to [Giachetta et al., 1997, Saunders, 1989], as most of the notion in this thesis is based on their work.

Chapter 3 deals with the geometric analysis of infinite dimensional Port-Hamiltonian systems which are based on the classical evolutionary approach. Therefore, we recapitulate the port based system description such as in [Ennsbrunner, 2006], for instance, where it must be emphasised that we confine ourselves to the first-order case only (the higher-order case can be found in [Ennsbrunner, 2006]). With regard to the formulation of (first-order) field theoretical applications the extension of this system representation by the use of appropriate differential operators is illustrated.

The formulation of field theories in the Port-Hamiltonian context is the main focus of **Chapter 4**, where it is worth noting that, in this work, we confine ourselves to first-order Hamiltonian field theoretic applications only. First, the Port-Hamiltonian modelling of beam models based on the Timoshenko beam theory is presented, where specifically the main differences between the presented approach and the one on the basis of the Stokes-Dirac structures are illustrated. The main part of this chapter deals with the (Port-)Hamiltonian formulation of the governing equations of fluid- and magnetohydrodynamics in a Lagrangian setting. In fact, the motion of a fluid continuum is analysed in detail, where the (Port-)Hamiltonian representation of an ideal and a Newtonian fluid which lead to a Port-Hamiltonian formulation of the well-known Navier-Stokes equations in a Lagrangian

setting is investigated. In fact, this point of view may be advantageously with respect to the modelling and the treatment of injection processes, for instance. Furthermore, this approach is directly extended such that electrically conducting fluids in the presence of external electromagnetic fields are taken into account which lead to a Port-Hamiltonian formulation of the governing equations of magnetohydrodynamics, however, on the condition of quasi-stationary electrodynamic relations.

Chapter 5 is dedicated to the controller design based on the illustrated Port-Hamiltonian framework, where we are mainly interested in the stabilisation of so-called *Hamiltonian boundary control systems*. In fact, the well-known *control via structural invariants* approach which is based on the control by interconnection methodology for finite dimensional Port-Hamiltonian systems is directly generalised to the presented infinite dimensional case, where specific criteria and conditions analogous to the lumped-parameter case which allow a systematic (boundary) controller design are derived. This approach is applied to the energy based boundary control of the Timoshenko beam in order to demonstrate the effectiveness of this control concept.

Finally, some proofs and detailed computations can be found in the **Appendix A**; these are omitted in the main parts of the thesis in order to enhance the readability. Nevertheless, the interested reader is asked to inspect these parts whenever they are referenced in the corresponding chapters.

Geometric Preliminaries

The purpose of this chapter is to present the main notions of differential geometry and to illustrate the geometric objects which will be used in the sequel. It is assumed that the reader is familiar with the basic geometric concepts of manifolds, bundles and tensors. In the sequel, tensor notation and, especially, Einstein's convention on sums will be used to keep the formulas short and readable. We use the standard symbol \otimes for the tensor product, d denotes the exterior derivative, \lrcorner the natural contraction between tensor fields and \wedge the exterior product. Moreover, the partial derivatives are abbreviated by ∂_A^B with respect to the coordinates with indices $\frac{A}{B}$ and, e.g., $[m_B^A]$ corresponds to the matrix representation of a (second-order) tensor m , for instance. The interested reader is referred to standard books dealing with differential geometry and Jet bundles such as [Boothby, 1986, Giachetta et al., 1997, Saunders, 1989] for more detailed information.

2.1 Bundles

This subsection is dedicated to the introduction of the necessary bundle constructions which will be of essential use throughout this thesis.

2.1.1 Tangent, Cotangent and Vertical Bundles

Let us introduce the bundle $\pi : \mathcal{E} \rightarrow \mathcal{B}$ with local coordinates (X^i) , $i = 1, \dots, m$ on \mathcal{B} and (X^i, x^α) , $\alpha = 1, \dots, n$ on \mathcal{E} . A (local) section $\Phi : \mathcal{B} \rightarrow \mathcal{E}$, or equivalently $\Phi \in \Gamma(\pi)$, which meets $\pi \circ \Phi = \text{id}_{\mathcal{B}}$ with respect to the identity map $\text{id}_{\mathcal{B}}$ on \mathcal{B} leads in local coordinates to $x^\alpha = \Phi^\alpha(X^i)$, where the set of all sections of the bundle $\pi : \mathcal{E} \rightarrow \mathcal{B}$ is denoted by $\Gamma(\pi)$. The tangent bundle $\tau_{\mathcal{E}} : \mathcal{T}(\mathcal{E}) \rightarrow \mathcal{E}$ (locally) equipped with coordinates $(X^i, x^\alpha, \dot{X}^i, \dot{x}^\alpha)$ and the cotangent bundle $\tau_{\mathcal{E}}^* : \mathcal{T}^*(\mathcal{E}) \rightarrow \mathcal{E}$ which possesses the coordinates $(X^i, x^\alpha, \dot{X}_i, \dot{x}_\alpha)$ can be introduced in a standard manner with respect to the holonomic bases $\{\partial_i, \partial_\alpha\}$ and $\{dX^i, dx^\alpha\}$ for the tangent and cotangent spaces, respectively. Typical elements of the tangent bundle $\tau_{\mathcal{E}} : \mathcal{T}(\mathcal{E}) \rightarrow \mathcal{E}$ are tangent vector fields $v : \mathcal{E} \rightarrow \mathcal{T}(\mathcal{E})$ taking in local coordinates the form of $v = v^i(X^i, x^\alpha)\partial_i + v^\alpha(X^i, x^\alpha)\partial_\alpha$ with $\dot{X}^i = v^i(X^i, x^\alpha)$ as well as $\dot{x}^\alpha = v^\alpha(X^i, x^\alpha)$ and elements of the cotangent bundle are 1-forms $\omega : \mathcal{E} \rightarrow \mathcal{T}^*(\mathcal{E}) =$

$\bigwedge^1 \mathcal{T}^*(\mathcal{E})$ locally given as $\omega = \omega_i(X^i, x^\alpha) dX^i + \omega_\alpha(X^i, x^\alpha) dx^\alpha$ with $\dot{X}_i = \omega_i(X^i, x^\alpha)$ as well as $\dot{x}_\alpha = \omega_\alpha(X^i, x^\alpha)$. These constructions can be summarised in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T}(\mathcal{E}) & \xrightarrow{\tau_{\mathcal{E}}} & \mathcal{E} \xleftarrow{\tau_{\mathcal{E}}^*} \mathcal{T}^*(\mathcal{E}) \\ \pi_* \downarrow & & \downarrow \pi \\ \mathcal{T}(\mathcal{B}) & \xrightarrow{\tau_{\mathcal{B}}} & \mathcal{B} \end{array}$$

In this context it is possible to introduce an important subbundle of the tangent bundle $\tau_{\mathcal{E}}$; the vertical bundle $\nu_{\mathcal{E}} : \mathcal{V}(\mathcal{E}) \rightarrow \mathcal{E}$ is equipped with local coordinates $(X^i, x^\alpha, \dot{x}^\alpha)$ with respect to the holonomic basis $\{\partial_\alpha\}$. Typical elements of the vertical bundle are vertical vector fields $v : \mathcal{E} \rightarrow \mathcal{V}(\mathcal{E})$ which meet $\pi_* \circ v = 0$ with respect to $\pi_* : \mathcal{T}(\mathcal{E}) \rightarrow \mathcal{T}(\mathcal{B})$ and take in local coordinates the form of $v = v^\alpha(X^i, x^\alpha) \partial_\alpha$, i.e., they are tangent to the fibres of the bundle π .

Throughout this thesis we make heavy use of the exterior algebra, where different operations are available. Exemplary, the exterior derivative d serves as a map $d : \bigwedge^k \mathcal{T}^*(\mathcal{E}) \rightarrow \bigwedge^{k+1} \mathcal{T}^*(\mathcal{E})$, for instance, the contraction or the interior product of a form with a vector field is denoted by $\lrcorner : \bigwedge^k \mathcal{T}^*(\mathcal{E}) \rightarrow \bigwedge^{k-1} \mathcal{T}^*(\mathcal{E})$ and the Lie derivative of a form $\omega : \mathcal{E} \rightarrow \bigwedge^k \mathcal{T}^*(\mathcal{E})$ along the tangent vector field $v : \mathcal{E} \rightarrow \mathcal{T}(\mathcal{E})$ is given by $v(\omega)$ and takes the form of $v(\omega) = v \lrcorner d\omega + d(v \lrcorner \omega)$. For example, a function $f \in C^\infty(\mathcal{E}) = \bigwedge^0 \mathcal{T}^*(\mathcal{E})$ is a 0-form, where $C^\infty(\mathcal{E})$ denotes the class of smooth functions on \mathcal{E} . The differential of f , a 1-form, reads as $df = \partial_i f dX^i + \partial_\alpha f dx^\alpha$ and, furthermore, $v(f) = v \lrcorner df = v^i \partial_i f + v^\alpha \partial_\alpha f$ denotes the Lie derivative of f along the tangent vector field $v : \mathcal{E} \rightarrow \mathcal{T}(\mathcal{E})$.

2.1.2 Bundle Morphisms and Pull-back Bundles

It is also of interest to consider maps between bundles, where we confine ourselves to so-called bundle morphisms. Let us consider the bundles $\pi : \mathcal{E} \rightarrow \mathcal{B}$ and $\bar{\pi} : \bar{\mathcal{E}} \rightarrow \bar{\mathcal{B}}$ equipped with local coordinates (X^i, x^α) and $(\bar{X}^{\bar{i}}, \bar{x}^{\bar{\alpha}})$ with $i, \bar{i} = 1, \dots, m$ as well as $\alpha, \bar{\alpha} = 1, \dots, n$. Then, a bundle morphism which maps fibres of $\pi : \mathcal{E} \rightarrow \mathcal{B}$ into fibres of $\bar{\pi} : \bar{\mathcal{E}} \rightarrow \bar{\mathcal{B}}$ is a pair (ψ, φ) which may be described by the commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\varphi} & \bar{\mathcal{E}} \\ \Phi \left(\begin{array}{c} \downarrow \pi \\ \downarrow \bar{\pi} \end{array} \right) & & \downarrow \bar{\pi} \\ \mathcal{B} & \xrightarrow{\psi} & \bar{\mathcal{B}} \end{array}$$

including the maps $\varphi : \mathcal{E} \rightarrow \bar{\mathcal{E}}$ and $\psi : \mathcal{B} \rightarrow \bar{\mathcal{B}}$ with respect to $\psi \circ \pi = \bar{\pi} \circ \varphi$. In local coordinates a bundle morphism takes the form of

$$\bar{X}^{\bar{i}} = \psi^{\bar{i}}(X^i), \quad \bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(X^i, x^\alpha) \quad (2.1)$$

and, consequently, for a (local) section $\Phi \in \Gamma(\pi)$ we obtain a (local) section $\bar{\Phi} \in \Gamma(\bar{\pi})$ according to $\bar{\Phi} = \varphi \circ \Phi \circ \psi^{-1}$ by means of the inverse map $\psi^{-1} : \bar{\mathcal{B}} \rightarrow \mathcal{B}$ which definitely exists whenever ψ is a diffeomorphism.

Throughout this thesis the concept of pull-back bundles is important for most of the forthcoming constructions.

Definition 2.1 (pull-back bundle) *Given the bundle $\pi : \mathcal{E} \rightarrow \mathcal{B}$ and a map $\rho : \mathcal{W} \rightarrow \mathcal{B}$ concerning the manifolds \mathcal{W} and \mathcal{B} . The pull-back of the bundle π by ρ is the bundle $\rho^*(\pi) : \rho^*(\mathcal{E}) \rightarrow \mathcal{W}$, where the total space is defined by $\rho^*(\mathcal{E}) = \{(z, x) \in \mathcal{W} \times \mathcal{E} \mid \pi(x) = \rho(z)\}$ and the projection $\rho^*(\pi)$ corresponds to $\rho^*(\pi)(z, x) = z$, see [Giachetta et al., 1997, Saunders, 1989].*

Roughly speaking, the typical fibre of the pull-back bundle $\rho^*(\pi)$ equals the typical fibre of the bundle π . Therefore, let us consider an adapted coordinate system on \mathcal{W} given by (z^ξ) , $\xi = 1, \dots, r$. Then the adapted coordinate system of the pull-back bundle $\rho^*(\pi)$ reads as (z^ξ, x^α) . In addition, a local section $\Phi \in \Gamma(\pi)$ yields a pull-back section $\rho^*(\Phi) \in \Gamma(\rho^*(\pi))$ locally given by

$$(z^\xi, x^\alpha) = (z^\xi, \Phi^\alpha(X^i) \circ \rho(z^\xi))$$

provided that the set $\rho^{-1}(\mathcal{Q})$ with $\mathcal{Q} \subset \mathcal{B}$ is non-empty, see [Giachetta et al., 1997]. These constructions may be visualised by the following commutative diagram:

$$\begin{array}{ccc} \rho^*(\mathcal{E}) & \xrightarrow{\bar{\rho}} & \mathcal{E} \\ \rho^*(\Phi) \left(\begin{array}{c} \downarrow \\ \rho^*(\pi) \end{array} \right) & & \left(\begin{array}{c} \downarrow \\ \pi \end{array} \right) \Phi \\ \mathcal{W} & \xrightarrow{\rho} & \mathcal{B} \end{array}$$

Moreover, it is remarkable that the pair $(\rho, \bar{\rho})$ characterises a bundle morphism which locally reads as¹

$$X^i = \rho^i(z^\xi), \quad x^\alpha = \bar{\rho}^\alpha(x^\alpha, z^\xi) = \delta_\beta^\alpha x^\beta,$$

and, therefore, we may write $\rho^*(\Phi) = \Phi \circ \rho$.

2.2 Jet Bundles

In order to handle partial differential equations (PDEs) the present framework must be extended such that partial derivatives of dependent coordinates with respect to independent coordinates can be considered. From a geometric point of view, this requirement leads us to the introduction of so-called Jet bundles, see, e.g., [Giachetta et al., 1997, Saunders, 1989].

2.2.1 First-order Jet Bundles

Let us consider again the bundle $\pi : \mathcal{E} \rightarrow \mathcal{B}$ with local coordinates (X^i) , $i = 1, \dots, m$ on \mathcal{B} , called the independent coordinates, and (X^i, x^α) , $\alpha = 1, \dots, n$ on \mathcal{E} including the dependent coordinates (x^α) . A (local) section $\Phi \in \Gamma(\pi)$ relates the dependent coordinates to the independent coordinates by $x^\alpha = \Phi^\alpha(X^i)$.

¹ δ_β^α denotes the Kronecker symbol with $\delta_\beta^\alpha = 1$ for $\alpha = \beta$ and $\delta_\beta^\alpha = 0$ otherwise.

Definition 2.2 (1-jet of a section) Two sections $\Phi, \Psi \in \Gamma(\pi)$ are 1-equivalent at $p \in \mathcal{B}$ if in some adapted coordinate system

$$\Phi^\alpha|_p = \Psi^\alpha|_p, \quad \partial_i \Phi^\alpha|_p = \partial_i \Psi^\alpha|_p$$

are fulfilled, i.e., two sections may be identified by their values and their first-order partial derivatives at $p \in \mathcal{B}$. The equivalence class containing Φ is called the 1-jet $j_p^1 \Phi$ of sections Φ at p , see [Saunders, 1989].

The set of all the 1-jets of local sections of the bundle π possesses a natural structure as a differentiable manifold denoted by $\mathcal{J}^1(\mathcal{E})$ called the *first Jet manifold*. Aside from the bundle π two additional bundles can be constructed which are given by

$$\pi^1 : \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{B}, \quad \pi_0^1 : \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{E}.$$

It is worth mentioning that the adapted coordinate system (X^i, x^α) on \mathcal{E} induces an adapted coordinate system on $\mathcal{J}^1(\mathcal{E})$ given by $(X^i, x^\alpha, x_i^\alpha)$ involving the *derivative coordinates* which are characterised by

$$x_i^\alpha(j_p^1 \Phi) = \partial_i \Phi^\alpha|_p,$$

where $j_p^1 \Phi$ denotes the 1-jet of the section Φ at $p \in \mathcal{B}$.

Definition 2.3 (first-order prolongation of sections) Given the bundle $\pi : \mathcal{E} \rightarrow \mathcal{B}$. The first-order prolongation of a section $\Phi \in \Gamma(\pi)$ is the section $j^1 \Phi : \mathcal{B} \rightarrow \mathcal{J}^1(\mathcal{E})$ defined by

$$j^1 \Phi(p) = j_p^1 \Phi$$

at $p \in \mathcal{B}$, see [Saunders, 1989].

In local coordinates a prolonged section $j^1 \Phi : \mathcal{B} \rightarrow \mathcal{J}^1(\mathcal{E})$ or, equivalently, $j^1 \Phi \in \Gamma(\pi^1)$ reads as

$$(X^i, x^\alpha, x_i^\alpha) = (X^i, \Phi^\alpha(X^i), \partial_i \Phi^\alpha(X^i)).$$

If we apply a bundle morphism (ψ, φ) from $\pi : \mathcal{E} \rightarrow \mathcal{B}$ to the bundle $\bar{\pi} : \bar{\mathcal{E}} \rightarrow \bar{\mathcal{B}}$ visualised by the commutative diagram

$$\begin{array}{ccc} \mathcal{J}^1(\mathcal{E}) & \xrightarrow{j^1 \varphi} & \mathcal{J}^1(\bar{\mathcal{E}}) \\ \left(\begin{array}{c} \pi_0^1 \downarrow \\ \mathcal{E} \xrightarrow{\varphi} \bar{\mathcal{E}} \\ \pi \downarrow \end{array} \right) & & \left(\begin{array}{c} \bar{\pi}_0^1 \downarrow \\ \bar{\mathcal{E}} \xrightarrow{\bar{\varphi}} \bar{\mathcal{B}} \\ \bar{\pi} \downarrow \end{array} \right) \\ \mathcal{B} & \xrightarrow{\psi} & \bar{\mathcal{B}} \end{array}$$

including a diffeomorphism ψ we have in local coordinates

$$\bar{X}^i = \psi^i(X^i), \quad \bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(X^i, x^\alpha)$$

and, therefore, the transition functions of the derivative coordinates follow as

$$\bar{x}_i^{\bar{\alpha}} = (\partial_i \varphi^{\bar{\alpha}} + \partial_\alpha \varphi^{\bar{\alpha}} x_i^\alpha) \partial_i (\psi^{-1})^i, \quad (2.2)$$

where $j^1 \varphi$ denotes the first prolongation of φ . From this construction it may be deduced that $\pi_0^1 : \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{E}$ is an affine bundle. Hence, a prolonged section $j^1 \Phi \in \Gamma(\pi^1)$ leads to a prolonged section $j^1 \bar{\Phi} \in \Gamma(\bar{\pi}^1)$ according to $j^1 \bar{\Phi} = j^1 \varphi \circ j^1 \Phi \circ \psi^{-1}$.

2.2.2 Higher-order Jet Bundles

In order to take higher-order derivative coordinates into account we have to introduce higher-order Jet bundles which may be constructed in an analogous manner as the first-order ones, see [Giachetta et al., 1997, Saunders, 1989], for instance. To keep the formulas short and readable we use the formal notion of an unordered multi-index J , where the k^{th} -order partial derivative is denoted by

$$\partial_J = \partial_{j_k} \circ \dots \circ \partial_{j_1}.$$

The unordered multi-index J denotes a collection of numbers according to (j_1, \dots, j_k) with $j_l \in \{1, \dots, m\}$ for $l = 1, \dots, k$, i.e., it specifies which derivatives are taken into account, and the order of the multi-index, denoted by $\#J = k$, characterises the number of derivatives which are needed (modulo permutations), see [Giachetta et al., 1997, Olver, 1993]. Especially, the notation J, i is an abbreviation for (j_1, \dots, j_k, i) and for the case $\#J = 0$ we have the identity $\partial_J \Phi = \Phi$ for $\Phi \in \Gamma(\pi)$.

Roughly speaking, we define the r -jet of a section $\Phi \in \Gamma(\pi)$ analogously to Definition 2.2, where the set of all the r -jets of local sections $\Phi \in \Gamma(\pi)$ leads to the introduction of the r^{th} Jet manifold $\mathcal{J}^r(\mathcal{E})$ which is equipped with adapted coordinates (X^i, x_j^α) with $0 \leq \#J \leq r$. In particular, for $\#J = 0$ we set $x_j^\alpha = x^\alpha$. Therefore, it is clear that we are able to state

$$\dots \xrightarrow{\pi_r^{r+1}} \mathcal{J}^r(\mathcal{E}) \xrightarrow{\pi_{r-1}^r} \mathcal{J}^{r-1}(\mathcal{E}) \rightarrow \dots \rightarrow \mathcal{J}^2(\mathcal{E}) \xrightarrow{\pi_1^2} \mathcal{J}^1(\mathcal{E}) \xrightarrow{\pi_0^1} \mathcal{J}^0(\mathcal{E}) = \mathcal{E} \xrightarrow{\pi} \mathcal{B},$$

where the additional bundles

$$\pi^r : \mathcal{J}^r(\mathcal{E}) \rightarrow \mathcal{B}, \quad \pi_s^r : \mathcal{J}^r(\mathcal{E}) \rightarrow \mathcal{J}^s(\mathcal{E}), \quad s < r,$$

can be constructed. In this context we can define the r -order prolongation of a (local) section $\Phi \in \Gamma(\pi)$ by $j^r \Phi : \mathcal{B} \rightarrow \mathcal{J}^r(\mathcal{E})$ or, equivalently, $j^r \Phi \in \Gamma(\pi^r)$ which takes in local coordinates the form of

$$(X^i, x^\alpha, x_j^\alpha) = (X^i, \Phi^\alpha(X^i), \partial_J \Phi^\alpha(X^i)), \quad 1 \leq \#J \leq r.$$

For the extension of a bundle morphism to higher-order Jet bundles a special operator must be introduced.

Definition 2.4 (total derivative) *The vector field $d_i : \mathcal{J}^{r+1}(\mathcal{E}) \rightarrow (\pi_r^{r+1})^*(\mathcal{T}(\mathcal{J}^r(\mathcal{E})))$ which reads as*

$$d_i = \partial_i + x_{j,i}^\alpha \partial_\alpha^J, \quad 0 \leq \#J \leq r,$$

is called the total derivative with respect to the independent coordinate X^i and meets

$$d_i(f) \circ j^{r+1} \Phi = \partial_i(f \circ j^r \Phi)$$

for $f \in C^\infty(\mathcal{J}^r(\mathcal{E}))$ and sections $\Phi \in \Gamma(\pi)$, see [Saunders, 1989].

The introduction of the total derivative allows the extension of the bundle morphism (2.1) with respect to high-order cases, where the transition functions are given by (2.2) and

$$\bar{x}_{\bar{I},\bar{j}}^{\bar{\alpha}} = d_j (\bar{x}_{\bar{I}}^{\bar{\alpha}}) \partial_{\bar{j}} (\psi^{-1})^j, \quad 1 \leq \#\bar{I} \leq r, \quad (2.3)$$

see [Giachetta et al., 1997]. If the bundle morphism (2.1) is induced by a 1-parameter transformation group then it is of particular interest to investigate the prolongation of the transformation group by defining the prolongation of the corresponding infinitesimal generators. In fact, we confine ourselves to the prolongation of infinitesimal generators represented by vertical vector fields only.

Definition 2.5 (prolongation of vertical vector fields) *Given a vertical vector field $v : \mathcal{E} \rightarrow \mathcal{V}(\mathcal{E})$ with local representation $v = v^\alpha(X^i, x^\alpha)\partial_\alpha$. The r -order prolongation of this vector field is given by $j^r v : \mathcal{J}^r(\mathcal{E}) \rightarrow \mathcal{V}(\mathcal{J}^r(\mathcal{E}))$ and takes in local coordinates the form of*

$$j^r v = v^\alpha \partial_\alpha + d_J (v^\alpha) \partial_\alpha^J, \quad 1 \leq \#J \leq r,$$

with respect to $d_J = d_{j_r} \circ \dots \circ d_{j_1}$, see [Olver, 1993].

2.2.3 Integration on Manifolds

In the sequel the integration on manifolds plays an important role. Therefore, we assume that the base manifold \mathcal{B} is an oriented compact manifold with (coherently oriented) boundary $\partial\mathcal{B}$, where it is of interest to integrate over certain differential forms. Thus, we briefly introduce the well-known Theorem of Stokes which will be of essential use for all further constructions. For more detailed information the reader is referred to, e.g., [Boothby, 1986, Frankel, 2004].

Theorem 2.1 (Stokes' Theorem) *Let \mathcal{B} be an oriented compact m -dimensional manifold with coherently oriented boundary $\partial\mathcal{B}$ and $\omega : \mathcal{B} \rightarrow \bigwedge^{m-1} \mathcal{T}^*(\mathcal{B})$ a continuously differentiable $(m-1)$ -form on \mathcal{B} . Then, we have²*

$$\int_{\mathcal{B}} d\omega = \int_{\partial\mathcal{B}} \iota^*(\omega)$$

with the inclusion mapping $\iota : \partial\mathcal{B} \rightarrow \mathcal{B}$, see [Boothby, 1986].

Having the total derivative at one's disposal we are able to introduce the horizontal differential in this context.

Definition 2.6 (horizontal differential) *Consider the form $\omega : \mathcal{J}^r(\mathcal{E}) \rightarrow \bigwedge \mathcal{T}^*(\mathcal{J}^r(\mathcal{E}))$. The horizontal differential is defined by*

$$d_h(\omega) = dX^i \wedge d_i(\omega),$$

see [Giachetta et al., 1997, Saunders, 1989].

²The boundary is called coherently oriented if $\partial\mathcal{B} = (-1)^m \widetilde{\partial\mathcal{B}}$ is met, where $\widetilde{\partial\mathcal{B}}$ denotes the boundary with respect to the orientation induced by \mathcal{B} see, e.g., [Boothby, 1986].

The horizontal differential and the exterior derivative are linked by the following lemma.

Lemma 2.1 *Given a section $\Phi \in \Gamma(\pi)$. The relation*

$$d \circ (j^k \Phi)^* = (j^{k+1} \Phi)^* \circ d_h$$

holds for every $k \geq 0$, see [Saunders, 1989].

In particular, for integrals over the oriented compact manifold \mathcal{B} involving horizontal differentials this result enables us to deduce

$$\int_{\mathcal{B}} (j^{r+1} \Phi)^* (d_h(\omega)) = \int_{\mathcal{B}} d((j^r \Phi)^* \omega) = \int_{\partial \mathcal{B}} \iota^* ((j^r \Phi)^* \omega).$$

Remark 2.1 *It is worth noting that the application of the horizontal differential of $(m-1)$ -forms $\omega : \mathcal{J}^r(\mathcal{E}) \rightarrow (\pi^r)^*(\wedge^{m-1} \mathcal{T}^*(\mathcal{B}))$ on \mathcal{B} which take in local coordinates the form of*

$$\omega = \omega^i \partial_i] dX, \quad dX = dX^1 \wedge \dots \wedge dX^m, \quad \omega^i \in C^\infty(\mathcal{J}^r(\mathcal{E})),$$

is equivalent to the divergence theorem, see [Marsden and Hughes, 1994, Olver, 1993], for instance.

2.3 Poisson Structures

Poisson structures play a prominent role for the characterisation and, especially, the analysis of finite dimensional Hamiltonian systems, see, e.g., [Giachetta et al., 1997, Marsden and Ratiu, 1994, Olver, 1993] and references therein. In this section we consider an n -dimensional (smooth) manifold \mathcal{M} locally equipped with coordinates (x^α) , $\alpha = 1, \dots, n$.

Definition 2.7 (Poisson bracket) *A manifold \mathcal{M} is called a Poisson manifold if it is equipped with a Poisson bracket which is a bilinear map $\{\cdot, \cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ satisfying*

1. *Skew-Symmetry*

$$\{F, W\} = -\{W, F\}$$

2. *Leibniz Rule*

$$\{F, W \cdot P\} = \{F, W\} \cdot P + W \cdot \{F, P\}$$

3. *Jacobi Identity*

$$\{\{F, W\}, P\} + \{\{P, F\}, W\} + \{\{W, P\}, F\} = 0$$

for $F, W, P \in C^\infty(\mathcal{M})$, i.e., $\{\cdot, \cdot\}$ is a derivation in each factor, see [Marsden and Ratiu, 1994, Olver, 1993].

Therefore, a Poisson bracket for $F, W \in C^\infty(\mathcal{M})$ can be uniquely defined as

$$\{F, W\} = (J]dW)]dF, \quad (2.4)$$

where J is a contravariant skew-symmetric tensor called the *structure tensor* in local coordinates given by

$$J = J^{\alpha\beta} \partial_\alpha \otimes \partial_\beta, \quad J^{\alpha\beta} \in C^\infty(\mathcal{M}), \quad \alpha, \beta = 1, \dots, n.$$

Moreover, in local coordinates (2.4) reads as

$$\{F, W\} = (\partial_\alpha F) J^{\alpha\beta} (\partial_\beta W).$$

The components of J are defined by the basic brackets $J^{\alpha\beta} = \{x^\alpha, x^\beta\}$ called the *structure functions* satisfying the condition of skew-symmetry

$$J^{\alpha\beta} = \{x^\alpha, x^\beta\} = -\{x^\beta, x^\alpha\} = -J^{\beta\alpha}$$

and the Jacobi Identity

$$\{\{x^\alpha, x^\beta\}, x^\gamma\} + \{\{x^\gamma, x^\alpha\}, x^\beta\} + \{\{x^\beta, x^\gamma\}, x^\alpha\} = 0$$

which takes the equivalent form of

$$J^{\varepsilon\gamma} (\partial_\varepsilon J^{\alpha\beta}) + J^{\varepsilon\beta} (\partial_\varepsilon J^{\gamma\alpha}) + J^{\varepsilon\alpha} (\partial_\varepsilon J^{\beta\gamma}) = 0 \quad (2.5)$$

due to

$$\begin{aligned} \{\{x^\alpha, x^\beta\}, x^\gamma\} &= \{J^{\alpha\beta}, x^\gamma\} = (\partial_\varepsilon J^{\alpha\beta}) J^{\varepsilon\delta} (\partial_\delta x^\gamma) = (\partial_\varepsilon J^{\alpha\beta}) J^{\varepsilon\gamma}, \\ \{\{x^\gamma, x^\alpha\}, x^\beta\} &= \{J^{\gamma\alpha}, x^\beta\} = (\partial_\varepsilon J^{\gamma\alpha}) J^{\varepsilon\delta} (\partial_\delta x^\beta) = (\partial_\varepsilon J^{\gamma\alpha}) J^{\varepsilon\beta}, \\ \{\{x^\beta, x^\gamma\}, x^\alpha\} &= \{J^{\beta\gamma}, x^\alpha\} = (\partial_\varepsilon J^{\beta\gamma}) J^{\varepsilon\delta} (\partial_\delta x^\alpha) = (\partial_\varepsilon J^{\beta\gamma}) J^{\varepsilon\alpha} \end{aligned}$$

with $\alpha, \beta, \gamma, \delta, \varepsilon = 1, \dots, n$, cf. [Olver, 1993], for instance. It is worth mentioning that the exterior derivative applied to functions on \mathcal{M} serves as a map $d : C^\infty(\mathcal{M}) \rightarrow T^*(\mathcal{M})$ and the structure tensor J is a skew-symmetric map of the form $J : T^*(\mathcal{M}) \rightarrow \mathcal{T}(\mathcal{M})$ since $J]dW = J^{\alpha\beta} (\partial_\beta W) \partial_\alpha$.

In this context the notion of a Poisson bracket leads to the definition of a Hamiltonian vector field, see [Giachetta et al., 1997, Marsden and Ratiu, 1994, Olver, 1993], for instance.

Definition 2.8 (Hamiltonian vector field) *Let us consider a Poisson manifold \mathcal{M} together with a smooth function $H \in C^\infty(\mathcal{M})$ called the Hamiltonian. A Hamiltonian vector field $v_H : \mathcal{M} \rightarrow \mathcal{T}(\mathcal{M})$ possesses the property*

$$v_H(F) = \{F, H\} = (J]dH)]dF$$

for an arbitrary smooth function $F \in C^\infty(\mathcal{M})$, where $v_H(F)$ denotes the Lie derivative of F along v_H .

In local coordinates a Hamiltonian vector field reads as $v_H = v_H^\alpha(x^\alpha)\partial_\alpha$ with $\dot{x}^\alpha = v_H^\alpha(x^\alpha)$ and, therefore, $v_H(F) = v_H^\alpha(x^\alpha)\partial_\alpha F = \{F, H\}$. Consequently, Hamilton's equations can be formulated as

$$\dot{x}^\alpha = v_H^\alpha(x^\alpha) = J^{\alpha\beta}\partial_\beta H$$

or, equivalently, in a coordinate free manner

$$\dot{x} = v_H = J|dH. \quad (2.6)$$

Remark 2.2 *If J locally satisfies $\text{rank}([J^{\alpha\beta}]) = 2k \leq \dim(\mathcal{M}) = n$ then it is possible to find (local) canonical coordinates such that the Poisson bracket for $F, W \in C^\infty(\mathcal{M})$ becomes*

$$\{F, W\} = (\partial_i F)(\partial^i W) - (\partial^i F)(\partial_i W), \quad \partial_i = \frac{\partial}{\partial q^i}, \quad \partial^i = \frac{\partial}{\partial p_i},$$

with respect to $x = (q, p, z)$ and $i = 1, \dots, k, j = 1, \dots, n - 2k$ whenever (2.5) is fulfilled. In this case, the Hamiltonian vector field is given as

$$v_H = (\partial^i H)\partial_i - (\partial_i H)\partial^i$$

and (2.6) takes (locally) the form of

$$\dot{q}^i = \partial^i H, \quad \dot{p}_i = -\partial_i H, \quad \dot{z}^j = 0.$$

Moreover, if $2k = n$ is (locally) fulfilled, i.e., $[J^{\alpha\beta}]$ has full rank, then the standard Poisson manifold becomes a symplectic manifold with even rank n and the equations are (locally) given by

$$\dot{q}^i = \partial^i H, \quad \dot{p}_i = -\partial_i H$$

which characterise the canonical form of Hamilton's equations. For more detailed information see, e.g., [Marsden and Ratiu, 1994, Olver, 1993].

Remark 2.3 *If in Definition 2.7 the properties are relaxed such that the Jacobi Identity is dropped then we speak about a generalised Poisson bracket on a generalised Poisson manifold. This fact is rather essential for the introduction of the Port-Hamiltonian system representation, see, e.g., [Dalsmo and van der Schaft, 1999, Stramigioli et al., 1998]. However, it is worth noting that for the case of a generalised Poisson bracket it is, in general, not ensured that a canonical representation exists.*

Port-Hamiltonian Systems

In the finite dimensional case the Hamiltonian formalism is well-known, where the governing equations are represented in an evolutionary first-order form. From a system theoretic point of view, whenever the Hamiltonian corresponds to the system's total energy, the resulting system equations describe, in general, an autonomous, lossless system, where the Hamiltonian serves as a conserved quantity. In order to generalise this framework with regard to dissipative effects and the definition of system in- and outputs the so-called Port-Controlled Hamiltonian system representation (with dissipation) was introduced. It has become an essential tool not only for modelling, system analysis and simulation purposes but also for the application of energy based control methods based on the underlying structural properties of this system class, see [Ortega et al., 2001, 2002, van der Schaft, 2000], for instance.

With respect to the extension of the Hamiltonian framework to the infinite dimensional case there exist several approaches; the polysymplectic approach going back to DeDonder/Weyl (e.g. [Giachetta et al., 1997, Kanatchikov, 1998] and references therein), a concept based on Stokes-Dirac structures (see [van der Schaft and Maschke, 2002]) and the classical evolutionary approach (see, e.g., [Marsden and Ratiu, 1994, Olver, 1993] and references therein). As mentioned before, in order to obtain a Port-Hamiltonian description, we confine ourselves to an extension of the classical evolutionary approach with regard to control purposes based on [Ennsbrunner, 2006], where it must be emphasised that we restrict ourselves to the first-order case only, i.e., we mainly focus our interests on first-order Hamiltonian field theory (for control purposes) as in [Ennsbrunner and Schlacher, 2005, Schlacher, 2007, 2008, Schöberl et al., 2008]. It is remarkable that this approach may be seen as a direct adaption of the classical evolutionary approach, where the main difference lies in the fact that the extended approach is able to consider non-trivial boundary conditions/terms which is crucial for concrete physical and engineering applications concerning control aspects. Furthermore, according to [Ennsbrunner, 2006] an infinite dimensional Port-Hamiltonian system representation can be introduced on the basis of specific multilinear maps by full analogy with the finite dimensional case.

Finally, it must be emphasised that we focus our interests on a geometric description in a coordinate system independent manner in order that system and structural properties which do not depend on the used coordinate system can be specified. This fact is rather

essential particularly with regard to subjects like physical based modelling and system analysis.

In Section 3.1 the well-known Port-Controlled Hamiltonian system class in the finite dimensional scenario is analysed in detail based on a geometric point of view. This part should be seen as the basis for the forthcoming section since the introduced geometric objects and concepts allow a generalisation to the distributed-parameter case; this topic is the main focus of section 3.2, where the extension of this system class to the distributed-parameter case on the basis of specific multilinear maps is discussed and analysed in detail based on [Ennsbrunner, 2006, Schlacher, 2007, 2008, Schöberl et al., 2008]. Particularly, with respect to the formulation of (first-order) field theoretical applications the extension of this system representation by means of appropriate differential operators is illustrated.

3.1 Finite Dimensional Port-Controlled Hamiltonian Systems

Let us consider an n -dimensional (smooth) manifold \mathcal{M} - called the state manifold - locally equipped with coordinates (x^α) , $\alpha = 1, \dots, n$ and a Hamiltonian $H \in C^\infty(\mathcal{M})$ which describes the total energy of the considered Hamiltonian system (2.6) for many applications. If we compute the total time change of the Hamiltonian along the solutions of (2.6) which equals the Lie derivative of H along the Hamiltonian vector field v_H we obtain $v_H(H) = 0$ in consideration of the skew-symmetry of the underlying Poisson structure. In this case, the Hamiltonian serves as a conserved quantity and, therefore, from a system theoretic point of view the equations (2.6) describe, in general, a lossless and autonomous system. Consequently, it is obvious to extend this system class with respect to dissipative effects and the introduction of appropriate system in- and outputs which leads to the definition of the Port-Controlled Hamiltonian system representation, see [van der Schaft, 2000].

Definition 3.1 (PCHD system) *A Port-Controlled Hamiltonian System (with dissipation), or PCH(D) system for short, is given as*

$$\begin{aligned} \dot{x} = v &= (J - R)]dH + u]G \\ y &= G^*]dH \end{aligned} \quad (3.1)$$

with the skew-symmetric interconnection map J , the symmetric positive semidefinite dissipation map R and the input map G as well as its adjoint map G^* with respect to the system input u and the collocated output y . Furthermore, the total change of the Hamiltonian along the solutions of (3.1) reads as

$$v(H) = -(R]dH)]dH + u]y \leq u]y. \quad (3.2)$$

Of course, for this setting the total derivative serves as a map $d : C^\infty(\mathcal{M}) \rightarrow \mathcal{T}^*(\mathcal{M})$ and the interconnection and the dissipation maps are maps of the form $J, R : \mathcal{T}^*(\mathcal{M}) \rightarrow \mathcal{T}(\mathcal{M})$, where the interconnection map J is skew-symmetric, i.e., it fulfils for arbitrary functions $W, F \in C^\infty(\mathcal{M})$ the relation

$$(J]dW)]dF + (J]dF)]dW = 0,$$

and the dissipation map R is a symmetric and positive semidefinite map according to

$$(R \rfloor dW) \rfloor dF - (R \rfloor dF) \rfloor dW = 0, \quad (R \rfloor dW) \rfloor dW \geq 0.$$

Thus, these maps are appropriate tensors in local coordinates given by

$$J = J^{\alpha\beta} \partial_\alpha \otimes \partial_\beta, \quad R = R^{\alpha\beta} \partial_\alpha \otimes \partial_\beta,$$

where the components satisfy $J^{\alpha\beta} = -J^{\beta\alpha}$, $R^{\alpha\beta} = R^{\beta\alpha}$ and $J^{\alpha\beta}, R^{\alpha\beta} \in C^\infty(\mathcal{M})$. Furthermore, we introduce the input vector bundle $v : \mathcal{U} \rightarrow \mathcal{M}$ (locally) equipped with coordinates (x^α, u^ξ) , $\xi = 1, \dots, m$, with respect to the holonomic basis $\{e_\xi\}$ as well as the dual vector bundle $v^* : \mathcal{Y} = \mathcal{U}^* \rightarrow \mathcal{M}$ – called the output vector bundle – which (locally) possesses the coordinates (x^α, y_ξ) and the basis $\{e^\xi\}$ for the fibres. In this context the input map is given by $G : \mathcal{U} \rightarrow \mathcal{T}(\mathcal{M})$ and its adjoint (dual) map corresponds to $G^* : \mathcal{T}^*(\mathcal{M}) \rightarrow \mathcal{U}^* = \mathcal{Y}$. Therefore, the relation

$$(u \rfloor G) \rfloor dH = u \rfloor (G^* \rfloor dH) = u \rfloor y$$

is fulfilled characterising the port with respect to the system input u and the corresponding collocated output y . Hence, the input map G as well as its adjoint map G^* can be both represented by an appropriate tensor which in local coordinates reads as

$$G = G_\xi^\alpha e^\xi \otimes \partial_\alpha, \quad G_\xi^\alpha \in C^\infty(\mathcal{M}).$$

Finally, it must be emphasised that the vector field v is not a tangent vector field on $\mathcal{T}(\mathcal{M})$ any more since it depends on the input u . In fact, it is a vector field of the pull-back bundle¹ $v^*(\tau_{\mathcal{M}}) : v^*(\mathcal{T}(\mathcal{M})) \rightarrow \mathcal{U}$ or, equivalently, it can be interpreted as a submanifold of $\mathcal{T}(\mathcal{M})$ parameterised by u .

It is worth mentioning that (3.2) states nothing else than the balance of energy principle, whenever the Hamiltonian H corresponds to the total energy of the system. In this case the change of the system's energy is equal to the difference of the power flow into the system characterised by the (energy) port $u \rfloor y$ and the dissipated power $(R \rfloor dH) \rfloor dH$.

Remark 3.1 *If the system (3.1) is modelled autonomous and no dissipation is considered then the interconnection map induces a generalised Poisson structure. If, in addition, the components $J^{\alpha\beta}$ meet (2.5) then J is equivalent to the structure tensor and the vector field v is defined as a Hamiltonian vector field v_H in a classical manner, see Definition 2.8.*

Finally, a PCHD system in local coordinates reads as

$$\begin{aligned} \dot{x}^\alpha &= v^\alpha(x^\alpha, u^\xi) = (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta H + G_\xi^\alpha u^\xi \\ y_\xi &= G_\xi^\alpha \partial_\alpha H, \end{aligned} \tag{3.3}$$

and (3.2) takes the form of

$$v(H) = -(\partial_\alpha H) R^{\alpha\beta} (\partial_\beta H) + y_\xi u^\xi$$

with respect to the corresponding vector field $v = v^\alpha(x^\alpha, u^\xi) \partial_\alpha$.

¹To enhance the readability the underlying pull-back structure is suppressed in the definition of the relevant maps.

Remark 3.2 *It is worth noting that the structure of a PCHD system is preserved by diffeomorphisms of the form $\bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x^{\alpha})$ with $\bar{\alpha} = 1, \dots, n$ and the transition functions for the input bundle read as $\bar{u}^{\bar{\xi}} = \psi^{\bar{\xi}}(x^{\alpha})u^{\xi}$, $\bar{\xi} = 1, \dots, m$, where $[\psi^{\bar{\xi}}(x^{\alpha})]$ is invertible. For the case of an affine input bundle which allows for affine input transformations see, e.g., [Schöberl and Schlacher, 2007b].*

Structural Invariants for PCHD Systems

Structural invariants or so-called Casimir functions play a prominent role for the analysis of Hamiltonian systems, see, e.g., [Marsden and Ratiu, 1994, Olver, 1993] and for the development of control concepts based on the Port-Hamiltonian framework, see [Ortega et al., 2001, van der Schaft, 2000], for instance.

Definition 3.2 (structural invariant, PCHD system) *A structural invariant $C \in C^{\infty}(\mathcal{M})$ for a PCHD system (3.1) satisfies in local coordinates the set of PDEs*

$$\partial_{\alpha}C (J^{\alpha\beta} - R^{\alpha\beta}) = 0 \quad (3.4)$$

implying that the total change of C along the solutions of (3.1) results in

$$v(C) = \dot{x} \rfloor dC = (u \rfloor G) \rfloor dC$$

which holds independently of the Hamiltonian H . If, additionally, $u = 0$ or $G^* \rfloor dC = 0$ is met, then the structural invariant serves as a conserved quantity for the PCHD system (3.1). In the case of $\text{rank}([J^{\alpha\beta} - R^{\alpha\beta}]) = n$ the structural invariant is called trivial, see [van der Schaft, 2000].

Consequently, in local coordinates Definition 3.2 implies

$$v(C) = \partial_{\alpha}C (J^{\alpha\beta} - R^{\alpha\beta}) \partial_{\beta}H + (\partial_{\alpha}C) G_{\xi}^{\alpha} u^{\xi} = (\partial_{\alpha}C) G_{\xi}^{\alpha} u^{\xi},$$

where for $u^{\xi} = 0$ or $(\partial_{\alpha}C) G_{\xi}^{\alpha} = 0$ the total change of the structural invariant C along the solutions of (3.1) vanishes, i.e., $v(C) = 0$ and in this case it serves as a conserved quantity for (3.1). It is worth mentioning that structural invariants are only characterised by the underlying structural properties of the system; i.e., they are completely determined by the interconnection and the dissipation map of the PCHD system and, thus, they do not depend on the system's Hamiltonian.

Remark 3.3 *For the autonomous and non-dissipative case the structural invariants are completely determined by the underlying (generalised) Poisson structure. In this case a structural invariant fulfils*

$$\{C, H\} = 0, \quad \forall H,$$

see, e.g., [Marsden and Ratiu, 1994, Olver, 1993], which locally implies $(\partial_{\alpha}C) J^{\alpha\beta} = 0$.

3.2 Infinite Dimensional Port-Controlled Hamiltonian Systems

This section is dedicated to the extension of the Port-Hamiltonian framework to the distributed-parameter case, where we are interested in an evolutionary representation of the governing equations. In fact, the underlying geometric concepts of the state manifold, etc. which are introduced in order to characterise a finite dimensional PCHD system must be replaced by appropriate geometric objects. Therefore, we take the Jet machinery into account, see section 2.2. Furthermore, we investigate the concept of an evolutionary vector field which characterises a certain set of PDEs, where the main objective is to find a Port-Hamiltonian formulation of these equations by generalising the relevant geometric concepts and objects from the finite dimensional scenario.

3.2.1 The Geometry of Distributed-Parameter Systems

The state of a distributed-parameter system is given by a certain set of functions on a compact manifold \mathcal{D} (with coherently oriented boundary $\partial\mathcal{D}$) locally equipped with coordinates (X^i) , $i = 1, \dots, m$, where the state may be described by a section of the bundle $\pi : \mathcal{X} \rightarrow \mathcal{D}$ - called the state bundle - which locally possesses the coordinates (X^i, x^α) , $\alpha = 1, \dots, n$. For this setting (X^i) denotes the independent spatial coordinates and (x^α) the dependent coordinates. Moreover, the time t plays the role of the curve (evolution) parameter and, thus, it is no coordinate in this context. Therefore, a section of the state bundle $\Phi \in \Gamma(\pi)$ describes in local coordinates the state of the infinite dimensional system by $x^\alpha = \Phi^\alpha(X^i)$.

In the sequel we need some important geometric structures which can be directly constructed from the state bundle. First of all, we introduce the r^{th} Jet manifold $\mathcal{J}^r(\mathcal{X})$ equipped with adapted coordinates $(X^i, x^\alpha, x_j^\alpha)$, $1 \leq \#J \leq r$ and all the required Jet bundles. Furthermore, we are able to construct the exterior pull-back bundle

$$(\pi^r)^* \left(\bigwedge^m \mathcal{T}^*(\mathcal{D}) \right) \rightarrow \mathcal{J}^r(\mathcal{X}),$$

with respect to the fibre basis $\{dX\}$, $dX = dX^1 \wedge \dots \wedge dX^m$, where the sections of this bundle are r -order densities of the form $\mathcal{F} dX$ with $\mathcal{F} \in C^\infty(\mathcal{J}^r(\mathcal{X}))$ and the (global) volume form dX , as well as the bundle

$$(\pi_0^r)^*(\mathcal{T}^*(\mathcal{X})) \wedge (\pi^r)^* \left(\bigwedge^m \mathcal{T}^*(\mathcal{D}) \right) \rightarrow \mathcal{J}^r(\mathcal{X}) \quad (3.5)$$

with the basis $\{dx^\alpha \wedge dX\}$ for the fibres, whose sections are given by $\chi_\alpha dx^\alpha \wedge dX$ with components $\chi_\alpha \in C^\infty(\mathcal{J}^r(\mathcal{X}))$. These sections are covector valued forms which may be interpreted as *densities with directions*. In this context the presented geometric framework allows to define a functional \mathfrak{F} as the integral over a r -order density on \mathcal{D} . More precisely, it serves as a map $\mathfrak{F} : \Gamma(\pi) \rightarrow \mathbb{R}$ and takes the form of

$$\mathfrak{F}(\Phi) = \int_{\mathcal{D}} (j^r \Phi)^*(\mathcal{F} dX), \quad \mathcal{F} \in C^\infty(\mathcal{J}^r(\mathcal{X})).$$

Finally, we consider the vertical bundle $\nu_{\mathcal{X}} : \mathcal{V}(\mathcal{X}) \rightarrow \mathcal{X}$ which possesses the coordinates $(X^i, x^\alpha, \dot{x}^\alpha)$ and which allows the introduction of a so-called evolutionary vector field.

Definition 3.3 (evolutionary vector field) *The vector field $v : \mathcal{J}^r(\mathcal{X}) \rightarrow (\pi_0^r)^*(\mathcal{V}(\mathcal{X}))$ is called an evolutionary vector field and is locally given by $v = v^\alpha \partial_\alpha$ with $v^\alpha \in C^\infty(\mathcal{J}^r(\mathcal{X}))$ which corresponds to the set of PDEs*

$$\dot{X}^i = 0, \quad \dot{x}^\alpha = v^\alpha, \quad v^\alpha \in C^\infty(\mathcal{J}^r(\mathcal{X})), \quad (3.6)$$

inclusive appropriate boundary conditions, see [Olver, 1993]. These equations describe a set of r -order evolution equations, where the curve (evolution) parameter (of the solution) is the time t .

It is worth noting that the evolutionary vector field does not generate a flow since it is no tangent vector field. However, on a time interval $[0, T] \subset \mathbb{R}_0^+$ together with appropriate boundary conditions it may generate a semi group according to

$$\gamma_t : [0, T] \times \Gamma(\pi) \rightarrow \Gamma(\pi), \quad t \in [0, T] \quad (3.7)$$

which maps sections to sections of the state bundle $\pi : \mathcal{X} \rightarrow \mathcal{D}$ such that

$$\Phi_t = \gamma_t(\Phi_0), \quad \Phi_{t_1+t_2} = \gamma_{t_2} \circ \gamma_{t_1}(\Phi_0)$$

hold with $\Phi_0, \Phi_t, \Phi_{t_1+t_2} \in \Gamma(\pi)$ and $t, t_1 + t_2 \in [0, T]$, where $\Phi_0 \in \Gamma(\pi)$ denotes the initial state/condition. In addition, the semi group satisfies

$$\partial_t \gamma_t^\alpha(\Phi_0) = v^\alpha \circ j^r(\gamma_t(\Phi_0))$$

and, especially,²

$$\partial_t \gamma_t^\alpha(\Phi_0)|_{t=0} = v^\alpha \circ j^r \Phi_0.$$

Finally, it must be mentioned that an evolutionary vector field can also be extended to a prolonged evolutionary vector field, where according to Definition 2.5 the s -order prolongation of an evolutionary vector field $v : \mathcal{J}^r(\mathcal{X}) \rightarrow (\pi_0^r)^*(\mathcal{V}(\mathcal{X}))$ is given by

$$j^s v : \mathcal{J}^{r+s}(\mathcal{X}) \rightarrow (\pi_s^{r+s})^*(\mathcal{V}(\mathcal{J}^s(\mathcal{X}))) \quad (3.8)$$

and takes in local coordinates the form of $j^s v = v^\alpha \partial_\alpha + d_J(v^\alpha) \partial_\alpha^J$ with $1 \leq \#J \leq s$, see also [Olver, 1993].

In the sequel, the main objective will be to investigate the concept of an evolutionary vector field in more detail in order to find a Port-Hamiltonian representation of a set of (r -order) evolution equations which are characterised by such a vector field.

²In particular, it is assumed that the given problem is well-posed in the sense of Hadamard, i.e., there exist suitable normed function spaces for the solution which is unique and varies continuously with the initial state, see [Curtain and Zwart, 1995]. This (rather strong) assumption must usually be investigated for each particular application.

3.2.2 First-order Hamiltonian Densities

In the infinite dimensional case we deal with a Hamiltonian functional of the form

$$\mathfrak{H}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^* (\mathcal{H} dX), \quad \mathcal{H} \in C^\infty(\mathcal{J}^1(\mathcal{X})), \quad (3.9)$$

where we confine ourselves in this thesis to the case of first-order Hamiltonian densities $\mathcal{H} dX$ with $\mathcal{H} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$ only. For the higher-order case the interested reader is referred to [Ennsbrunner, 2006]. In the finite dimensional scenario the total change of the Hamiltonian $H \in C^\infty(\mathcal{M})$ along the solutions of (3.1) – given by (3.2) – has turned out to play a crucial role on the one hand for the characterisation of the dissipative effects and on the other hand for the introduction of the (energy) ports. Therefore, in this subsection we are mainly interested in the analysis of the formal change of (3.9) along (3.7) in order to obtain an analogous expression in terms of an evolutionary vector field which will allow the characterisation of an infinite dimensional Port-Hamiltonian system afterwards.

Formal Change of the Hamiltonian Functional

The change of (3.9) along (3.7) is *formally* given by^{3,4}

$$v(\mathfrak{H}(\Phi)) = \int_{\mathcal{D}} (j^{r+1}\Phi)^* (j^1v(\mathcal{H} dX)) = \int_{\mathcal{D}} (j^{r+1}\Phi)^* (j^1v]d(\mathcal{H} dX)),$$

with respect to the first-order case and in consideration of the evolutionary vector field $v : \mathcal{J}^r(\mathcal{X}) \rightarrow (\pi_0^r)^*(\mathcal{V}(\mathcal{X}))$ with $r \geq 2$, where its first prolongation takes in local coordinates the form of

$$j^1v = v^\alpha \partial_\alpha + d_i(v^\alpha) \partial_\alpha^i.$$

Consequently, we locally obtain

$$v(\mathfrak{H}(\Phi)) = \int_{\mathcal{D}} (j^r\Phi)^* (v^\alpha \partial_\alpha \mathcal{H} dX) + \int_{\mathcal{D}} (j^{r+1}\Phi)^* (d_i(v^\alpha) \partial_\alpha^i \mathcal{H} dX)$$

and integration by parts leads to

$$v(\mathfrak{H}(\Phi)) = \int_{\mathcal{D}} (j^r\Phi)^* (v^\alpha \delta_\alpha \mathcal{H} dX) + \int_{\mathcal{D}} (j^{r+1}\Phi)^* (d_i(v^\alpha \partial_\alpha^i \mathcal{H} dX)),$$

where we have introduced the variational derivative $\delta_\alpha(\cdot) = \partial_\alpha(\cdot) - d_i(\partial_\alpha^i(\cdot))$, see, e.g., [Olver, 1993]. It is worth noting that, in this case, the variational derivative serves as a map

$$\delta : (\pi^1)^* \left(\bigwedge^m \mathcal{T}^*(\mathcal{D}) \right) \rightarrow (\pi_0^2)^* (\mathcal{T}^*(\mathcal{X})) \wedge (\pi^2)^* \left(\bigwedge^m \mathcal{T}^*(\mathcal{D}) \right) \quad (3.10)$$

³In the sequel, this construction will be called *the formal change* of the Hamiltonian functional.

⁴If the semi group (3.7) parameterised in t exists, then the *formal change* of the Hamiltonian functional (involving the pull-back of the Hamiltonian density by the semi group) equals the time derivative of the functional provided that all applied operations are admissible.

and its application takes in local coordinates the form of⁵

$$\delta(\mathcal{H} dX) = \delta_\alpha \mathcal{H} dx^\alpha \wedge dX. \quad (3.11)$$

In terms of the horizontal differential, see Appendix A.1, we are able to state

$$v(\mathfrak{H}(\Phi)) = \int_{\mathcal{D}} (j^r \Phi)^* (v^\alpha \delta_\alpha \mathcal{H} dX) + \int_{\mathcal{D}} (j^{r+1} \Phi)^* (d_h (v^\alpha \partial_\alpha^i \mathcal{H} \partial_i] dX))$$

which is equivalent to

$$v(\mathfrak{H}(\Phi)) = \int_{\mathcal{D}} (j^r \Phi)^* (v^\alpha \delta_\alpha \mathcal{H} dX) + \int_{\partial \mathcal{D}} \iota^* ((j^r \Phi)^* (v^\alpha \partial_\alpha^i \mathcal{H} \partial_i] dX))$$

by applying Lemma 2.1 and Stokes' Theorem. Therefore, it is obvious to introduce a boundary map, see, e.g., [Schlacher, 2007, Schöberl et al., 2008], of the form

$$\delta^\partial : (\pi^1)^* \left(\bigwedge^m \mathcal{T}^*(\mathcal{D}) \right) \rightarrow (\pi_0^1)^* (\mathcal{T}^*(\mathcal{X})) \wedge (\pi^1)^* \left(\bigwedge^{m-1} \mathcal{T}^*(\mathcal{D}) \right) \quad (3.12)$$

whose application takes in local coordinates the form of

$$\delta^\partial(\mathcal{H} dX) = \partial_\alpha^i \mathcal{H} dx^\alpha \wedge \partial_i] dX. \quad (3.13)$$

Finally, we are able to end up with the formal change of the Hamiltonian functional (3.9) along (3.7) in a coordinate free manner

$$v(\mathfrak{H}(\Phi)) = \int_{\mathcal{D}} (j^r \Phi)^* (v] \delta(\mathcal{H} dX)) + \int_{\partial \mathcal{D}} \iota^* ((j^r \Phi)^* (v] \delta^\partial(\mathcal{H} dX))) . \quad (3.14)$$

This important result is the basis for all further investigations with respect to the generalisation of the Port-Hamiltonian framework to the distributed-parameter case. In fact, the formal change of the Hamiltonian functional splits into two parts; the first part is defined inside the domain involving the variational derivative which serves as the map (3.10) and the second part degenerates to a boundary term with respect to the introduced boundary map (3.12). Furthermore, it is clear that both parts – on the domain as well as on the boundary – are characterised by certain pairings involving the evolutionary vector field and the terms (3.11), (3.13) respectively. Therefore, with regard to the introduction of an infinite dimensional Port-Hamiltonian system representation it is obvious that by an appropriate choice of the evolution equations characterised by the evolutionary vector field it will be possible on the one hand to distinguish structural properties not only inside the domain but also on the boundary and on the other hand these specific pairings will allow the introduction of (energy) ports acting inside the domain as well as through the boundary. Before we proceed with the extension of the Port-Hamiltonian framework to the considered distributed-parameter case we intend to investigate the boundary term in more detail.

⁵In fact, these expressions are covector valued forms which are sections of the bundle (3.5) for $r = 2$.

Boundary Term

In order to find a more manageable expression for the boundary term in local coordinates we introduce the boundary pull-back bundle $\iota^*(\pi) : \iota^*(\mathcal{X}) \rightarrow \partial\mathcal{D}$ equipped with coordinates $(X_\partial^{i_\partial}, x^\alpha)$, $i_\partial = 1, \dots, m-1$, where the inclusion mapping $\iota : \partial\mathcal{D} \rightarrow \mathcal{D}$ is assumed to be given by

$$\iota : (X_\partial^{i_\partial}) \rightarrow (X^{i_\partial} = X_\partial^{i_\partial}, X^m = \text{const.}) , \quad i_\partial = 1, \dots, m-1 , \quad (3.15)$$

see, [Ennsbrunner and Schlacher, 2005, Ennsbrunner, 2006, Schöberl et al., 2008]. In this context the coordinates $(X_\partial^{i_\partial})$ on $\partial\mathcal{D}$ are called adapted to the boundary if (3.15) is met. Therefore, we are able to introduce a corresponding boundary volume form

$$dX_\partial = \partial_m \rfloor dX = (-1)^{m-1} dX_\partial^1 \wedge \dots \wedge dX_\partial^{m-1}$$

and a boundary section $\Phi_\partial \in \Gamma(\iota^*(\pi))$ which is related to a section $\Phi \in \Gamma(\pi)$ according to $\Phi_\partial = \iota^*(\Phi) = \Phi \circ \iota$ or, equivalently, $\iota^* \circ \Phi^* = \Phi_\partial^*$. Furthermore, we are also able to pull-back certain Jet bundles to the boundary. Therefore, we consider the bundle $\iota^*(\pi^r) : \iota^*(\mathcal{J}^r(\mathcal{X})) \rightarrow \partial\mathcal{D}$ with adapted coordinates $(X_\partial^{i_\partial}, x^\alpha, x_j^\alpha)$, $1 \leq \#J \leq r$, where a prolonged section $j^r\Phi \in \Gamma(\pi^r)$ leads to $\iota^*(j^r\Phi) = j^r\Phi \circ \iota$ which is abbreviated by⁶ $\Phi_\partial^r = \iota^*(j^r\Phi)$ or, equivalently, $(\Phi_\partial^r)^* = \iota^* \circ (j^r\Phi)^*$.

Having this machinery at one's disposal the boundary term can be reformulated in local coordinates as

$$\int_{\partial\mathcal{D}} \iota^* ((j^r\Phi)^* (v^\alpha \partial_\alpha^i \mathcal{H} \partial_i \rfloor dX)) = \int_{\partial\mathcal{D}} (\Phi_\partial^r)^* ((v^\alpha \circ \iota) (\partial_\alpha^m \mathcal{H} \circ \iota) dX_\partial) ,$$

or, equivalently,

$$\int_{\partial\mathcal{D}} \iota^* ((j^r\Phi)^* (v \rfloor \delta^\partial (\mathcal{H} dX))) = \int_{\partial\mathcal{D}} (\Phi_\partial^r)^* (\iota^* (v \rfloor \delta^\partial (\mathcal{H} dX))) \quad (3.16)$$

with

$$\iota^* (v) = (v^\alpha \circ \iota) \partial_\alpha , \quad (v^\alpha \circ \iota) \in C^\infty(\iota^*(\mathcal{J}^r(\mathcal{X}))) ,$$

as well as

$$\iota^* (\delta^\partial (\mathcal{H} dX)) = \iota^* (\partial_\alpha^i \mathcal{H} dx^\alpha \wedge \partial_i \rfloor dX) = (\partial_\alpha^m \mathcal{H} \circ \iota) dx^\alpha \wedge dX_\partial$$

with

$$(\partial_\alpha^m \mathcal{H} \circ \iota) \in C^\infty(\iota^*(\mathcal{J}^1(\mathcal{X}))) .$$

3.2.3 Hamiltonian Evolution Equations I

The investigations from the last subsection and, especially, the important result (3.14) enable us to propose a direct generalisation of Definition 3.1 to the distributed-parameter

⁶Note that $\Phi_\partial^r \neq j^r\Phi_\partial$, in general, since the pull-back boundary bundles are not equipped with an underlying Jet bundle structure.

case based on [Ennsbrunner, 2006, Schlacher, 2007, 2008], for instance. Therefore, we introduce a coordinate-free version of an infinite dimensional PCHD system, where it is worth mentioning that we restrict ourselves to the case that the proposed system class may describe a set of second-order evolution equations. In fact, for this case, the interconnection, the dissipation as well as the input map are represented by appropriate multilinear maps by direct analogy with the finite dimensional case. It is worth noting that we will often denote this case as the so-called non-differential operator case in order to avoid confusions since in the next subsection we will further generalise this system class by replacing the relevant multilinear maps by appropriate differential operators. Furthermore, it is remarkable that the proposed system class enables us to directly characterise the main structural properties known from the lumped-parameter case concerning the (physical) interpretation of the interconnection and the dissipation map and for the introduction of the (energy) ports we have two possibilities, i.e., we consider on the one hand distributed ports and on the other hand so-called boundary ports which describe for many applications the influence of the boundary conditions. Moreover, this system class also allows the introduction of structural invariants together with the derivation of the necessary conditions in an analogous manner as in the finite dimensional case, where it is worth noting that the variational derivative will play a crucial role.

The iPCHD System Class (the Non-Differential Operator Case)

Based on the investigations from the last subsection and, in particular, with respect to the formal change of the functional (3.14) we are able to introduce a direct generalisation of Definition 3.1 to the distributed-parameter case based on [Ennsbrunner, 2006, Schlacher, 2007, 2008], for instance, by an appropriate choice of the considered evolution equations.

Definition 3.4 (iPCHD system, non-differential operator case) *An infinite dimensional PCHD system, or iPCHD system for short, with the Hamiltonian functional (3.9) is given as*

$$\begin{aligned} \dot{x} = v &= (\mathcal{J} - \mathcal{R})(\delta(\mathcal{H} dX)) + u \rfloor \mathcal{G} \\ y &= \mathcal{G}^* \rfloor \delta(\mathcal{H} dX) \end{aligned} \quad (3.17)$$

inclusive appropriate boundary conditions together with $\dot{X} = 0$ and with the skew-symmetric interconnection map \mathcal{J} , the symmetric positive semidefinite dissipation map \mathcal{R} , the input map \mathcal{G} as well as its adjoint map \mathcal{G}^ with respect to the distributed system input u and the distributed collocated output y . Furthermore, the formal change of the Hamiltonian functional (3.9) along (3.7) takes the form of*

$$\begin{aligned} v(\mathfrak{H}(\Phi)) &= - \int_{\mathcal{D}} (j^2 \Phi)^* (\mathcal{R}(\delta(\mathcal{H} dX)) \rfloor \delta(\mathcal{H} dX)) + \int_{\mathcal{D}} (j^2 \Phi)^* (u \rfloor y) \\ &\quad + \int_{\partial \mathcal{D}} \iota^* ((j^2 \Phi)^* (v \rfloor \delta^\partial(\mathcal{H} dX))) , \end{aligned} \quad (3.18)$$

with respect to the evolutionary vector field v ⁷.

In this context the variational derivative which serves as a map according to (3.10) now plays the analogous role of the exterior derivative in the lumped-parameter case and the interconnection and the dissipation maps are maps of the form

$$\mathcal{J}, \mathcal{R} : (\pi_0^2)^* (T^*(\mathcal{X})) \wedge (\pi^2)^* \left(\bigwedge^m T^*(\mathcal{D}) \right) \rightarrow (\pi_0^2)^* (\mathcal{V}(\mathcal{X})), \quad (3.19)$$

where the interconnection map \mathcal{J} serves as a skew-symmetric map according to

$$\mathcal{J}(\omega)]\varpi + \mathcal{J}(\varpi)]\omega = 0$$

for $\omega = \omega_\alpha dx^\alpha \wedge dX$ and $\varpi = \varpi_\alpha dx^\alpha \wedge dX$ with $\omega_\alpha, \varpi_\alpha \in C^\infty(\mathcal{J}^2(\mathcal{X}))$ and the dissipation map \mathcal{R} is symmetric and positive semidefinite, i.e.,

$$\mathcal{R}(\omega)]\varpi - \mathcal{R}(\varpi)]\omega = 0, \quad \mathcal{R}(\omega)]\omega \geq 0.$$

Furthermore, in local coordinates these maps read as

$$\mathcal{J}(\omega) = \mathcal{J}^{\alpha\beta} \omega_\beta \partial_\alpha, \quad \mathcal{R}(\omega) = \mathcal{R}^{\alpha\beta} \omega_\beta \partial_\alpha, \quad \beta = 1, \dots, n,$$

with respect to the components $\mathcal{J}^{\alpha\beta} = -\mathcal{J}^{\beta\alpha}$, $\mathcal{R}^{\alpha\beta} = \mathcal{R}^{\beta\alpha}$ and $\mathcal{J}^{\alpha\beta}, \mathcal{R}^{\alpha\beta} \in C^\infty(\mathcal{J}^2(\mathcal{X}))$. Moreover, the input map \mathcal{G} as well as its adjoint map \mathcal{G}^* are defined by

$$\mathcal{G} : \mathcal{U} \rightarrow (\pi_0^2)^* (\mathcal{V}(\mathcal{X})), \quad \mathcal{G}^* : (\pi_0^2)^* (T^*(\mathcal{X})) \wedge (\pi^2)^* \left(\bigwedge^m T^*(\mathcal{D}) \right) \rightarrow \mathcal{Y}, \quad (3.20)$$

where $v : \mathcal{U} \rightarrow \mathcal{J}^2(\mathcal{X})$ denotes the input vector bundle (locally) equipped with coordinates $(X^i, x^\alpha, x_j^\alpha, u^\xi)$ with $1 \leq \#J \leq 2$ and $\xi = 1, \dots, n_u$ with respect to the holonomic basis $\{e_\xi\}$. Therefore, the output vector bundle can be defined as the dual bundle $v^* : \mathcal{Y} = \mathcal{U}^* \rightarrow \mathcal{J}^2(\mathcal{X})$ which possesses the local coordinates $(X^i, x^\alpha, x_j^\alpha, y_\xi)$ as well as the fibre basis $\{e^\xi \otimes dX\}$. Furthermore, it is dual to the input vector bundle with respect to the bilinear map⁸

$$\mathcal{Y} \times_{\mathcal{J}^2(\mathcal{X})} \mathcal{U} \rightarrow \bigwedge^m T^*(\mathcal{D})$$

in local coordinates given by the interior product

$$u]y = (u^\xi e_\xi)](y_\eta e^\eta \otimes dX) = y_\xi u^\xi dX, \quad \eta = 1, \dots, n_u.$$

Consequently, we are able to derive the relation

$$(u] \mathcal{G}] \delta(\mathcal{H} dX) = u] (\mathcal{G}^*] \delta(\mathcal{H} dX)) = u] y$$

⁷It is remarkable that the evolutionary vector field v is not a tangent vector field on $(\pi_0^2)^* (\mathcal{V}(\mathcal{X}))$ any more since it depends on the distributed input u . However, in order to enhance the readability we suppress the underlying pull-back structure in the definition of the relevant maps in the sequel.

⁸In order to enhance the readability we suppress the underlying pull-back constructions in the definition of the bilinear map.

characterising the port distributed over \mathcal{D} . Thus, the input map \mathcal{G} and its adjoint map \mathcal{G}^* can both be represented by the tensor

$$\mathcal{G} = \mathcal{G}_\xi^\alpha e^\xi \otimes \partial_\alpha, \quad \mathcal{G}_\xi^\alpha \in C^\infty(\mathcal{J}^2(\mathcal{X})),$$

and, therefore, in local coordinates we obtain

$$u] \mathcal{G} = (u^\xi e_\xi)] (\mathcal{G}_\eta^\alpha e^\eta \otimes \partial_\alpha) = \mathcal{G}_\xi^\alpha u^\xi \partial_\alpha$$

as well as

$$\mathcal{G}^*] \delta(\mathcal{H} dX) = (\mathcal{G}_\xi^\alpha e^\xi \otimes \partial_\alpha)] (\delta_\beta \mathcal{H} dx^\beta \wedge dX) = \mathcal{G}_\xi^\alpha \delta_\alpha \mathcal{H} e^\xi \otimes dX = y_\xi e^\xi \otimes dX.$$

Hence, it is clear that in local coordinates the proposed iPCHD system representation (3.17) reads as

$$\begin{aligned} \dot{x}^\alpha &= v^\alpha = (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) \delta_\beta \mathcal{H} + \mathcal{G}_\xi^\alpha u^\xi \\ y_\xi &= \mathcal{G}_\xi^\alpha \delta_\alpha \mathcal{H} \end{aligned}$$

and (3.18) locally takes the form of

$$\begin{aligned} v(\mathfrak{H}(\Phi)) &= - \int_{\mathcal{D}} (j^2 \Phi)^* ((\delta_\alpha \mathcal{H}) \mathcal{R}^{\alpha\beta} (\delta_\beta \mathcal{H}) dX) + \int_{\mathcal{D}} (j^2 \Phi)^* (y_\xi u^\xi dX) \\ &\quad + \int_{\partial \mathcal{D}} \iota^* ((j^2 \Phi)^* (v^\alpha \partial_\alpha^i \mathcal{H} \partial_i] dX)), \end{aligned}$$

with respect to the evolutionary vector field $v = v^\alpha \partial_\alpha$.

Remark 3.4 *It is worth noting that the structure of an iPCHD system is preserved by bundle morphisms of the form (2.1) which possess the transition functions (2.2) as well as (2.3) and the transition functions for the input bundle read as $\bar{u}^\xi = \phi_\xi^{\bar{\xi}} u^\xi$, $\bar{\xi} = 1, \dots, n_w$, with $\phi_\xi^{\bar{\xi}} \in C^\infty(\mathcal{J}^2(\mathcal{X}))$, where $[\phi_\xi^{\bar{\xi}}]$ is invertible. For more detailed information see [Schlacher, 2008].*

Remark 3.5 *In order to emphasise the main differences between the presented Port-Hamiltonian framework and the classical evolutionary approach (e.g., [Marsden and Ratiu, 1994, Olver, 1993]) it is worth noting that in the infinite dimensional case a Poisson bracket may be defined as a bilinear map according to*

$$\{\mathfrak{W}, \mathfrak{Q}\}(\Phi) = \int_{\mathcal{D}} (j^2 \Phi)^* (\mathcal{J}(\delta(\mathfrak{Q} dX))] \delta(\mathfrak{W} dX) = \int_{\mathcal{D}} (j^2 \Phi)^* ((\delta_\alpha \mathfrak{W}) \mathcal{J}^{\alpha\beta} (\delta_\beta \mathfrak{Q}) dX)$$

for the functionals

$$\mathfrak{W}(\Phi) = \int_{\mathcal{D}} (j^1 \Phi)^* (\mathfrak{W} dX), \quad \mathfrak{Q}(\Phi) = \int_{\mathcal{D}} (j^1 \Phi)^* (\mathfrak{Q} dX)$$

with $\mathfrak{W}, \mathfrak{Q} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$, satisfying the condition of skew-symmetry

$$\{\mathfrak{W}, \mathfrak{Q}\}(\Phi) = -\{\mathfrak{Q}, \mathfrak{W}\}(\Phi)$$

and the Jacobi Identity

$$\{\{\mathfrak{W}, \mathfrak{Q}\}, \mathfrak{P}\}(\Phi) + \{\{\mathfrak{P}, \mathfrak{W}\}, \mathfrak{Q}\}(\Phi) + \{\{\mathfrak{Q}, \mathfrak{P}\}, \mathfrak{W}\}(\Phi) = 0$$

for all functionals $\mathfrak{W}, \mathfrak{Q}, \mathfrak{P}$ with $\mathfrak{P}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^* (\mathcal{P} \, dX)$, $\mathcal{P} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$, see, e.g., [Marsden and Ratiu, 1994, Olver, 1993]. Thus, the map \mathcal{J} is defined according to (3.19). It is worth mentioning that the Leibniz' Rule has no counterpart in this setting. Furthermore, a Hamiltonian (evolutionary) vector field $v_{\mathcal{H}}$ may be defined by

$$v_{\mathcal{H}}(\mathfrak{F}(\Phi)) = \{\mathfrak{F}, \mathfrak{H}\}(\Phi) + \int_{\partial\mathcal{D}} \iota^* ((j^2\Phi)^* (v_{\mathcal{H}}] \delta^\partial (\mathcal{F} \, dX))) \quad (3.21)$$

which in local coordinates reads as

$$v_{\mathcal{H}}(\mathfrak{F}(\Phi)) = \int_{\mathcal{D}} (j^2\Phi)^* ((\delta_\alpha \mathcal{F}) \mathcal{J}^{\alpha\beta} (\delta_\beta \mathcal{H}) \, dX) + \int_{\partial\mathcal{D}} \iota^* ((j^2\Phi)^* (v_{\mathcal{H}}^\alpha \partial_\alpha \mathcal{F} \, \partial_i] \, dX))$$

for an arbitrary functional $\mathfrak{F}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^* (\mathcal{F} \, dX)$, $\mathcal{F} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$, splitting into a term defined on the domain and an appropriate boundary term, cf. (3.14), with respect to the Hamiltonian functional (3.9). Therefore, Hamilton's equations may be defined by $\dot{x} = v_{\mathcal{H}} = \mathcal{J}(\delta(\mathcal{H} \, dX))$.

These considerations may be seen as a direct link to the classical evolutionary approach (applied to the non-differential operator case), see, e.g., [Marsden and Ratiu, 1994, Olver, 1993], where the classical approach is only able to consider trivial boundary conditions/terms and, thus, no boundary term is necessary for the definition of a Hamiltonian (evolutionary) vector field $v_{\mathcal{H}}$. Hence, if in (3.21) the boundary term vanishes, then $v_{\mathcal{H}}$ corresponds to the classical definition of a Hamiltonian (evolutionary) vector field as in [Marsden and Ratiu, 1994, Olver, 1993], for instance.

Remark 3.6 If in Remark 3.5 the Jacobi Identity is dropped then we may speak about a generalised Poisson bracket. Therefore, if the system (3.17) is a lossless system, i.e., $\mathcal{R} = 0$, and we have no distributed port then the map \mathcal{J} induces a generalised Poisson structure and the evolutionary vector field v of Definition 3.4 may be interpreted as a Hamiltonian (evolutionary) vector field $v_{\mathcal{H}}$ according to (3.21).

Next, we intend to analyse the boundary term, where we are mainly interested in determining appropriate boundary in- and outputs which will lead us to the introduction of so-called boundary ports.

Boundary Ports

The remaining task will be to investigate the boundary term in more detail which allows the introduction of (energy) ports acting through the boundary $\partial\mathcal{D}$ for many applications provided that the physical meaning is apparent. Thus, in consideration of (3.16) the boundary term can be reformulated as

$$\int_{\partial\mathcal{D}} \iota^* ((j^2\Phi)^* (v] \delta^\partial (\mathcal{H} \, dX))) = \int_{\partial\mathcal{D}} (\Phi_\partial^2)^* (\iota^* (v] \delta^\partial (\mathcal{H} \, dX))) \quad (3.22)$$

which in local coordinates is equivalent to

$$\int_{\partial\mathcal{D}} (\Phi_{\partial}^2)^* ((\dot{x}^\alpha \circ \iota) (\partial_\alpha^m \mathcal{H} \circ \iota) \, dX_{\partial})$$

with $\Phi_{\partial}^2 = j^2 \Phi \circ \iota$. With regard to the introduction of the (energy) ports acting through the boundary it must be emphasised that due to the pairing in (3.22) the determination of the boundary in- and outputs clearly is not unique. Therefore, we are interested in deriving a relation of the form⁹

$$(\dot{x}^\alpha \circ \iota) (\partial_\alpha^m \mathcal{H} \circ \iota) \, dX_{\partial} = y_{\partial, \xi_{\partial}} u_{\partial}^{\xi_{\partial}} \, dX_{\partial} = u_{\xi_{\partial}}^{\partial} y^{\partial, \xi_{\partial}} \, dX_{\partial},$$

where it is clear that there are, in general, two main possibilities for the choices of the boundary in- and outputs (or even combinations of them). For the investigation of the first possibility we introduce the boundary input vector bundle $\nu_{\partial} : \mathcal{U}_{\partial} \rightarrow \iota^*(\mathcal{J}^2(\mathcal{X}))$ equipped with local coordinates $(X_{\partial}^{i_{\partial}}, x^\alpha, x_J^\alpha, u_{\partial}^{\xi_{\partial}})$ with $1 \leq \#J \leq 2$, $\xi_{\partial} = 1, \dots, n_u^{\partial}$ and the holonomic basis $\{e_{\partial, \xi_{\partial}}\}$ as well as the dual boundary vector bundle $\nu_{\partial}^* : \mathcal{Y}_{\partial} = \mathcal{U}_{\partial}^* \rightarrow \iota^*(\mathcal{J}^2(\mathcal{X}))$ – the boundary output vector bundle – which possesses the local coordinates $(X_{\partial}^{i_{\partial}}, x^\alpha, x_J^\alpha, y_{\partial, \xi_{\partial}})$ and the fibre basis $\{e_{\partial}^{\xi_{\partial}} \otimes dX_{\partial}\}$ with respect to the bilinear map

$$\mathcal{Y}_{\partial} \times_{\iota^*(\mathcal{J}^2(\mathcal{X}))} \mathcal{U}_{\partial} \rightarrow \bigwedge^{m-1} T^*(\mathcal{D})$$

in local coordinates given by the interior product

$$u_{\partial} \rfloor y_{\partial} = \left(u_{\partial}^{\xi_{\partial}} e_{\partial, \xi_{\partial}} \right) \rfloor (y_{\partial, \eta_{\partial}} e_{\partial}^{\eta_{\partial}} \otimes dX_{\partial}) = y_{\partial, \xi_{\partial}} u_{\partial}^{\xi_{\partial}} \, dX_{\partial}, \quad \eta_{\partial} = 1, \dots, n_u^{\partial}.$$

Therefore, we introduce the boundary map \mathcal{G}_{∂} as well as the adjoint boundary map \mathcal{G}_{∂}^* both represented by the tensor

$$\mathcal{G}_{\partial} = \mathcal{G}_{\partial, \xi_{\partial}}^{\alpha} e_{\partial}^{\xi_{\partial}} \otimes \partial_{\alpha}, \quad \mathcal{G}_{\partial, \xi_{\partial}}^{\alpha} \in C^{\infty}(\iota^*(\mathcal{J}^2(\mathcal{X}))),$$

and choose $u_{\partial} \rfloor \mathcal{G}_{\partial} = \iota^*(v)$ which in local coordinates reads as

$$u_{\partial} \rfloor \mathcal{G}_{\partial} = \left(u_{\partial}^{\xi_{\partial}} e_{\partial, \xi_{\partial}} \right) \rfloor (\mathcal{G}_{\partial, \eta_{\partial}}^{\alpha} e_{\partial}^{\eta_{\partial}} \otimes \partial_{\alpha}) = \mathcal{G}_{\partial, \xi_{\partial}}^{\alpha} u_{\partial}^{\xi_{\partial}} \, \partial_{\alpha} = (\dot{x}^\alpha \circ \iota) \, \partial_{\alpha}.$$

Thus, we obtain the boundary port

$$(u_{\partial} \rfloor \mathcal{G}_{\partial}) \rfloor \iota^*(\delta^{\partial}(\mathcal{H} \, dX)) = u_{\partial} \rfloor (\mathcal{G}_{\partial}^* \rfloor \iota^*(\delta^{\partial}(\mathcal{H} \, dX))) = u_{\partial} \rfloor y_{\partial} \quad (3.23)$$

with respect to the collocated boundary output $y_{\partial} = \mathcal{G}_{\partial}^* \rfloor \iota^*(\delta^{\partial}(\mathcal{H} \, dX))$ including the adjoint boundary map \mathcal{G}_{∂}^* , where in local coordinates we obtain

$$\begin{aligned} \mathcal{G}_{\partial}^* \rfloor \iota^*(\delta^{\partial}(\mathcal{H} \, dX)) &= \left(\mathcal{G}_{\partial, \xi_{\partial}}^{\alpha} e_{\partial}^{\xi_{\partial}} \otimes \partial_{\alpha} \right) \rfloor ((\partial_{\beta}^m \mathcal{H} \circ \iota) \, dx^{\beta} \wedge dX_{\partial}) \\ &= \mathcal{G}_{\partial, \xi_{\partial}}^{\alpha} (\partial_{\alpha}^m \mathcal{H} \circ \iota) \, e_{\partial}^{\xi_{\partial}} \otimes dX_{\partial} = y_{\partial, \xi_{\partial}} e_{\partial}^{\xi_{\partial}} \otimes dX_{\partial}. \end{aligned}$$

⁹Note the abuse of notation. In the sequel, we write $\partial\mathcal{D}$ even when the boundary ports are only defined on a part of $\partial\mathcal{D}$.

For the other choice of the boundary ports we introduce the boundary input vector bundle $\nu^\partial : \mathcal{U}^\partial \rightarrow \iota^*(\mathcal{J}^2(\mathcal{X}))$ with local coordinates $(X_\partial^{i\partial}, x^\alpha, x_J^\alpha, u_{\xi_\partial}^\partial)$ and the holonomic basis $\{e^{\partial, \xi_\partial}\}$ as well as the dual boundary vector bundle $\nu^{\partial, *} : \mathcal{Y}^\partial = \mathcal{U}^{\partial, *} \rightarrow \iota^*(\mathcal{J}^2(\mathcal{X}))$ equipped with local coordinates $(X_\partial^{i\partial}, x^\alpha, x_J^\alpha, y^{\partial, \xi_\partial})$ and the basis $\{dX_\partial \otimes e_{\xi_\partial}^\partial\}$ for the fibres with respect to the bilinear map

$$\mathcal{U}^\partial \times_{\iota^*(\mathcal{J}^2(\mathcal{X}))} \mathcal{Y}^\partial \rightarrow \bigwedge^{m-1} \mathcal{T}^*(\mathcal{D})$$

which is locally given by the interior product

$$y^\partial \rfloor u^\partial = (y^{\partial, \xi_\partial} dX_\partial \otimes e_{\xi_\partial}^\partial) \rfloor (u_{\eta_\partial}^\partial e^{\partial, \eta_\partial}) = u_{\xi_\partial}^\partial y^{\partial, \xi_\partial} dX_\partial.$$

Hence, the boundary map \mathcal{G}^∂ as well as its adjoint map $\mathcal{G}^{\partial, *}$ can be introduced which are both given by the tensor

$$\mathcal{G}^\partial = \mathcal{G}_\alpha^{\partial, \xi_\partial} dx^\alpha \wedge dX_\partial \otimes e_{\xi_\partial}^\partial, \quad \mathcal{G}_\alpha^{\partial, \xi_\partial} \in C^\infty(\iota^*(\mathcal{J}^2(\mathcal{X}))),$$

and we choose $\mathcal{G}^\partial \rfloor u^\partial = \iota^*(\delta^\partial(\mathcal{H} dX))$ locally given as

$$\begin{aligned} \mathcal{G}^\partial \rfloor u^\partial &= (\mathcal{G}_\alpha^{\partial, \xi_\partial} dx^\alpha \wedge dX_\partial \otimes e_{\xi_\partial}^\partial) \rfloor (u_{\eta_\partial}^\partial e^{\partial, \eta_\partial}) \\ &= \mathcal{G}_\alpha^{\partial, \xi_\partial} u_{\xi_\partial}^\partial dx^\alpha \wedge dX_\partial = (\partial_\alpha^m \mathcal{H} \circ \iota) dx^\alpha \wedge dX_\partial. \end{aligned}$$

Therefore, we obtain the boundary port

$$\iota^*(v) \rfloor (\mathcal{G}^\partial \rfloor u^\partial) = (\iota^*(v) \rfloor \mathcal{G}^{\partial, *}) \rfloor u^\partial = y^\partial \rfloor u^\partial \quad (3.24)$$

with respect to the collocated boundary output $y^\partial = \iota^*(v) \rfloor \mathcal{G}^{\partial, *}$, where in local coordinates we obtain

$$\begin{aligned} \iota^*(v) \rfloor \mathcal{G}^{\partial, *} &= (\dot{x}^\alpha \circ \iota) \partial_\alpha \rfloor (\mathcal{G}_\beta^{\partial, \xi_\partial} dx^\beta \wedge dX_\partial \otimes e_{\xi_\partial}^\partial) \\ &= \mathcal{G}_\alpha^{\partial, \xi_\partial} (\dot{x}^\alpha \circ \iota) dX_\partial \otimes e_{\xi_\partial}^\partial = y^{\partial, \xi_\partial} dX_\partial \otimes e_{\xi_\partial}^\partial. \end{aligned}$$

It is worth noting that we have only investigated the two main possibilities for the choice of the boundary ports, although, a combination of them is possible since the exact choice depends on the considered problem. For a more general discussion on this topic the interested reader is referred to [Ennsbrunner, 2006].

In conclusion, it may be said that the formal change of the Hamiltonian functional (3.9) along (3.7) consists of the dissipative effects inside the domain, the distributed port and the boundary port which may be defined by the relations (3.23) or (3.24) respectively (or even combinations of them) and which characterises for many applications the influence of the boundary conditions. Consequently, it is worth mentioning that (3.18) states nothing else than the balance of energy principle, whenever the Hamiltonian functional (3.9) corresponds to the total energy of the system.

Structural Invariants for iPCHD Systems

Analogously to the lumped-parameter case we are also able to define structural invariants for the distributed-parameter case, where in the distributed-parameter scenario the definition of the boundary ports play a crucial role. Therefore, we confine ourselves to the two main parameterisations of the boundary ports stated in (3.23) as well as (3.24).

Definition 3.5 (structural invariant, iPCHD system) *A structural invariant for an iPCHD system (3.17) with $\mathcal{H} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$ is given by*

$$\mathfrak{E}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^* (\mathcal{C} \, dX) , \quad \mathcal{C} \in C^\infty(\mathcal{J}^1(\mathcal{X})) .$$

First, for the boundary port parameterisation (3.23) it satisfies in local coordinates the set of PDEs

$$\delta_\alpha \mathcal{C} (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0$$

implying that the formal change of \mathfrak{E} along (3.7) results in

$$v(\mathfrak{E}(\Phi)) = \int_{\mathcal{D}} (j^2\Phi)^* ((u]\mathcal{G})] \delta(\mathcal{C} \, dX)) + \int_{\partial\mathcal{D}} (\Phi_\partial^2)^* ((u_\partial]\mathcal{G}_\partial)] \iota^* (\delta^\partial(\mathcal{C} \, dX)) .$$

If, additionally, $u = 0$ or $\mathcal{G}^*] \delta(\mathcal{C} \, dX) = 0$ as well as $u_\partial = 0$ or $\mathcal{G}_\partial^*] \iota^* (\delta^\partial(\mathcal{C} \, dX)) = 0$, then the structural invariant serves as a conserved quantity for the iPCHD system (3.17) concerning the case (3.23).

Second, for the boundary port parameterisation (3.24) it satisfies the set of PDEs

$$\delta_\alpha \mathcal{C} (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0 , \quad (\dot{x}^\alpha \circ \iota) (\partial_\alpha^m \mathcal{C} \circ \iota) = 0$$

and the formal change of \mathfrak{E} along (3.7) results in

$$v(\mathfrak{E}(\Phi)) = \int_{\mathcal{D}} (j^2\Phi)^* ((u]\mathcal{G})] \delta(\mathcal{C} \, dX) .$$

If, additionally, $u = 0$ or $\mathcal{G}^*] \delta(\mathcal{C} \, dX) = 0$ is met, then the structural invariant serves as a conserved quantity for (3.17) with respect to the case (3.24).

In the case of $\text{rank}([\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}]) = n$, \mathcal{C} is a total derivative (for both cases) only.

These conditions follow by a direct computation and may be seen as a direct adaption of Definition 3.2 to the introduced iPCHD system representation. Especially, for the case $\text{rank}([\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}]) = n$ it is clear that $\delta_\alpha \mathcal{C} = 0$ must be satisfied which is fulfilled for total derivatives of the form $\mathcal{C} = d_i(\bar{C}^i)$ with $\bar{C}^i \in C^\infty(\mathcal{X})$ since

$$\delta_\alpha (d_i(\bar{C}^i)) = \partial_\alpha (d_i(\bar{C}^i)) - d_j \partial_\alpha^j (d_i(\bar{C}^i)) = \partial_\alpha (d_i(\bar{C}^i)) - d_i(\partial_\alpha(\bar{C}^i)) = 0 .$$

For this case, it is worth mentioning that a structural invariant simplifies to

$$\mathfrak{E}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^* (d_i(\bar{C}^i) \, dX) = \int_{\partial\mathcal{D}} \iota^* (\bar{C}^i \circ \Phi \, \partial_i] \, dX)$$

by applying the horizontal differential.

finite dimensional PCHD system	iPCHD system, non-differential operator case
a set of ordinary differential equations	a set of second-order evolution equations incl. appropriate boundary conditions
state manifold \mathcal{M} tangent bundle $\tau_{\mathcal{M}}$ cotangent bundle $\tau_{\mathcal{M}}^*$	state bundle π , Jet bundles (order 2) vertical bundle $\nu_{\mathcal{X}}$ construction of certain pull-back bundles
Hamiltonian $H \in C^\infty(\mathcal{M})$	Hamiltonian functional $\mathfrak{H}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^*(\mathcal{H} dX)$ with $\Phi \in \Gamma(\pi)$
total derivative $d : C^\infty(\mathcal{M}) \rightarrow T^*(\mathcal{M})$	variational derivative δ , see (3.10) boundary map δ^∂ , see (3.12)
$J, R : T^*(\mathcal{M}) \rightarrow \mathcal{T}(\mathcal{M})$	multilinear maps \mathcal{J}, \mathcal{R} see (3.19)
$G : \mathcal{U} \rightarrow \mathcal{T}(\mathcal{M}), G^* : T^*(\mathcal{M}) \rightarrow \mathcal{Y}$	multilinear maps $\mathcal{G}, \mathcal{G}^*$ see (3.20) multilinear maps $\mathcal{G}_\partial, \mathcal{G}_\partial^*$ and $\mathcal{G}^\partial, \mathcal{G}^{\partial,*}$ see, e.g., (3.23), (3.24)
port $u \rfloor y$ with $y = G^* \rfloor dH$	distributed port with $y = \mathcal{G}^* \rfloor \delta(\mathcal{H} dX)$ two main possibilities for the boundary ports (3.23), (3.24)
conditions for structural invariants $C \in C^\infty(\mathcal{M})$ see Definition 3.2	conditions for structural invariants $\mathfrak{C}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^*(C dX)$ see Definition 3.5

Table 3.1: The correspondences of finite and infinite dimensional Port-Hamiltonian systems (the non-differential operator case)

Conclusions

In order to emphasise the analogies of the proposed iPCHD system representation with the Port-Hamiltonian representation from the finite dimensional case we are able to propose the table 3.1, where the direct correspondences can be found.

3.2.4 Hamiltonian Evolution Equations II

With regard to the Port-Hamiltonian formulation of field theories which is the main part of the next chapter it will become apparent that the proposed iPCHD representation of Definition 3.4 is not general enough. Therefore, this subsection is dedicated to a further generalisation of the Port-Hamiltonian framework to the distributed-parameter case. In fact, we intend to extend the introduced iPCHD system representation of Definition 3.4 by replacing the relevant multilinear maps (3.19), (3.20) respectively by appropriate linear vector valued differential operators (the so-called differential operator case), where in the

sequel both iPCHD representations and even combinations of them will turn out to be the adequate description for field theories based on the Port-Hamiltonian framework. Nevertheless, it must be emphasised that the forthcoming iPCHD representation concerning the differential operator case is too general regarding the introduction of the boundary ports and, therefore, after the general definition of the system class we take two specific operators which will play a crucial role for the applications in the next chapter into account.

Linear Vector Valued Differential Operators

Before we proceed we have to introduce the considered differential operators. Roughly speaking, a differential operator serves as a map from a jet bundle (or a specific pull-back bundle) to a manifold; in fact, we restrict ourselves to specific linear differential operators which are maps from a pull-back bundle to a specific vector space.

Definition 3.6 (linear vector valued differential operator) *An r -order linear vector valued differential operator is a map of the form*

$$\mathfrak{D} : (\pi_0^p)^* (\mathcal{T}^* (\mathcal{X})) \wedge (\pi^p)^* \left(\bigwedge^m \mathcal{T}^* (\mathcal{D}) \right) \rightarrow (\pi_0^{p+r})^* (\mathcal{V} (\mathcal{X})) , \quad p \geq 0, r > 0,$$

which is locally given by

$$\mathfrak{D} (\omega) = \mathfrak{D}^{\alpha\beta J} d_J (\omega_\beta) \partial_\alpha, \quad 0 \leq \#J \leq r, \quad d_J = d_{j_r} \circ \dots \circ d_{j_1},$$

with respect to $\omega = \omega_\alpha dx^\alpha \wedge dX$ with $\omega_\alpha \in C^\infty (\mathcal{J}^p (\mathcal{X}))$ and the components $\mathfrak{D}^{\alpha\beta J} \in C^\infty (\mathcal{J}^p (\mathcal{X}))$. Furthermore, its (formal) adjoint takes the form of

$$\mathfrak{D}^* (\omega) = (-1)^{\#J} d_J (\mathfrak{D}^{\beta\alpha J} \omega_\beta) \partial_\alpha.$$

It is worth mentioning that the adjoint operator can be easily obtained by integration by parts leading to

$$\mathfrak{D} (\omega) \rfloor \varpi = \mathfrak{D}^* (\varpi) \rfloor \omega + d_h (\mathfrak{d}), \quad (3.25)$$

with respect to $\omega = \omega_\alpha dx^\alpha \wedge dX$ as well as $\varpi = \varpi_\alpha dx^\alpha \wedge dX$ with $\omega_\alpha, \varpi_\alpha \in C^\infty (\mathcal{J}^p (\mathcal{X}))$ and $\mathfrak{d} = \mathfrak{d}^i \partial_i \rfloor dX$, where the components \mathfrak{d}^i are bilinear expressions involving the components $\omega_\alpha, \varpi_\alpha$ and total derivatives of them up to order $r - 1$, see, e.g., [Olver, 1993]. Furthermore, the specification of the adjoint operator allows an important characterisation of the operator itself.

Definition 3.7 (skew-, self-adjoint operator) *An r -order linear vector valued differential operator \mathfrak{D} is called skew-adjoint if $\mathfrak{D}^* = -\mathfrak{D}$ and it is self-adjoint if $\mathfrak{D}^* = \mathfrak{D}$, respectively, see [Olver, 1993].*

In the sequel the objective is to extend the iPCHD system representation by means of appropriate linear vector valued differential operators, where, in particular, the characterisation of the operators of Definition 3.7 plays an important role.

The iPCHD System Class (the Differential Operator Case)

Having this framework at one's disposal the iPCHD system representation can be further generalised, where the interconnection and the dissipation map are represented by r -order linear vector valued differential operators according to Definition 3.6 and the input map also serves as a linear differential operator in this context.

Definition 3.8 (iPCHD system, differential operator case) *An iPCHD system with respect to the differential operator case with the Hamiltonian functional (3.9) reads as*

$$\begin{aligned} \dot{x} = v &= (\mathfrak{J} - \mathfrak{R})(\delta(\mathcal{H} dX)) + \mathfrak{G}(u) \\ y &= \mathfrak{G}^*(\delta(\mathcal{H} dX)) \end{aligned} \quad (3.26)$$

inclusive appropriate boundary conditions together with $\dot{X} = 0$ and with the skew-adjoint operator \mathfrak{J} , the self-adjoint non-negative operator \mathfrak{R} as well as the input operator \mathfrak{G} and its (formal) adjoint \mathfrak{G}^ with respect to the distributed input u and the distributed collocated output y . Furthermore, the formal change of the Hamiltonian functional (3.9) along (3.7) takes the form of*

$$\begin{aligned} v(\mathfrak{H}(\Phi)) &= \int_{\mathcal{D}} (j^{2+r}\Phi)^* ((\mathfrak{J} - \mathfrak{R})(\delta(\mathcal{H} dX)) \rfloor \delta(\mathcal{H} dX)) + \int_{\mathcal{D}} (j^{2+r}\Phi)^* (\mathfrak{G}(u) \rfloor \delta(\mathcal{H} dX)) \\ &\quad + \int_{\partial\mathcal{D}} \iota^* ((j^{2+r}\Phi)^* (v) \rfloor \delta^\partial(\mathcal{H} dX)) . \end{aligned} \quad (3.27)$$

For this case the operators \mathfrak{J} and \mathfrak{R} are r -order linear vector valued differential operators which are maps of the form

$$\mathfrak{J}, \mathfrak{R} : (\pi_0^2)^*(\mathcal{T}^*(\mathcal{X})) \wedge (\pi^2)^* \left(\bigwedge^m \mathcal{T}^*(\mathcal{D}) \right) \rightarrow (\pi_0^{2+r})^*(\mathcal{V}(\mathcal{X})), \quad r > 0, \quad (3.28)$$

where \mathfrak{J} is a skew-adjoint operator according to

$$\mathfrak{J}(\omega) \rfloor \varpi + \mathfrak{J}(\varpi) \rfloor \omega = d_h(j), \quad j = j^i \partial_i \rfloor dX, \quad (3.29)$$

with $\omega = \omega_\alpha dx^\alpha \wedge dX$, $\varpi = \varpi_\alpha dx^\alpha \wedge dX$ as well as $\omega_\alpha, \varpi_\alpha \in C^\infty(\mathcal{J}^2(\mathcal{X}))$ and \mathfrak{R} is a non-negative self-adjoint operator, i.e.,

$$\mathfrak{R}(\omega) \rfloor \varpi - \mathfrak{R}(\varpi) \rfloor \omega = d_h(\mathfrak{r}), \quad \mathfrak{r} = \mathfrak{r}^i \partial_i \rfloor dX, \quad \mathfrak{R}(\omega) \rfloor \omega \geq 0. \quad (3.30)$$

The input operator \mathfrak{G} and its adjoint operator \mathfrak{G}^* are maps according to

$$\mathfrak{G} : \mathcal{U} \rightarrow (\pi_0^{2+r})^*(\mathcal{V}(\mathcal{X})), \quad \mathfrak{G}^* : (\pi_0^2)^*(\mathcal{T}^*(\mathcal{X})) \wedge (\pi^2)^* \left(\bigwedge^m \mathcal{T}^*(\mathcal{D}) \right) \rightarrow \mathcal{Y} \quad (3.31)$$

and they are linear r -order differential operators with respect to the relation

$$\mathfrak{G}(u) \rfloor \omega = u \rfloor \mathfrak{G}^*(\omega) + d_h(\mathfrak{g}), \quad \mathfrak{g} = \mathfrak{g}^i \partial_i \rfloor dX, \quad (3.32)$$

where \mathfrak{G} and its adjoint \mathfrak{G}^* are locally given by

$$\mathfrak{G}(u) = \mathfrak{G}_\xi^{\alpha J} d_J(u^\xi) \partial_\alpha, \quad \mathfrak{G}^*(\omega) = (-1)^{\#J} d_J(\mathfrak{G}_\xi^{\alpha J} \omega_\alpha) e^\xi \otimes dX.$$

These operators may also be interpreted as r -order linear vector valued differential operators, though, with respect to the corresponding in- and output vector bundles¹⁰. In this context the components j^i and \mathfrak{r}^i are bilinear expressions involving the components $\omega_\alpha, \varpi_\alpha$ and total derivatives of them up to order $r - 1$ as well as the components g^i which are bilinear expressions, too, and which contain the components u^ξ, ω_α and total derivatives of them up to order $r - 1$. Consequently, from (3.32) we are able to derive the relation

$$\mathfrak{G}(u)]\delta(\mathcal{H}dX) = u]\mathfrak{G}^*(\delta(\mathcal{H}dX)) + d_h(\mathfrak{g}) = u]y + d_h(\mathfrak{g})$$

characterising the port distributed over \mathcal{D} . Hence, it is obvious that an iPCHD system of the form (3.26) may describe a set of $(r + 2)$ -order evolution equations. Nevertheless, the definition of the boundary ports for this system class is, in general, more sophisticated than for the non-differential operator case (Definition 3.4) since (3.27) leads – besides (3.16) – to additional boundary terms due to the applications of the operators according to (3.29), (3.30) and (3.32). Thus, the definition of the boundary ports depends on the considered application and cannot be explicitly defined for the general case which has been presented so far. Furthermore, this fact has serious consequences for the determination of the structural invariants for this system class, where due to the applications of the operators it is no longer possible to derive the necessary conditions for the structural invariants in this general setting¹¹.

Specific Operators

As mentioned before, in the sequel we will consider two types of operators motivated by the forthcoming applications. First of all, we introduce a second-order non-negative self-adjoint operator \mathfrak{R} locally given by

$$\mathfrak{R}(\omega) = d_i(\mathfrak{R}^{\alpha\beta ij} d_j(\omega_\beta)) \partial_\alpha, \quad \mathfrak{R}^{\alpha\beta ij} = \mathfrak{R}^{\beta\alpha ji}, \quad (3.33)$$

with, in general, $\mathfrak{R}^{\alpha\beta ij} \in C^\infty(\mathcal{J}^3(\mathcal{X}))$, which satisfies the relation

$$\mathfrak{R}(\omega)]\varpi - \mathfrak{R}(\varpi)]\omega = d_h(\bar{\mathfrak{R}}(\omega)]\varpi_\partial - \bar{\mathfrak{R}}(\varpi)]\omega_\partial) \quad (3.34)$$

with respect to

$$\omega = \omega_\alpha dx^\alpha \wedge dX, \quad \varpi = \varpi_\alpha dx^\alpha \wedge dX, \quad \omega_\alpha, \varpi_\alpha \in C^\infty(\mathcal{J}^2(\mathcal{X})),$$

and $\omega_\partial = -\partial_i]\omega, \varpi_\partial = -\partial_i]\varpi$ as well as

$$\bar{\mathfrak{R}}(\omega) = \mathfrak{R}^{\alpha\beta ij} d_j(\omega_\beta) \partial_\alpha.$$

¹⁰In fact, the in- and output vector bundles on the domain as well as on the boundary can be introduced in an analogous manner as before with respect to the corresponding bilinear maps concerning the specific duality properties, though, with respect to the higher-order case. Therefore, their introduction is omitted at this stage.

¹¹Therefore, the treatment of the structural invariants for the iPCHD system class concerning the differential operator case is completely omitted.

Thus, (3.34) locally reads as

$$\begin{aligned} \varpi_\alpha d_i (\mathfrak{R}^{\alpha\beta ij} d_j (\omega_\beta)) dX - \omega_\alpha d_i (\mathfrak{R}^{\alpha\beta ij} d_j (\varpi_\beta)) dX \\ = d_h (\varpi_\alpha \mathfrak{R}^{\alpha\beta ij} d_j (\omega_\beta) \partial_i] dX - \omega_\alpha \mathfrak{R}^{\alpha\beta ij} d_j (\varpi_\beta) \partial_i] dX) . \end{aligned}$$

The operator \mathfrak{R} is non-negative by means of the relation

$$\begin{aligned} \mathfrak{R}(\omega)]\omega = \omega_\alpha d_i (\mathfrak{R}^{\alpha\beta ij} d_j (\omega_\beta)) dX = \\ - d_i (\omega_\alpha) \mathfrak{R}^{\alpha\beta ij} d_j (\omega_\beta) dX + d_h (\omega_\alpha \mathfrak{R}^{\alpha\beta ij} d_j (\omega_\beta) \partial_i] dX) , \end{aligned}$$

where the non-negativity of the operator follows if

$$- d_i (\omega_\alpha) \mathfrak{R}^{\alpha\beta ij} d_j (\omega_\beta) \geq 0 \quad (3.35)$$

is met.

Furthermore, we define a first-order input operator \mathfrak{G} which corresponds to

$$\mathfrak{G}(u)]\omega = u] \mathfrak{G}^*(\omega) + d_h (\bar{\mathfrak{G}}(u)]\omega_\partial) \quad (3.36)$$

with

$$\mathfrak{G}(u) = \mathfrak{G}(u^\xi e_\xi) = \mathfrak{G}_\xi^{\alpha i} d_i (u^\xi) \partial_\alpha \quad (3.37)$$

and, in general, $\mathfrak{G}_\xi^{\alpha i} \in C^\infty(\mathcal{J}^2(\mathcal{X}))$ as well as

$$\bar{\mathfrak{G}}(u) = \mathfrak{G}_\xi^{\alpha i} u^\xi \partial_\alpha, \quad \omega_\partial = -\partial_i] \omega ,$$

where (3.36) locally takes the form of

$$\omega_\alpha \mathfrak{G}_\xi^{\alpha i} d_i (u^\xi) dX = -u^\xi d_i (\omega_\alpha \mathfrak{G}_\xi^{\alpha i}) dX + d_h (\omega_\alpha \mathfrak{G}_\xi^{\alpha i} u^\xi \partial_i] dX) . \quad (3.38)$$

Conclusions

In order to emphasise the analogies of the iPCHD system representation concerning the differential operator case with the Port-Hamiltonian representation from the finite dimensional case we are able to propose the table 3.2, where the direct correspondences can be found.

3.2.5 Concluding Remarks

Finally, it must be emphasised that the iPCHD system representations of Definition 3.4 and 3.8 will be combined in the sequel, i.e., combinations of the maps (3.19), (3.20) as well as (3.28), (3.31) will appear in the iPCHD formulation of certain applications. In fact, the Port-Hamiltonian framework has been introduced rather generally in order to cover a wide range of applications concerning the formulation of field theories which is the main part of the next chapter.

finite dimensional PCHD system	iPCHD system, differential operator case
a set of ordinary differential equations	a set of $(r + 2)$ -order evolution equations incl. appropriate boundary conditions
state manifold \mathcal{M} tangent bundle $\tau_{\mathcal{M}}$ cotangent bundle $\tau_{\mathcal{M}}^*$	state bundle π , Jet bundles (order r) vertical bundle $\nu_{\mathcal{X}}$ construction of certain pull-back bundles
Hamiltonian $H \in C^\infty(\mathcal{M})$	Hamiltonian functional $\mathfrak{H}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^*(\mathcal{H} dX)$ with $\Phi \in \Gamma(\pi)$
total derivative $d : C^\infty(\mathcal{M}) \rightarrow T^*(\mathcal{M})$	variational derivative δ , see (3.10) boundary map δ^∂ , see (3.12)
$J, R : T^*(\mathcal{M}) \rightarrow T(\mathcal{M})$	operators $\mathfrak{J}, \mathfrak{R}$ see (3.28), (3.29) and (3.30)
$G : \mathcal{U} \rightarrow T(\mathcal{M}), G^* : T^*(\mathcal{M}) \rightarrow \mathcal{Y}$	operators $\mathfrak{G}, \mathfrak{G}^*$ see (3.31), (3.32) multilinear maps $\mathcal{G}_\partial, \mathcal{G}_\partial^*$ and $\mathcal{G}^\partial, \mathcal{G}^{\partial,*}$ see, e.g., (3.23), (3.24)
port $u y$ with $y = G^*]dH$	distributed ports with $y = \mathfrak{G}^*(\delta(\mathcal{H} dX))$ boundary ports depend on the application of the corresponding operators together with (3.23), (3.24)
conditions for structural invariants $C \in C^\infty(\mathcal{M})$ see Definition 3.2	conditions for structural invariants $\mathfrak{C}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^*(\mathcal{C} dX)$ depend on the application of the corresponding operators

Table 3.2: The correspondences of finite and infinite dimensional Port-Hamiltonian systems (the differential operator case)

Port-Hamiltonian Formulation of Field Theories

In order to point out the effectiveness of the Port-Hamiltonian framework with respect to the formulation of field theories the main focus of this chapter is dedicated to the Port-Hamiltonian representation of three physical applications; namely, we investigate the Port-Hamiltonian description of the governing equations of beams modelled according to the Timoshenko theory and of fluid mechanical as well as magnetohydrodynamic applications. These applications have become established more and more in the control community over the past years, see [Kim and Renardy, 1987, Luo et al., 1999, Macchelli and Melchiorri, 2004a,b, Vazquez and Krstic, 2008, Zhang, 2007], for instance.

In section 4.1 we investigate the Port-Hamiltonian modelling of the Timoshenko beam mainly based on [Schöberl and Schlacher, 2011, Siuka et al., 2011], where we motivate the Port-Hamiltonian modelling task on the basis of the introduced system representations of chapter 3. Afterwards, the gained formulation will be compared to another Port-Hamiltonian representation based on the concept of the Stokes-Dirac structures, see [Macchelli and Melchiorri, 2004a,b], and the main differences concerning the mechanical aspects will be discussed. Section 4.2 deals with the Port-Hamiltonian formulation of fluid dynamical applications in a Lagrangian setting. Therefore, we will extensively analyse the governing equations from a geometric point of view in order to gain enough insights to achieve a Port-Hamiltonian formulation of the basic fluid equations – namely the Navier-Stokes equations in a Lagrangian setting. This point of view may be advantageously with respect to the modelling of injection processes, for instance. On the basis of these considerations this formulation will be extended in order to also take the interaction of free currents and electromagnetic fields with fluid matter into account which will lead us to the Port-Hamiltonian formulation of the governing equations of magnetohydrodynamics (in a Lagrangian setting) based on [Schöberl et al., 2010, Siuka et al., 2010]; this is the main focus of section 4.3, where we will investigate the so-called inductionless magnetohydrodynamic case. Roughly speaking, in the inductionless magnetohydrodynamic case we consider the macroscopic behaviour of an electrically conducting fluid (continuum) in the presence of external electromagnetic fields, where it is assumed that the dynamic of the additionally induced electromagnetic parts can be neglected which is the case for many

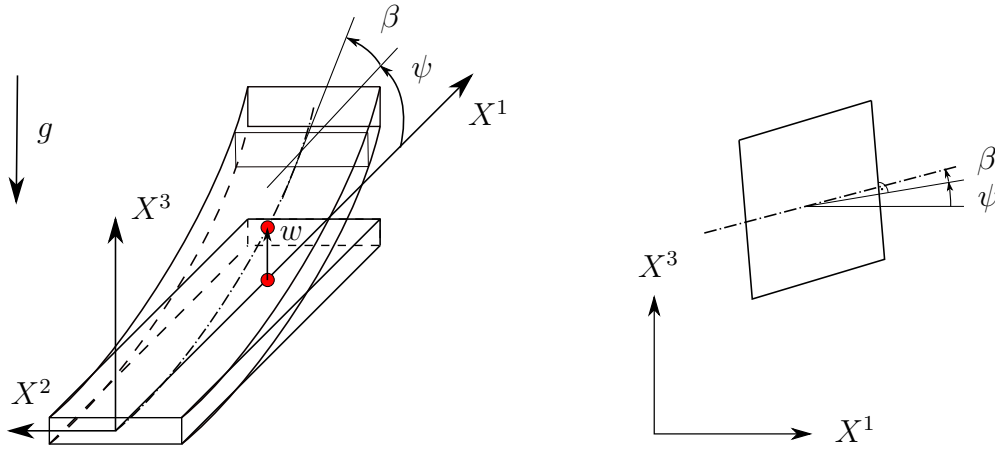


Figure 4.1: The Timoshenko beam and a beam element

industrial applications, see, e.g., [Davidson, 2001, Eringen and Maugin, 1990, Sutton and Sherman, 2006].

4.1 Port-Hamiltonian Modelling of the Timoshenko Beam

This section is dedicated to the derivation of the iPCH representation of the governing equations of the Timoshenko beam. In general, the Timoshenko beam model is based on linearised geometric as well as linear constitutive relations and it takes the shear deformation effects and the rotatory inertia of the beam into account. More precisely, we consider the beam configuration of Figure 4.1, where w denotes the deflection and ψ characterises the angle of rotation due to bending. The spatial coordinate along the beam axis in its (undeformed) initial configuration ($w = \psi = 0$) is given by $X^1 \in [0, L]$, $L \in \mathbb{R}^+$. Furthermore, the motion of the beam is restricted to the (X^1, X^3) -plane and we assume no beam elongation. Therefore, the governing equations of the Timoshenko beam in the case under consideration are given by the set of coupled second-order PDEs, see, e.g., [Meirovitch, 1997, Ziegler, 1998],

$$\begin{aligned} \rho \ddot{w} &= d_1 (kAG(w_1 - \psi)) - \rho g, \\ I_m \ddot{\psi} &= d_1 (EI_a \psi_1) + kAG(w_1 - \psi), \end{aligned} \quad (4.1)$$

where the derivative coordinates with respect to the independent spatial coordinate X^1 are denoted by w_1 and ψ_1 as well as w_{11} and ψ_{11} characterising the first- and the second-order spatial derivatives¹. Moreover, the relation $w_1 = \psi + \beta$ is met with respect to the angle of distortion due to shear denoted by β . Thus, the (one-dimensional) spatial domain is represented by $\mathcal{D} = [0, L]$ equipped with the spatial coordinate X^1 and the appropriate volume form $dX = dX^1$, i.e., $m = 1$. The boundary $\partial\mathcal{D}$ is characterised by $X^1 = 0$ as well

¹It is worth noting that (4.1) corresponds to a set of second-order evolution equations, cf. (3.6), with respect to $x = (w, \psi, v_w, v_\psi)$, for instance, by rewriting the equations in terms of the deflection velocity $\dot{w} = v_w$ as well as the rotational velocity $\dot{\psi} = v_\psi$ and by considering the (trivial) equation $\dot{X}^1 = 0$ concerning the independent spatial coordinate X^1 .

as $X^1 = L$. The beam parameters are given by the mass per unit length $0 < \rho \in C^\infty(\mathcal{D})$, the shear modulus $0 < G \in C^\infty(\mathcal{D})$, the mass moment of inertia $0 < I_m \in C^\infty(\mathcal{D})$, the area moment of inertia $0 < I_a \in C^\infty(\mathcal{D})$, the gravitational constant $g \in \mathbb{R}^+$, the numerical factor $k \in \mathbb{R}^+$ depending on the shape of the cross section as well as the cross-sectional area $0 < A \in C^\infty(\mathcal{D})$ and the elasticity module $0 < E \in C^\infty(\mathcal{D})$. For further details see [Meirovitch, 1997, Ziegler, 1998], for instance.

In order to obtain an equivalent iPCH representation of the governing equations we choose the displacements w, ψ and the temporal momenta which read as

$$p_w = \rho \dot{w}, \quad p_\psi = I_m \dot{\psi}$$

as dependent coordinates. In this context the state bundle $\pi : \mathcal{X} \rightarrow \mathcal{D}$ possesses the coordinates (X^1, x^α) , $\alpha = 1, \dots, 4$, with $x = (w, \psi, p_w, p_\psi)$ and the required Jet bundles can be constructed in a standard manner. Furthermore, the kinetic energy of the beam takes the form of

$$\mathfrak{T}(\Phi) = \frac{1}{2} \int_0^L \Phi^* \left(\left(\frac{1}{\rho} (p_w)^2 + \frac{1}{I_m} (p_\psi)^2 \right) dX \right), \quad \Phi \in \Gamma(\pi),$$

in terms of the temporal momenta and the potential energy reads as

$$\mathfrak{V}(\Phi) = \int_0^L (j^1 \Phi)^* \left(\left(\frac{1}{2} E I_a (\psi_1)^2 + \frac{1}{2} k A G (w_1 - \psi)^2 + \rho g w \right) dX \right), \quad \Phi \in \Gamma(\pi),$$

including a gravitational potential related to the initial configuration. The Hamiltonian functional is equivalent to the sum of the kinetic and the potential energy of the beam

$$\mathfrak{H}(\Phi) = \mathfrak{T}(\Phi) + \mathfrak{V}(\Phi) = \int_0^L (j^1 \Phi)^* (\mathcal{H} dX) \quad (4.2)$$

with respect to the first-order Hamiltonian density

$$\mathcal{H} dX = \left(\frac{1}{2\rho} (p_w)^2 + \frac{1}{2I_m} (p_\psi)^2 + \frac{1}{2} E I_a (\psi_1)^2 + \frac{1}{2} k A G (w_1 - \psi)^2 + \rho g w \right) dX \quad (4.3)$$

which corresponds to the sum of the kinetic and the potential energy density.

Proposition 4.1 *Consider the Hamiltonian functional (4.2) with the first-order Hamiltonian density (4.3). The iPCH system representation of the Timoshenko beam takes (in matrix representation) the form of*

$$\dot{x} = v = \begin{bmatrix} \dot{w} \\ \dot{\psi} \\ \dot{p}_w \\ \dot{p}_\psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_w \mathcal{H} \\ \delta_\psi \mathcal{H} \\ \delta_{p_w} \mathcal{H} \\ \delta_{p_\psi} \mathcal{H} \end{bmatrix} = \mathcal{J}(\delta(\mathcal{H} dX)), \quad (4.4)$$

and the formal change of (4.2) reads as

$$\begin{aligned} v(\mathfrak{H}(\Phi)) &= \iota_L^* \left((j^1 \Phi)^* \left(\frac{1}{\rho} p_w k A G (w_1 - \psi) + \frac{1}{I_m} p_\psi E I_a \psi_1 \right) \right) \\ &\quad - \iota_0^* \left((j^1 \Phi)^* \left(\frac{1}{\rho} p_w k A G (w_1 - \psi) + \frac{1}{I_m} p_\psi E I_a \psi_1 \right) \right) \end{aligned} \quad (4.5)$$

with respect to the inclusion mappings $\iota_0 : \{0\} \rightarrow \{X^1 = 0\}$ and $\iota_L : \{L\} \rightarrow \{X^1 = L\}$.

In order to show the equivalence of (4.1) and (4.4) we consider the first set of the equations (4.4) which read as

$$\dot{w} = \delta_{p_w} \mathcal{H} = \partial_{p_w} \mathcal{H} = \frac{1}{\rho} p_w, \quad \dot{\psi} = \delta_{p_\psi} \mathcal{H} = \partial_{p_\psi} \mathcal{H} = \frac{1}{I_m} p_\psi$$

since the variational derivatives with respect to the temporal momenta degenerate to partial ones. The second set of the equations follows by a direct computation to

$$\begin{aligned} \dot{p}_w &= -\delta_w \mathcal{H} = -\partial_w \mathcal{H} + d_1 (\partial_w^1 \mathcal{H}) = -\rho g + d_1 (kAG (w_1 - \psi)), \\ \dot{p}_\psi &= -\delta_\psi \mathcal{H} = -\partial_\psi \mathcal{H} + d_1 (\partial_\psi^1 \mathcal{H}) = kAG (w_1 - \psi) + d_1 (EI_a (\psi_1)). \end{aligned}$$

Substituting with the first set of the equations shows the equivalence with (4.1). The formal change of the Hamiltonian functional follows by direct computation.

It is worth mentioning that the formal change of the Hamiltonian functional (4.5) allows for a clear physical interpretation since the formal change involves the collocation between the deflection velocity \dot{w} and the shearing force $kGA (w_1 - \psi)$ as well as the rotational velocity $\dot{\psi}$ due to bending and the bending moment $EI_a \psi_1$ on the boundary. Therefore, the formal change of the functional is characterised by the geometric as well as the natural boundary conditions which often appear in mechanics. Furthermore, it is obvious how to introduce the boundary ports which may be parameterised according to the general cases (3.23) and (3.24) which allow to consider, e.g., the forces and moments as boundary inputs and the velocities as appropriate boundary outputs or conversely.

Example 4.1 *Let us consider a beam cantilevered at $X^1 = L$ with iPCH representation (4.4). The boundary conditions at the clamped end (at $X^1 = L$) are equivalent to the geometric boundary conditions*

$$\iota_L^* \left(\Phi^* \left(\frac{1}{\rho} p_w \right) \right) = 0, \quad \iota_L^* \left(\Phi^* \left(\frac{1}{I_m} p_\psi \right) \right) = 0$$

and the free end (at $X^1 = 0$) is characterised by the natural boundary conditions

$$\iota_0^* \left((j^1 \Phi)^* (kAG (w_1 - \psi)) \right) = 0, \quad \iota_0^* \left((j^1 \Phi)^* (EI_a \psi_1) \right) = 0,$$

expressing the fact that the shearing force as well as the bending moment must vanish at the free end. In this case (4.5) takes the form of $v(\mathcal{H}(\Phi)) = 0$ and the Hamiltonian functional serves as a conserved quantity (the total energy is conserved).

With regard to Definition 3.5 it is clear that the structural invariants for this configuration are total derivatives only since \mathcal{J} has full rank resulting in

$$\delta_w \mathcal{C} = \delta_\psi \mathcal{C} = \delta_{p_w} \mathcal{C} = \delta_{p_\psi} \mathcal{C} = 0.$$

Due to the free and the clamped end the remaining conditions are given by

$$\begin{aligned} \partial_w^1 \mathcal{C} \circ \iota_0 &= 0 \\ \partial_\psi^1 \mathcal{C} \circ \iota_0 &= 0 \\ \partial_{p_w}^1 \mathcal{C} \circ \iota_0 = \partial_{p_w}^1 \mathcal{C} \circ \iota_L &= 0 \\ \partial_{p_\psi}^1 \mathcal{C} \circ \iota_0 = \partial_{p_\psi}^1 \mathcal{C} \circ \iota_L &= 0 \end{aligned}$$

together with arbitrary $\partial_w^1 \mathcal{C} \circ \iota_L$ as well as $\partial_\psi^1 \mathcal{C} \circ \iota_L$. Therefore, it is easily verified that the two structural invariants of the form

$$\begin{aligned} \mathfrak{e}^1(\Phi) &= \frac{1}{L} \int_0^L (j^1 \Phi)^* ((w + X^1 w_1) \, dX) = \frac{1}{L} \int_0^L (j^1 \Phi)^* (d_1 (X^1 w) \, dX) = \iota_L^* (w \circ \Phi) , \\ \mathfrak{e}^2(\Phi) &= \frac{1}{L} \int_0^L (j^1 \Phi)^* ((\psi + X^1 \psi_1) \, dX) = \frac{1}{L} \int_0^L (j^1 \Phi)^* (d_1 (X^1 \psi) \, dX) = \iota_L^* (\psi \circ \Phi) \end{aligned}$$

fulfil the conditions from above and, additionally, they serve as conserved quantities for the considered beam configuration clearly reflecting the fact that the deflection and the angle of rotation at the clamped end are fixed.

It must be emphasised that the iPCH representation of the Timoshenko beam is not unique. In [Macchelli and Melchiorri, 2004a,b] a Port-Hamiltonian formulation of the governing beam equations is presented based on the concept of the Stokes-Dirac structures, where for the dependent coordinates, besides the temporal momenta, the deformations $w_1 - \psi$, ψ_1 are used in order to deal with the duality properties of the underlying Stokes-Dirac structures (so-called energy variables are used, see [van der Schaft and Maschke, 2002]). As a consequence, no derivative coordinates appear directly in the Hamiltonian density. Thus, in this case the variational derivative degenerates to a partial one, though, the interconnection map must be replaced by an appropriate differential operator, where the boundary ports are derived by the integration by parts methodology. Nevertheless, the choice of the dependent coordinates has essential physical consequences. In [Macchelli and Melchiorri, 2004a,b] no displacement coordinates are used and, therefore, the gravity force density may be considered by a distributed input only since no gravitational potential can be assigned. Furthermore, due to the choice of the deformations it is not possible to describe the location of the beam with respect to an inertial system which could be a drawback with regard to control purposes whenever position control is the objective. Moreover – with respect to the modelling of plates, etc. – this choice of the coordinates does not allow a direct application to (spatially) higher dimensional applications, where we deal with more than one independent spatial coordinate, since restrictions appear as additional PDEs which would lead to the class of restricted iPCH(D) systems. However, the choice of the displacements, as it is the case for the illustrated Port-Hamiltonian approach, can be applied directly to higher dimensional cases.

Example 4.2 (continued) Consider again the configuration of the beam of Example 4.1 but the clamped end is replaced by an actuated boundary and the shearing force and the bending moment at $X^1 = L$ are considered as boundary system inputs with respect to the case (3.24), i.e., the boundary map takes (in matrix representation) the form of

$$[\mathcal{G}_\alpha^{\partial, \xi_\partial}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{G}_\alpha^{\partial, \xi_\partial} = \delta_\alpha^{\xi_\partial}, \quad \xi_\partial = 1, 2.$$

Then, (4.5) reads as

$$v(\mathfrak{H}(\Phi)) = (\Phi_{\partial_L}^1)^* (u_1^\partial y^{\partial, 1} + u_2^\partial y^{\partial, 2}), \quad \Phi_{\partial_L}^1 = j^1 \Phi \circ \iota_L,$$

containing the boundary system inputs

$$u_1^\partial = (kGA(w_1 - \psi)) \circ \iota_L, \quad u_2^\partial = (EI_a \psi_1) \circ \iota_L,$$

and the corresponding boundary outputs

$$y^{\partial,1} = \left(\frac{1}{\rho} p_w \right) \circ \iota_L, \quad y^{\partial,2} = \left(\frac{1}{I_m} p_\psi \right) \circ \iota_L.$$

4.2 Port-Hamiltonian Formulation of Fluid Dynamics

In this section we will derive a Port-Hamiltonian representation of the governing equations of fluid dynamical applications in a Lagrangian setting. First of all, we will investigate in detail the underlying concepts which are necessary for deriving the Navier-Stokes equations based on a purely geometric point of view in order to obtain a formulation, where we switch from the usual Eulerian description to a description on the basis of the Lagrangian point of view. This task will be essential for a Port-Hamiltonian representation based on the former introduced approach. In this context we will make heavy use of the continuum mechanical relationships concerning both descriptions. Therefore, in the Lagrangian picture we are able to characterise the motion of a continuum with fluid matter, where we first investigate the Port-Hamiltonian formulation of the so-called ideal fluid, where we consider no dissipative effects due to viscous stresses, in order to obtain an iPCH formulation of the governing equations. In the sequel, this framework will be extended with respect to the consideration of viscous stresses leading to an appropriate iPCHD representation of the Navier-Stokes equations in a Lagrangian view.

4.2.1 The Geometry of Lagrangian Fluid Dynamics

For Lagrangian fluid dynamics the concept of a reference manifold is important in contrast to fluid dynamics based on the Eulerian point of view. In this context the reference manifold serves as a label for the fluid particle positions at the initial point of time. Therefore, we introduce the (trivial) reference bundle $\pi_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{I}$ with $\mathcal{R} = \mathcal{I} \times \mathcal{B}$ equipped with coordinates² (t^0, X^i) , where \mathcal{B} denotes the reference manifold which possesses the so-called material coordinates (X^i) and \mathcal{I} is the time manifold with the coordinate (t^0) . The reference manifold \mathcal{B} is supposed to be a (compact) Riemannian manifold (with coherently oriented boundary $\partial\mathcal{B}$) equipped with a (positive definite) metric

$$G = G_{ij} dX^i \otimes dX^j, \quad G_{ij} = G_{ji} \in C^\infty(\mathcal{B}), \quad (4.6)$$

serving as a map $G : \mathcal{T}(\mathcal{B}) \rightarrow \mathcal{T}^*(\mathcal{B})$. The inverse of the metric is a map $\hat{G} : \mathcal{T}^*(\mathcal{B}) \rightarrow \mathcal{T}(\mathcal{B})$ given by

$$\hat{G} = G^{ij} \partial_i \otimes \partial_j, \quad G^{ij} = G^{ji} \in C^\infty(\mathcal{B}),$$

where the components fulfil $G^{ij} G_{jk} = \delta_k^i$. Furthermore, the associated volume form reads as

$$\text{VOL} = \sqrt{\det[G_{ij}]} dX^1 \wedge \dots \wedge dX^{m_x}.$$

²In this section the Latin indices vary from 1 to m_x and the Greek indices from 1 to n_q .

In order to characterise a configuration of a continuum with fluid matter we introduce a configuration manifold \mathcal{Q} which possesses the local coordinates (q^α) which may be interpreted as spatial coordinates, see, e.g., [Marsden and Hughes, 1994]. Since the configuration manifold is also a (compact) Riemannian manifold (with coherently oriented boundary) we equip it with the (positive definite) metric

$$g = g_{\alpha\beta} dq^\alpha \otimes dq^\beta, \quad g_{\alpha\beta} = g_{\beta\alpha} \in C^\infty(\mathcal{Q}), \quad (4.7)$$

with inverse

$$\hat{g} = g^{\alpha\beta} \partial_\alpha \otimes \partial_\beta, \quad g^{\alpha\beta} = g^{\beta\alpha} \in C^\infty(\mathcal{Q}),$$

according to $g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha$ and the associated volume form reads as

$$\text{vol} = \sqrt{\det [g_{\alpha\beta}]} dq^1 \wedge \dots \wedge dq^{n_a}.$$

In this context we are able to introduce the (trivial) configuration bundle $\pi_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{I}$ with $\mathcal{C} = \mathcal{I} \times \mathcal{Q}$ equipped with coordinates (t^0, q^α) which can be extended to the bundle $\pi_L : \mathcal{E} \rightarrow \mathcal{R}$ with $\mathcal{E} = \mathcal{C} \times_{\mathcal{I}} \mathcal{R}$ which possesses the coordinates (t^0, X^i, q^α) . A section $\Phi \in \Gamma(\pi_L)$ (locally) leads to $q^\alpha = \Phi^\alpha(t^0, X^i)$ which is called a motion in the Lagrangian setting. Roughly speaking, the motion maps a reference state $\mathcal{S} \subset \mathcal{B}$ – where an element of \mathcal{S} corresponds to a fluid particle in a unique manner – to a configuration $\Phi(t^0, \mathcal{S}) \subset \mathcal{Q}$ at a certain point of time t^0 .

Remark 4.1 *Since the reference manifold labels the fluid particle positions at the initial point of time, all subsequent configurations of the fluid particles are described by the motion $\Phi \in \Gamma(\pi_L)$ which characterises the particle paths leading to a particle placement field. Thus, for a particle labelled as X with coordinates (X^i) the motion $\Phi(t^0, X^i)$ represents the position of the particle at the time t^0 . In this context the spatial coordinates (q^α) may be interpreted as the fluid particle positions and \mathcal{Q} characterises the region in which the fluid flows. In this context, let us consider the reference state \mathcal{S} . Then, $\Phi(t^0, \mathcal{S})$ at a fixed point of time t^0 characterises the volume which is moving with the fluid. Therefore, for fluid dynamics we may identify the labels and the initial positions for $t^0 = 0$ and, hence, \mathcal{B} and \mathcal{Q} as well as G and g may coincide. Nevertheless, we strictly distinguish between the reference and the configuration manifold in order to separate the independent and dependent coordinates. For more detailed information see [Aris, 1989, Bennett, 2006, Chorin and Marsden, 1990, Marsden et al., 2001].*

Remark 4.2 *For fluid dynamics it is also convenient to call the motion $\Phi \in \Gamma(\pi_L)$ the fluid flow map, see [Chorin and Marsden, 1990].*

It is worth mentioning that the special structure of the configuration bundle may be characterised by the tensor

$$\Lambda = dt^0 \otimes \partial_0$$

– a so-called (trivial) reference frame – which represents a trivial connection on \mathcal{C} corresponding to an inertial frame, see, e.g., [Schöberl, 2007, Schöberl and Schlacher, 2007a] and references therein. In fact, all metric coefficients are time independent due to the considered case of the inertial frame. Furthermore, we are able to introduce the first Jet

manifold $\mathcal{J}^1(\mathcal{E})$, where the first prolongation of a section $\Phi \in \Gamma(\pi_L)$ leads to $q_0^\alpha = \partial_0 \Phi^\alpha = V_0^\alpha$ and $q_i^\alpha = \partial_i \Phi^\alpha = F_i^\alpha$. Therefore, V_0^α denote the components of the material velocity and F_i^α represent the components of the deformation gradient which are well-known quantities in elasticity theory, see [Marsden and Hughes, 1994]. In the sequel, we confine ourselves to the case $m_x = n_q$ and, therefore, we assume $\dim \mathcal{B} = \dim \mathcal{Q}$. In addition, it is supposed that the motion is smooth enough and, therefore, it is assumed that the motion is an invertible mapping in order that for an open set $\Phi(t^0, \mathcal{B})$ at a fixed point of time t^0 we are able to define the inverse of the motion by the map $\hat{\Phi} : \Phi(t^0, \mathcal{B}) \rightarrow \mathcal{B}$ leading to $X^i = \hat{\Phi}^i(t^0, q^\alpha)$.

In this context we are able to define the velocity as the tangent vector field $v : \mathcal{J}^1(\mathcal{C}) \rightarrow (\pi_{\mathcal{C},0}^1)^*(\mathcal{V}(\mathcal{C}))$ which takes the form of $v = q_0^\alpha \partial_\alpha$. In consideration of the motion this vector field enables us to introduce two important concepts, see [Marsden and Hughes, 1994]. On the one hand, by restricting v to the motion we obtain the material velocity $v = V_0^\alpha \partial_\alpha$ including the former introduced components $V_0^\alpha = \partial_0 \Phi^\alpha$ and on the other hand, since we assume that the motion is invertible, we deduce

$$v = v^\alpha \partial_\alpha = (V_0^\alpha \circ \hat{\Phi}) \partial_\alpha$$

which is called the spatial velocity with components $v^\alpha = V_0^\alpha \circ \hat{\Phi} \in C^\infty(\mathcal{C})$. It is worth mentioning that the spatial velocity may be interpreted as a vertical tangent vector field $v : \mathcal{C} \rightarrow \mathcal{V}(\mathcal{C})$. In this context the corresponding connection on the vertical bundle $\mathcal{V}(\mathcal{C}) \rightarrow \mathcal{C}$ reads as

$$\Lambda_c = dt^0 \otimes \partial_0 + dq^\alpha \otimes \left(\partial_\alpha - \gamma_{\alpha\gamma}^\beta \dot{q}^\gamma \partial_\beta \right), \quad \dot{\partial}_\beta = \frac{\partial}{\partial \dot{q}^\beta}, \quad (4.8)$$

including the Christoffel symbols of the second kind given by

$$\gamma_{\alpha\gamma}^\beta = \gamma_{\gamma\alpha}^\beta = \frac{1}{2} g^{\beta\delta} (\partial_\alpha g_{\gamma\delta} + \partial_\gamma g_{\alpha\delta} - \partial_\delta g_{\alpha\gamma}) \in C^\infty(\mathcal{Q}).$$

Remark 4.3 *Roughly speaking, in mechanics the concept of connections is indispensable for the formulation of velocities and accelerations in an intrinsic manner. For detailed information see, e.g., [Giachetta et al., 1997, Schöberl, 2007, Schöberl and Schlacher, 2007a].*

In the sequel, we will often pull-back certain forms such as $\omega \text{ vol}$ with $\omega \in C^\infty(\mathcal{Q})$ with the help of the motion according to

$$\int_{\Phi(t^0, \mathcal{S})} \omega \text{ vol} = \int_{\mathcal{S}} \Phi^* (\omega \text{ vol}) = \int_{\mathcal{S}} J (\omega \circ \Phi) \text{ VOL},$$

where the integral has to be evaluated for a fixed time t^0 . The expression

$$J = \det [F_i^\alpha] \sqrt{\frac{\det [g_{\alpha\beta} \circ \Phi]}{\det [G_{ij}]}} \in C^\infty(\mathcal{R}) \quad (4.9)$$

is called the Jacobian of the motion, see [Marsden and Hughes, 1994], for instance.

Remark 4.4 *More precisely, when we pull-back the form $\omega \text{ vol}$ with $\omega \in C^\infty(\mathcal{Q})$ we obtain, in general,*

$$\Phi^*(\text{vol}) = J(\omega \circ \Phi) (dX^1 - \Gamma_0^1 dt^0) \wedge \dots \wedge (dX^{m_x} - \Gamma_0^{m_x} dt^0), \quad \Gamma_0^i \in C^\infty(\mathcal{R}).$$

Since the reference bundle is used for labelling the fluid particle positions at a fixed initial point of time the coefficients $\Gamma_0^i \in C^\infty(\mathcal{R})$ are not explicitly required for all further calculations. In particular, when we integrate over such forms on the fibres of \mathcal{R} at a fixed time t^0 then we have to consider the restriction $dt^0 = 0$. Therefore, we do not consider these parts and instead of the former expression we write $\Phi^(\text{vol}) = J \text{ VOL}$ which corresponds to the Change of Variables Theorem in [Marsden and Hughes, 1994]. For a more general discussion about this topic the interested reader is referred to [Schöberl, 2007].*

It is worth noting that the Jacobian describes the ratio of an elementary volume in the configuration to its initial volume in the reference state. The former assumption of the invertibility of the motion now corresponds to the requirement $0 < J < \infty$, see [Aris, 1989].

4.2.2 Conservation of Mass

The Jacobian (4.9) plays an important role for the principle of conservation of mass. Therefore, we assume the existence of the mass density $\rho \in C^\infty(\mathcal{C})$. Then, the mass $m(\mathcal{S}) \in \mathbb{R}^+$ of a continuum filled with fluid matter is defined as

$$m(\mathcal{S}) = \int_{\Phi(t^0, \mathcal{S})} \rho \text{ vol},$$

where the integral has to be evaluated at a fixed time t^0 for a configuration $\Phi(t^0, \mathcal{S}) \subset \mathcal{Q}$. The pull-back of this expression results in

$$m(\mathcal{S}) = \int_{\Phi(t^0, \mathcal{S})} \rho \text{ vol} = \int_{\mathcal{S}} J(\rho \circ \Phi) \text{ VOL}$$

and, consequently, the mass is conserved if

$$\int_{\mathcal{S}} \partial_0 (J(\rho \circ \Phi) \text{ VOL}) = 0 \tag{4.10}$$

is met. Since this relation must hold for every domain of integration and every point of time t^0 the equation of continuity in the Lagrangian description takes the form of

$$\partial_0 (J(\rho \circ \Phi)) = \partial_0 (\rho_{\mathcal{R}}) = 0, \tag{4.11}$$

where we have introduced the mass density in the reference state $\rho_{\mathcal{R}} \in C^\infty(\mathcal{B})$ according to $\rho_{\mathcal{R}} = J(\rho \circ \Phi)$. Otherwise, from (4.10) we obtain the equivalent result

$$\int_{\mathcal{S}} \partial_0 (J(\rho \circ \Phi) \text{ VOL}) = \int_{\Phi(t^0, \mathcal{S})} v_{\Phi}(\rho \text{ vol}) = 0$$

with³ $v_\Phi = \Phi_*(\partial_0) = \partial_0 + v^\alpha \partial_\alpha$, where Φ_* denotes the Push-forward of Φ . The evaluation of the Lie derivative results in (for fixed t^0)

$$\int_{\Phi(t^0, S)} (v_\Phi(\rho) + \rho \operatorname{div}(v)) \operatorname{vol} = 0$$

with respect to

$$\operatorname{div}(v) = \frac{1}{\sqrt{\det[g_{\alpha\beta}]}} \partial_\alpha \left(v^\alpha \sqrt{\det[g_{\alpha\beta}]} \right) \operatorname{vol}.$$

Hence, the equation of continuity in the Eulerian description takes the usual form of

$$v_\Phi(\rho) + \rho \operatorname{div}(v) = 0. \quad (4.12)$$

4.2.3 Stress Forms and Constitutive Relations in Fluid Dynamics

For the investigation of the constitutive relations in fluid dynamics we introduce the corresponding stress forms and stress tensors in the Eulerian as well as in the Lagrangian description based on [Frankel, 2004, Marsden and Hughes, 1994, Schlacher et al., 2004, Schöberl, 2007, Schöberl and Schlacher, 2007a].

Stress Forms

In the sequel the main object of interest will be the Cauchy stress form represented by the vector valued form

$$\sigma = \sigma^{\alpha\beta} \partial_\alpha \rfloor \operatorname{vol} \otimes \partial_\beta, \quad \sigma^{\alpha\beta} \in C^\infty(\mathcal{C}),$$

which characterises the effect of the surface forces; for fluid dynamics those are characterised by the hydrostatic pressure and viscous effects. Therefore, for a continuum with fluid matter the Cauchy stress form reads as

$$\sigma = -\mathcal{P} g^{\alpha\beta} \partial_\alpha \rfloor \operatorname{vol} \otimes \partial_\beta + \bar{\sigma}, \quad (4.13)$$

see, e.g., [Aris, 1989, Chorin and Marsden, 1990], including the hydrostatic pressure $\mathcal{P} \in C^\infty(\mathcal{C})$ and the viscous stress form

$$\bar{\sigma} = \bar{\sigma}^{\alpha\beta} \partial_\alpha \rfloor \operatorname{vol} \otimes \partial_\beta.$$

In order to obtain the corresponding expression in the Lagrangian setting we have to pull-back the form part by the motion $\Phi \in \Gamma(\pi_L)$. Therefore, we carry out a so-called Piola transformation leading to the first Piola-Kirchhoff stress form according to

$$P = \Phi^* (\sigma^{\alpha\beta} \partial_\alpha \rfloor \operatorname{vol}) \otimes \partial_\beta = P^{i\beta} \partial_i \rfloor \operatorname{VOL} \otimes \partial_\beta, \quad P^{i\beta} \in C^\infty(\mathcal{R}), \quad (4.14)$$

with the components

$$P^{i\beta} = J \left(\hat{F}_\alpha^i \sigma^{\alpha\beta} \right) \circ \Phi = J \left(-\mathcal{P} \hat{F}_\alpha^i g^{\alpha\beta} + \hat{F}_\alpha^i \bar{\sigma}^{\alpha\beta} \right) \circ \Phi.$$

³This vector field can be interpreted as the infinitesimal generator of an isomorphism $\phi_\tau : \mathcal{Q} \rightarrow \mathcal{Q}$ which maps a configuration at t^0 to a configuration at $t^0 + \tau$, see, e.g., [Schlacher et al., 2004].

In order to verify this result it is worth noting that we already have used the components of the inverse of the deformation gradient which read as $\hat{F}_\alpha^i = \partial_\alpha \hat{\Phi}^i$, where $\hat{\Phi}$ denotes the inverse of the motion according to $X^i = \hat{\Phi}^i(t^0, q^\alpha)$. It is obvious that the relation $\hat{F}_\alpha^i F_j^\alpha = \delta_j^i$ is met. Instead of investigating the pull-back of ∂_α by Φ we consider the equivalent relation of the push-forward of ∂_α by $\hat{\Phi}$ which takes the form of $\hat{\Phi}_*(\partial_\alpha) = (\hat{F}_\alpha^i \circ \Phi) \partial_i$. From this relation the former result may be directly derived. In an analogous manner we obtain the second Piola-Kirchhoff stress form

$$S = \Phi^* (\sigma^{\alpha\beta} \partial_\alpha \rfloor \text{vol} \otimes \partial_\beta) = S^{ij} \partial_i \rfloor \text{VOL} \otimes \partial_j, \quad S^{ij} \in C^\infty(\mathcal{R}),$$

with the components

$$S^{ij} = J \left(\hat{F}_\alpha^i \hat{F}_\beta^j \sigma^{\alpha\beta} \right) \circ \Phi = J \left(-\mathcal{P} \hat{F}_\alpha^i \hat{F}_\beta^j g^{\alpha\beta} + \hat{F}_\alpha^i \hat{F}_\beta^j \bar{\sigma}^{\alpha\beta} \right) \circ \Phi, \quad (4.15)$$

where the components of the Piola tensors are related by $S^{ij} F_j^\beta = P^{i\beta}$. It is worth mentioning that if the Cauchy stress form is symmetric, i.e, $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$ then S is also symmetric and, thus, in this case the components meet $S^{ij} = S^{ji}$. For computational reasons we additionally introduce the so-called first viscous Piola-Kirchhoff stress form resulting from the pull-back of the form part of $\bar{\sigma}$ by the motion Φ according to

$$\bar{P} = \Phi^* (\bar{\sigma}^{\alpha\beta} \partial_\alpha \rfloor \text{vol}) \otimes \partial_\beta = \bar{P}^{i\beta} \partial_i \rfloor \text{VOL} \otimes \partial_\beta, \quad \bar{P}^{i\beta} = J \left(\hat{F}_\alpha^i \bar{\sigma}^{\alpha\beta} \right) \circ \Phi,$$

where it is obvious that we can write $P^{i\beta} = J(-\mathcal{P} \hat{F}_\alpha^i g^{\alpha\beta}) \circ \Phi + \bar{P}^{i\beta}$. Analogously, the second viscous Piola-Kirchhoff stress form takes the form of

$$\bar{S} = \Phi^* (\bar{\sigma}^{\alpha\beta} \partial_\alpha \rfloor \text{vol} \otimes \partial_\beta) = \bar{S}^{ij} \partial_i \rfloor \text{VOL} \otimes \partial_j, \quad \bar{S}^{ij} = J \left(\hat{F}_\alpha^i \hat{F}_\beta^j \bar{\sigma}^{\alpha\beta} \right) \circ \Phi.$$

Viscous Stresses and the Stored Energy in Fluid Dynamics

Before we will analyse the constitutive relations concerning the viscous stress form in detail we introduce the Cauchy Green tensor which is obtained by the pull-back of the metric tensor (4.7) by the motion $\Phi \in \Gamma(\pi_L)$ resulting in⁴

$$C = \Phi^*(g) = C_{ij} dX^i \otimes dX^j, \quad C_{ij} = (g_{\alpha\beta} \circ \Phi) F_i^\alpha F_j^\beta \in C^\infty(\mathcal{R}). \quad (4.16)$$

From the Cauchy Green tensor the so-called Lagrangian or material rate of deformation tensor can be derived according to

$$D = \frac{1}{2} \partial_0(C) = D_{ij} dX^i \otimes dX^j, \quad D_{ij} \in C^\infty(\mathcal{R}),$$

with the components

$$D_{ij} = D_{ji} = \frac{1}{2} (g_{\alpha\beta} \circ \Phi) \left(F_j^\beta (\partial_i V_0^\alpha + \gamma_{\delta\gamma}^\alpha V_0^\gamma F_i^\delta) + F_i^\alpha (\partial_j V_0^\beta + \gamma_{\delta\gamma}^\beta V_0^\gamma F_j^\delta) \right) \circ \Phi$$

⁴More precisely, the pull-back of the motion generates additional terms involving dt^0 . Due to similar reasons already stated in Remark 4.4 these terms are neglected throughout this section.

which play an important role for the characterisation of the viscous stresses in fluid dynamics. The exact computation can be found in Appendix A.2. Of course, we deduce that

$$D = \frac{1}{2} \partial_0 (\Phi^* (g)) = \Phi^* \left(\frac{1}{2} v_\Phi (g) \right) = \Phi^* (d)$$

with $v_\Phi = \Phi_* (\partial_0) = \partial_0 + v^\alpha \partial_\alpha$ is met, where the Eulerian or spatial rate of deformation tensor takes the form of

$$d = \frac{1}{2} v_\Phi (g) = d_{\alpha\beta} dq^\alpha \otimes dq^\beta, \quad d_{\alpha\beta} \in C^\infty(\mathcal{C}),$$

with the components

$$d_{\alpha\beta} = d_{\beta\alpha} = \frac{1}{2} (g_{\gamma\beta} (\partial_\alpha v^\gamma + \gamma_{\alpha\delta}^\gamma v^\delta) + g_{\alpha\gamma} (\partial_\beta v^\gamma + \gamma_{\beta\delta}^\gamma v^\delta)). \quad (4.17)$$

Again, the exact computation can also be found in Appendix A.2. From the definition of the material and the spatial rate of deformation tensor it is obvious that the components are related by $D_{ij} = (d_{\alpha\beta} \circ \Phi) F_i^\alpha F_j^\beta$.

Now we are able to introduce the constitutive relations for the viscous stress form, where in the sequel we will confine ourselves to Newtonian fluid dynamics which is the basis for the Navier-Stokes equations. In fact, for a Newtonian fluid the components of the viscous stress form depend linearly on the components of the rate of deformation tensor and the fluid flow is isotropic, i.e., there exists no preferred direction. Therefore, we assume the existence of the relation

$$\bar{\sigma} = \mathcal{K} \rfloor d = \mathcal{K}^{\alpha\beta\gamma\delta} d_{\gamma\delta} \partial_\alpha \rfloor \text{vol} \otimes \partial_\beta, \quad (4.18)$$

including the fourth-order tensor

$$\mathcal{K} = \mathcal{K}^{\alpha\beta\gamma\delta} \partial_\alpha \rfloor \text{vol} \otimes \partial_\beta \otimes \partial_\gamma \otimes \partial_\delta$$

with components

$$\mathcal{K}^{\alpha\beta\gamma\delta} = \lambda g^{\alpha\beta} g^{\gamma\delta} + \mu g^{\alpha\gamma} g^{\beta\delta} + \mu g^{\alpha\delta} g^{\beta\gamma}, \quad \lambda, \mu \in \mathbb{R}^+, \quad (4.19)$$

see [Aris, 1989], where it is easily verified that the symmetry properties

$$\mathcal{K}^{\alpha\beta\gamma\delta} = \mathcal{K}^{\beta\alpha\gamma\delta} = \mathcal{K}^{\alpha\beta\delta\gamma} = \mathcal{K}^{\gamma\delta\alpha\beta} \quad (4.20)$$

are met. Since the components are given by $\bar{\sigma}^{\alpha\beta} = \mathcal{K}^{\alpha\beta\gamma\delta} d_{\gamma\delta}$ the symmetry condition $\bar{\sigma}^{\alpha\beta} = \bar{\sigma}^{\beta\alpha}$ is fulfilled and, therefore, we are able to conclude $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$ in consideration of (4.13). Finally, the first viscous Piola-Kirchhoff stress form takes the form of

$$\bar{P} = J \left(\hat{F}_\alpha^i \mathcal{K}^{\alpha\beta\gamma\delta} d_{\gamma\delta} \circ \Phi \right) \partial_i \rfloor \text{VOL} \otimes \partial_\beta = J \left(\hat{F}_\alpha^i \mathcal{K}^{\alpha\beta\gamma\delta} \hat{F}_\gamma^k \hat{F}_\delta^l \circ \Phi \right) D_{kl} \partial_i \rfloor \text{VOL} \otimes \partial_\beta \quad (4.21)$$

and the second viscous Piola-Kirchhoff stress form reads as

$$\bar{S} = J \left(\hat{F}_\alpha^i \hat{F}_\beta^j \mathcal{K}^{\alpha\beta\gamma\delta} \hat{F}_\gamma^k \hat{F}_\delta^l \circ \Phi \right) D_{kl} \partial_i \rfloor \text{VOL} \otimes \partial_j. \quad (4.22)$$

Remark 4.5 *It is worth noting that in subsection 4.3.1 we will drop the requirement of an inertial frame and we will introduce the so-called convected picture, see, e.g., [Aris, 1989, Marsden and Hughes, 1994, Simo et al., 1988], where we use a frame which is attached to the continuum of fluid matter and, thus, may be interpreted such that the coordinate lines are fixed to the deforming medium. From a more intuitive point of view it may make sense to introduce the constitutive relations in such a frame and then transform them back to the inertial frame in order to obtain the corresponding spatial and material quantities. Nevertheless, for the considered case of a Newtonian fluid this approach leads to the same constitutive relations as already introduced in (4.13) and (4.18). However, special care must be taken for the case of electrically conducting fluids which will be extensively treated in section 4.3. In fact, for a conducting fluid which is in motion the additional electrodynamic constitutive relations are only valid in the so-called fluid frame (a frame which is attached to the continuum of fluid matter, see [Burke, 1994]) and, therefore, the use of convective coordinates will be indispensable. For a profound discussion and more detailed information see [Aris, 1989, Burke, 1994, Simo et al., 1988], for instance.*

For fluid dynamics the stresses may be divided into two types. The first type of stress is dedicated to a reversible interchange with the strain energy and the other type causes dissipative effects. Therefore, for the considered case of a Newtonian fluid the part involving the hydrostatic pressure corresponds to the first type while the viscous stress form (4.18) to the other. In order to characterise and specify the first type of stress we assume the existence of a stored energy function E_{st} , see [Marsden and Hughes, 1994], which meets

$$S^{ij} - \bar{S}^{ij} = -J \left(\mathcal{P} \hat{F}_\alpha^i \hat{F}_\beta^j g^{\alpha\beta} \right) \circ \Phi = 2\rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial C_{ij}}, \quad (4.23)$$

where the stored energy E_{st} usually depends on the material coordinates (X^i), the metric coefficients of g as well as G and of the components of the deformation gradient given by $q_i^\alpha = \partial_i \Phi^\alpha = F_i^\alpha$. In fact, the relation (4.23) is rather general and, therefore, we make a further restriction. In the sequel we will confine ourselves to so-called barotropic fluids, where the fluid motion is such that the pressure and the density are directly related, e.g., the pressure is a function of the density only⁵. For this case – in consideration of the Lagrangian description – the stored energy only depends on the Jacobian (4.9) and, thus, on the fluid's *deformation*, see [Marsden et al., 2001]. Finally, we are able to end up with the result

$$\mathcal{P} \circ \Phi = -\rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial J}, \quad (4.24)$$

where the exact computation can be found in Appendix A.3. Therefore, it must be emphasised that – with regard to (4.24) – the pressure $\mathcal{P} \circ \Phi$ depends on the material density $\rho_{\mathcal{R}}$ and on the motion Φ since the stored energy is a function of the Jacobian.

⁵It is worth noting that the barotropy also incorporates the case, where the temperature or the entropy may only depend on the density (or on the pressure, respectively). E.g., the well-known polytropic relation $\mathcal{P} = c(\rho)^n$ with the polytropic index $n \geq 0$ and $c = \text{const.}$ serves as a barotropic relation. For more detailed information see, e.g., [Truckenbrodt, 1989].

4.2.4 The Balance of Linear Momentum

In this section we intend to briefly recapitulate the governing equations concerning balance of linear momentum in the spatial (Eulerian) as well as material (Lagrangian) description based on, e.g., [Marsden and Hughes, 1994, Schlacher et al., 2004, Schöberl, 2007] with respect to a tensorial formulation.

The Spatial Picture

In the spatial picture balance of momentum is equivalent to⁶

$$\frac{d}{d\tau} \left(\int_{\phi_\tau(t^0, \mathcal{K})} \rho \text{vol} \otimes v \right) = \int_{\phi_\tau(t^0, \mathcal{K})} \rho \text{vol} \otimes b + \int_{\phi_\tau(t^0, \partial\mathcal{K})} \sigma$$

evaluated at a fixed time t^0 , where $\mathcal{K} \subset \mathcal{Q}$ denotes a configuration and $\phi_\tau : \mathcal{Q} \rightarrow \mathcal{Q}$ is an isomorphism which maps a configuration at t^0 to a configuration at $t^0 + \tau$. It is worth noting that the infinitesimal generator of ϕ_τ is the vector field $v_\Phi = \partial_0 + v^\alpha \partial_\alpha$ which we have already used for the derivation of the conservation of mass in the spatial picture. Furthermore, the expression

$$\rho \text{vol} \otimes b = \rho \text{vol} \otimes b^\alpha \partial_\alpha$$

represents the volume density of the body forces. With regard to the investigation of the infinitesimal version we introduce the covariant differential associated with Λ_c (see [Giachetta et al., 1997]) of the Cauchy stress form as the map

$$d_{\Lambda_c}(\sigma) : \sigma \rightarrow d_{\Lambda_c}(\sigma) \in \Gamma \left(\bigwedge^{n_q} \mathcal{T}^*(\mathcal{C}) \otimes \mathcal{V}(\mathcal{C}) \right),$$

with respect to the connection (4.8), which in local coordinates reads as

$$\begin{aligned} d_{\Lambda_c}(\sigma) &= d(\sigma^{\alpha\beta} \partial_\alpha \rfloor \text{vol}) \otimes \partial_\beta + \sigma^{\alpha\delta} \gamma_{\alpha\delta}^\beta \text{vol} \otimes \partial_\beta \\ &= \left(\partial_0 \sigma^{\alpha\beta} dt^0 + \partial_\gamma \sigma^{\alpha\beta} dq^\gamma + \sigma^{\alpha\beta} \frac{\partial_\gamma (\sqrt{\det [g_{\alpha\beta}]})}{\sqrt{\det [g_{\alpha\beta}]}} dq^\gamma \right) \wedge \partial_\alpha \rfloor \text{vol} \otimes \partial_\beta \\ &\quad + \sigma^{\alpha\delta} \gamma_{\alpha\delta}^\beta \text{vol} \otimes \partial_\beta, \end{aligned}$$

see [Schlacher et al., 2004, Schöberl, 2007]. Finally, with the help of the covariant differential the infinitesimal version of balance of momentum takes the form of

$$\rho \text{vol} \otimes \left(\partial_0 v^\beta + v^\delta \partial_\delta v^\beta + \gamma_{\gamma\delta}^\beta v^\gamma v^\delta \right) \partial_\beta = \rho \text{vol} \otimes b + d_{\Lambda_c}(\sigma) \wedge dt^0 \quad (4.25)$$

with

$$d_{\Lambda_c}(\sigma) \wedge dt^0 = \left(\partial_\alpha \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \gamma_{\alpha\delta}^\delta + \sigma^{\alpha\delta} \gamma_{\alpha\delta}^\beta \right) \text{vol} \otimes \partial_\beta$$

⁶In order to enhance the readability the inclusion mappings are omitted in the corresponding integral expressions.

where we have

$$\gamma_{\alpha\delta}^\delta = \frac{1}{\sqrt{\det [g_{\alpha\beta}]}} \partial_\alpha \left(\sqrt{\det [g_{\alpha\beta}]} \right).$$

Hence, the spatial form of balance of momentum (in local coordinates) is equivalent to

$$\rho \left(\partial_0 v^\beta + v^\delta \partial_\delta v^\beta + \gamma_{\gamma\delta}^\beta v^\gamma v^\delta \right) = \rho b^\beta + \partial_\alpha \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \gamma_{\alpha\delta}^\delta + \sigma^{\alpha\delta} \gamma_{\alpha\delta}^\beta. \quad (4.26)$$

The Material Picture

In order to obtain the material or Lagrangian counterpart to (4.25) we investigate the pull-back of the form part of $d_{\Lambda_c}(\sigma)$ by the motion Φ resulting in

$$\Phi^* (d_{\Lambda_c}(\sigma)) = d_{\Lambda_c}^\Phi(P)$$

which includes

$$\Phi^* (d(\sigma^{\alpha\beta} \partial_\alpha] \text{vol})) = d(\Phi^*(\sigma^{\alpha\beta} \partial_\alpha] \text{vol})) = d(P^{i\beta} \partial_i] \text{VOL})$$

as well as

$$\Phi^* (\sigma^{\alpha\delta} \gamma_{\alpha\delta}^\beta \text{vol}) = \Phi^* (\sigma^{\alpha\epsilon} \hat{F}_\epsilon^i \gamma_{\alpha\delta}^\beta) J F_i^\delta \text{VOL} = P^{i\alpha} F_i^\delta (\gamma_{\alpha\delta}^\beta \circ \Phi) \text{VOL}.$$

Therefore, we derive the covariant differential of the first Piola-Kichhoff stress form which reads as

$$\begin{aligned} d_{\Lambda_c}^\Phi(P) &= d(P^{i\beta} \partial_i] \text{VOL}) \otimes \partial_\beta + P^{i\alpha} F_i^\delta (\gamma_{\alpha\delta}^\beta \circ \Phi) \text{VOL} \otimes \partial_\beta \\ &= \left(\partial_0 P^{i\beta} dt^0 + \partial_k P^{i\beta} dX^k + P^{i\beta} \frac{\partial_k (\sqrt{\det [G_{ij}]})}{\sqrt{\det [G_{ij}]}} dX^k \right) \wedge \partial_i] \text{VOL} \otimes \partial_\beta \\ &\quad + P^{i\alpha} F_i^\delta (\gamma_{\alpha\delta}^\beta \circ \Phi) \text{VOL} \otimes \partial_\beta \end{aligned}$$

and, furthermore, we are able to conclude

$$d_{\Lambda_c}^\Phi(P) \wedge dt^0 = \left(\partial_i P^{i\beta} + P^{i\beta} \Gamma_{ik}^k + P^{i\alpha} F_i^\delta (\gamma_{\alpha\delta}^\beta \circ \Phi) \right) \text{VOL} \otimes \partial_\beta,$$

where Γ_{jk}^i denote the components of the Christoffel symbols of the second kind with respect to the metric G leading to

$$\Gamma_{ik}^k = \frac{1}{\sqrt{\det [G_{ij}]}} \partial_i \left(\sqrt{\det [G_{ij}]} \right).$$

Consequently, the balance of momentum in the Lagrangian setting reads as

$$\rho_{\mathcal{R}} \text{VOL} \otimes \left(\partial_0 V_0^\beta + (\gamma_{\gamma\delta}^\beta \circ \Phi) V_0^\gamma V_0^\delta \right) \partial_\beta = \rho_{\mathcal{R}} \text{VOL} \otimes B + d_{\Lambda_c}^\Phi(P) \wedge dt^0 \quad (4.27)$$

with $B = (b^\alpha \circ \Phi) \partial_\alpha$ which locally corresponds to

$$\rho_{\mathcal{R}} \left(\partial_0 V_0^\beta + (\gamma_{\gamma\delta}^\beta \circ \Phi) V_0^\gamma V_0^\delta \right) = \rho_{\mathcal{R}} (b^\beta \circ \Phi) + \partial_i P^{i\beta} + P^{i\beta} \Gamma_{ik}^k + P^{i\alpha} F_i^\delta (\gamma_{\alpha\delta}^\beta \circ \Phi). \quad (4.28)$$

4.2.5 Port-Hamiltonian Formulation of the Ideal Fluid

This section is dedicated to the Hamiltonian representation of the (compressible) ideal fluid based on the Lagrangian point of view. The ideal fluid characterises a so-called inviscid fluid flow, where the Cauchy stress form takes the form of

$$\sigma = -\mathcal{P}g^{\alpha\beta}\partial_\alpha\rfloor\text{vol} \otimes \partial_\beta, \quad (4.29)$$

i.e., $\bar{\sigma}^{\alpha\beta} = 0$, see [Aris, 1989, Chorin and Marsden, 1990]. Of course, such a fluid flow has no practical relevance, however, we intend to find an appropriate Hamiltonian formulation of the ideal fluid in a Lagrangian setting since a Newtonian fluid – which is the basis for the Navier-Stokes equations – may be treated in an analogous manner by incorporating the viscous stresses.

First of all, we will introduce the well-known governing equations of an ideal fluid in the usual Eulerian description, where we rewrite these equations by the use of the concept of the motion in order to obtain the equivalent equations in the material or Lagrangian picture based on the results of the former subsection, i.e., we will take the motion of an ideal fluid continuum into account. On the basis of these computations the main objective is to find an infinite dimensional (Port-)Hamiltonian representation of the governing equations which describe the ideal fluid continuum in motion, where we intend to represent these equations in the form

$$\dot{x} = \mathcal{J}(\delta(\mathcal{H}dX))$$

– restricting ourselves to the non-differential operator case – with respect to appropriate choices of the dependent coordinates x , of the Hamiltonian density $\mathcal{H}dX$ and of the interconnection map \mathcal{J} . Furthermore, we are interested in deriving the formal change of the considered Hamiltonian functional which leads to an appropriate boundary term only (i.e., a term defined on the surface of the fluid continuum).

The Ideal Fluid

In the spatial or Eulerian setting the covariant differential associated with Λ_c of the Cauchy stress form simplifies to

$$d_{\Lambda_c}(\sigma) \wedge dt^0 = -(\partial_\alpha \mathcal{P})g^{\alpha\beta}\text{vol} \otimes \partial_\beta$$

for the case of an inviscid flow. Thus, the governing equations for a compressible ideal fluid are given by⁷

$$\rho \left(\partial_0 v^\beta + v^\delta \partial_\delta v^\beta + \gamma_{\gamma\delta}^\beta v^\gamma v^\delta \right) = -(\partial_\alpha \mathcal{P})g^{\alpha\beta}. \quad (4.30)$$

– the well-known Euler equations – together with (4.12). In consideration of the motion we are able to conclude

$$\Phi^*(d_{\Lambda_c}(\sigma)) \wedge dt^0 = d_{\Lambda_c}^\Phi(P) \wedge dt^0 = -J \left(\hat{F}_\alpha^i g^{\alpha\beta} \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi) \text{VOL} \otimes \partial_\beta \quad (4.31)$$

⁷For simplicity the body force density is neglected.

and, consequently, the Lagrangian or material form of (4.30) reads as

$$\left(\rho \partial_0 V_0^\beta + \rho \gamma_{\gamma\delta}^\beta V_0^\gamma V_0^\delta \right) \circ \Phi = - \left(\hat{F}_\alpha^i g^{\alpha\beta} \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi), \quad (4.32)$$

together with (4.11), which characterises as set of PDEs for the motion since

$$V_0^\beta = \partial_0 \Phi^\beta, \quad \partial_0 (V_0^\beta) = \partial_{00} \Phi^\beta, \quad \hat{F}_\alpha^i = \partial_\alpha \hat{\Phi}^i = (\partial_i \Phi^\alpha)^{-1}.$$

For more detailed information the interested reader is referred to [Bennett, 2006, Marsden et al., 2001], for instance.

Example 4.3 *It is worth noting that the ideal fluid in the compressible case incorporates a so-called isentropic (adiabatic reversible) flow, where the relation*

$$\mathcal{P} = A(\rho)^\kappa, \quad \kappa > 1, \quad A \in \mathbb{R}^+,$$

is met, i.e., \mathcal{P} is clearly a function of the density ρ . Hence, an isentropic flow may be seen as a special case of a barotropic flow with the adiabatic exponent $\kappa > 1$. Now, for this case we intend to compute the stored energy function by evaluating the relation (4.24). First of all by plugging in the motion Φ we are able to deduce

$$\mathcal{P} \circ \Phi = (\rho_{\mathcal{R}})^\kappa \frac{A}{(J)^\kappa},$$

where $\mathcal{P} \circ \Phi$ clearly depends on the material density $\rho_{\mathcal{R}}$ and on the Jacobian J , i.e., on the motion Φ . Due to the former results this expression must be equivalent to (4.24) and, therefore, the stored energy function takes the form of

$$E_{st} = \frac{(\rho_{\mathcal{R}})^{\kappa-1} A}{\kappa - 1} \frac{1}{(J)^{\kappa-1}} + c, \quad c \in \mathbb{R}.$$

In terms of spatial quantities we obtain

$$E_{st} = \frac{(\rho)^{\kappa-1} A}{\kappa - 1} + c = \frac{\mathcal{P}}{\rho(\kappa - 1)} + c$$

which corresponds to the well-known result as in [Chorin and Marsden, 1990, Eringen and Maugin, 1990], for instance.

Port-Hamiltonian Formulation of the Ideal Fluid

As mentioned before, the objective is to represent the equations (4.32) in the (Port-) Hamiltonian form

$$\dot{x} = \mathcal{J}(\delta(\mathcal{H} dX)).$$

Before we clarify the choices for the dependent coordinates, the Hamiltonian density and the interconnection map we have to introduce a bit more notation. So far, for the Lagrangian picture it has turned out that the motion $\Phi \in \Gamma(\pi_L)$ is the crucial fact for the

representation of the appropriate (material) quantities. Therefore, it is clear that the material quantities such as the first Piola-Kirchhoff stress form P or the material rate of the deformation tensor D may only be derived if the solution or, equivalently, the motion $\Phi \in \Gamma(\pi_L)$ of the considered application is known. Especially, with regard to the definition of the Jacobian (4.9) we may introduce the expression

$$\check{J} = \det [F_i^\alpha] \sqrt{\frac{\det [g_{\alpha\beta}]}{\det [G_{ij}]}} \in C^\infty(\mathcal{J}^1(\mathcal{E})),$$

where this expression does not incorporate the knowledge of the motion⁸. If the motion Φ – or equivalently the solution of the considered problem – is known then we deduce that the relation

$$\check{J} \circ j^1\Phi = J$$

is met. In an analogous manner it is easily seen that we are able to state

$$\begin{aligned} \check{P}^{i\beta} \circ j^1\Phi &= P^{i\beta}, & \check{P}^{i\beta} &\in C^\infty(\mathcal{J}^1(\mathcal{E})), \\ \check{S}^{ij} \circ j^1\Phi &= S^{ij}, & \check{S}^{ij} &\in C^\infty(\mathcal{J}^1(\mathcal{E})), \\ \check{C}_{ij} \circ j^1\Phi &= C_{ij}, & \check{C}_{ij} &\in C^\infty(\mathcal{J}^1(\mathcal{E})), \\ \check{D}_{ij} \circ j^1\Phi &= D_{ij}, & \check{D}_{ij} &\in C^\infty(\mathcal{J}^1(\mathcal{E})) \end{aligned}$$

which also imply

$$\check{\bar{P}}^{i\beta} \circ j^1\Phi = \bar{P}^{i\beta}, \quad \check{\bar{S}}^{ij} \circ j^1\Phi = \bar{S}^{ij}$$

in consideration of the definitions of the appropriate material quantities. In this context it is worth noting that we may write

$$S^{ij} F_j^\beta = \left(\check{S}^{ij} q_j^\beta \right) \circ j^1\Phi = \check{P}^{i\beta} \circ j^1\Phi = P^{i\beta}.$$

In fact, the stored energy which fulfils (4.24) may be – without the knowledge of the motion – interpreted as a function of the metric coefficients $G_{ij} \in C^\infty(\mathcal{B})$ as well as $g_{\alpha\beta} \in C^\infty(\mathcal{Q})$ and the derivative coordinates (q_i^α) . Consequently, we formally have

$$\check{\mathcal{P}} = -\rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial \check{J}}$$

which is related to the hydrostatic pressure by

$$\check{\mathcal{P}} \circ j^1\Phi = \mathcal{P} \circ \Phi.$$

With regard to a Hamiltonian representation of the ideal fluid we introduce the kinetic energy for a continuum with fluid matter according to

$$\mathfrak{T} = \int_{\Phi(t^0, S)} \frac{1}{2} (v]v]g) \rho \text{ vol},$$

⁸In this context the components F_i^α should be seen as a place holder for $\partial_i \Phi^\alpha(t^0, X^i)$ whenever $\Phi \in \Gamma(\pi_L)$ is known.

where the integral has to be evaluated at a fixed time t^0 for the configuration $\Phi(t^0, \mathcal{S}) \subset \mathcal{Q}$ with respect to the reference state $\mathcal{S} \subset \mathcal{B}$. In consideration of the motion we obtain the equivalent expression

$$\mathfrak{T} = \int_{\Phi(t^0, \mathcal{S})} \frac{1}{2} \rho g_{\alpha\beta} v^\alpha v^\beta \text{vol} = \int_{\mathcal{S}} \frac{1}{2} \rho_{\mathcal{R}} (g_{\alpha\beta} \circ \Phi) V_0^\alpha V_0^\beta \text{VOL}. \quad (4.33)$$

The potential energy of the fluid continuum takes the form of

$$\mathfrak{V} = \int_{\mathcal{S}} \rho_{\mathcal{R}} E_{st} \text{VOL} \quad (4.34)$$

including the stored energy E_{st} .

In order to cope with the Port-Hamiltonian framework we choose the material coordinates (X^i) which serve as the labelling coordinates as independent coordinates on \mathcal{D} such that \mathcal{D} and \mathcal{S} coincide. As dependent coordinates we choose $x = (q^\alpha, p_\alpha)$ including the fluid particle positions (q^α) and the momenta with respect to time (p_α) which correspond to

$$p_\alpha = \rho_{\mathcal{R}} g_{\alpha\beta} \dot{q}^\beta \sqrt{\det [G_{ij}]}. \quad (4.35)$$

Therefore, the state bundle $\pi : \mathcal{X} \rightarrow \mathcal{D} = \mathcal{S}$ is equipped with the coordinates $(X^i, q^\alpha, p_\alpha)$. It is worth noting that the time coordinate t^0 only plays the role of the evolution parameter for the presented Port-Hamiltonian framework and, therefore, the identification $\dot{q}^\alpha = q_0^\alpha$ is met. Consequently, in consideration of the motion the momenta with respect to time may be interpreted as so-called material momenta P_α according to

$$P_\alpha = p_\alpha \circ j^1\Phi = \left(\rho_{\mathcal{R}} g_{\alpha\beta} q_0^\beta \sqrt{\det [G_{ij}]} \right) \circ j^1\Phi = \rho_{\mathcal{R}} (g_{\alpha\beta} \circ \Phi) V_0^\beta \sqrt{\det [G_{ij}]}. \quad (4.36)$$

Furthermore, if we compare the former introduced quantities we are able to conclude

$$\begin{aligned} \rho_{\mathcal{R}}, G_{ij} &\in C^\infty(\mathcal{D}), \\ g_{\alpha\beta} &\in C^\infty(\mathcal{X}), \\ \check{P}, \check{J}, E_{st} &\in C^\infty(\mathcal{J}^1(\mathcal{X})). \end{aligned}$$

The Hamiltonian functional is equivalent to the sum of the kinetic and the potential energy of the fluid continuum (4.33), (4.34) respectively and reads as⁹

$$\mathfrak{H}(\Phi) = \mathfrak{T}(\Phi) + \mathfrak{V}(\Phi) = \int_{\mathcal{S}} (j^1\Phi)^* (\mathcal{H} \text{d}X) \quad (4.36)$$

with respect to the first-order Hamiltonian density

$$\mathcal{H} \text{d}X = \left(\frac{1}{2\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} g^{\alpha\beta} p_\alpha p_\beta + \rho_{\mathcal{R}} E_{st} \sqrt{\det [G_{ij}]} \right) \text{d}X, \quad (4.37)$$

⁹Since the identification $\dot{q}^\alpha = q_0^\alpha$ is met and in consideration of (4.35) the motion $\Phi \in \Gamma(\pi_L)$ (and its prolongation with respect to t^0) may be identified with a section of the state bundle $\pi : \mathcal{X} \rightarrow \mathcal{D}$ for a fixed point of time t^0 . Therefore, in the sequel we will consider the motion instead of sections of $\pi : \mathcal{X} \rightarrow \mathcal{D}$ in the relevant expressions in order to enhance the readability.

where we have used the relation

$$\dot{q}^\beta = \frac{1}{\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} g^{\alpha\beta} p_\alpha$$

resulting from (4.35). Due to the former explanations we have $\mathcal{H} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$. It is worth noting that the Hamiltonian density corresponds to the sum of the kinetic and the potential energy density of the fluid continuum.

Proposition 4.2 *Consider the Hamiltonian functional (4.36) with the first-order Hamiltonian density (4.37). The Port-Hamiltonian representation of the governing equations of the ideal fluid in a Lagrangian description (4.32) is given by*

$$\begin{aligned} \partial_0 \Phi^\alpha &= \delta^\alpha \mathcal{H} \circ j^1 \Phi = \frac{(g^{\alpha\beta} \circ \Phi)}{\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} P_\beta, \\ \partial_0 P_\alpha &= -\delta_\alpha \mathcal{H} \circ j^2 \Phi = -\frac{(\partial_\alpha g^{\beta\gamma}) \circ \Phi}{2\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} P_\beta P_\gamma - \sqrt{\det [G_{ij}]} J \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi) \end{aligned}$$

and the formal change of (4.36) reads as¹⁰

$$v(\mathfrak{H}(\Phi)) = - \int_{\partial \mathcal{S}} J \left(\frac{\mathcal{P} g^{\alpha\beta} \hat{F}_\alpha^i}{\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} \circ \Phi \right) P_\beta \partial_i \text{VOL}. \quad (4.38)$$

The exact computation can be found in the Appendix A.4. First of all, it can be easily verified that the equations of Proposition 4.2 are equivalent to

$$\dot{x} = \begin{bmatrix} \dot{q}^\alpha \\ \dot{p}_\alpha \end{bmatrix} = \begin{bmatrix} 0 & \delta_\beta^\alpha \\ -\delta_\alpha^\beta & 0 \end{bmatrix} \begin{bmatrix} \delta_\beta^\alpha \mathcal{H} \\ \delta^\beta \mathcal{H} \end{bmatrix} = \mathcal{J}(\delta(\mathcal{H} \text{d}X)) \quad (4.39)$$

by suppressing the motion. In order to show that the equations of Proposition 4.2 and (4.32) coincide we substitute the first set into the second set of the equations leading to

$$\rho_{\mathcal{R}} \partial_0 \left((g_{\alpha\beta} \circ \Phi) V_0^\beta \right) = -\frac{1}{2} \rho_{\mathcal{R}} (g_{\delta\beta} (\partial_\alpha g^{\beta\gamma}) g_{\varepsilon\gamma}) \circ \Phi V_0^\delta V_0^\varepsilon - J \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi)$$

with $V_0^\alpha = \partial_0 \Phi^\alpha$. Furthermore, we obtain

$$\begin{aligned} \rho_{\mathcal{R}} (g_{\alpha\beta} \circ \Phi) \partial_0 \left(V_0^\beta \right) + \rho_{\mathcal{R}} V_0^\beta V_0^\gamma (\partial_\gamma g_{\alpha\beta} \circ \Phi) \\ = -\frac{1}{2} \rho_{\mathcal{R}} (g_{\delta\beta} (\partial_\alpha g^{\beta\gamma}) g_{\varepsilon\gamma}) \circ \Phi V_0^\delta V_0^\varepsilon - J \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi) \end{aligned}$$

and in consideration of

$$\begin{aligned} \partial_\alpha (g_{\beta\delta} g^{\beta\gamma}) &= 0 \\ g_{\beta\delta} \partial_\alpha (g^{\beta\gamma}) &= -g^{\beta\gamma} \partial_\alpha (g_{\beta\delta}) \\ g_{\beta\delta} \partial_\alpha (g^{\beta\gamma}) g_{\gamma\varepsilon} &= -\partial_\alpha (g_{\varepsilon\delta}) \end{aligned}$$

¹⁰To enhance the readability the inclusion mapping is omitted.

we are able to state

$$\begin{aligned} \rho_{\mathcal{R}} (g_{\alpha\beta} \circ \Phi) \partial_0 \left(V_0^\beta \right) + \rho_{\mathcal{R}} V_0^\beta V_0^\gamma (\partial_\gamma g_{\alpha\beta} \circ \Phi) \\ = \frac{1}{2} \rho_{\mathcal{R}} (\partial_\alpha g_{\varepsilon\delta}) \circ \Phi V_0^\delta V_0^\varepsilon - J \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi) . \end{aligned}$$

Finally, we obtain

$$(\rho g_{\alpha\beta} \circ \Phi) \partial_0 \left(V_0^\beta \right) + (\rho g_{\alpha\delta} \gamma_{\beta\gamma}^\delta \circ \Phi) V_0^\beta V_0^\gamma = - \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi)$$

from which the desired result follows directly.

In the end, it is worth noting that (4.38) can be rewritten as

$$- \int_{\partial\mathcal{S}} J \left(\mathcal{P} \hat{F}_\alpha^i \circ \Phi \right) V_0^\alpha \partial_i \rfloor \text{VOL} = - \int_{\Phi(t^0, \partial\mathcal{S})} v^\alpha \mathcal{P} \partial_\alpha \rfloor \text{vol} = \int_{\Phi(t^0, \partial\mathcal{S})} \sigma \rfloor v \rfloor g \quad (4.40)$$

with respect to the inviscid case, where the Cauchy stress form is given by (4.29). For an ideal fluid in a Lagrangian setting this term completely reflects the influence of the boundary conditions, cf. [Bennett, 2006].

4.2.6 Port-Hamiltonian Formulation of the Navier-Stokes Equations

This section is dedicated to the derivation of the Hamiltonian formulation of the Navier-Stokes equations in a Lagrangian setting, where the objective is to obtain a formal Port-Hamiltonian representation of these equations. In fact, to keep the forthcoming calculations short and readable we confine ourselves to the case of a trivial metric, where we have $g_{\alpha\beta} = \delta_{\alpha\beta}$ as well as $G_{ij} = \delta_{ij}$; i.e., we intend to find a Port-Hamiltonian representation of the Navier-Stokes equations restricted to Cartesian coordinates.

First of all, we will investigate the viscous stress forms in the Eulerian and Lagrangian picture in order to obtain the governing equations, again, in the Eulerian and Lagrangian formulation. As we will see later on, the main objective of this subsection is to extend the (Port-)Hamiltonian formulation of the ideal fluid (of course, restricted to the case of a trivial metric) by means of the viscous stresses in order to obtain a formal iPCHD representation of the Navier-Stokes equations in a Lagrangian setting, i.e., we take a Newtonian fluid continuum in motion into account. It is worth noting that this point of view may be advantageously with respect to the modelling of injection processes, for instance. In fact, we intend to represent the governing equations in the form

$$\dot{x} = (\mathcal{J} - \mathfrak{R}) (\delta (\mathcal{H} \text{d}X)) ,$$

where we extend the Hamiltonian formulation of the ideal fluid continuum by an appropriate non-negative self-adjoint differential operator \mathfrak{R} according to (3.33), (3.34) respectively. Roughly speaking, in order to obtain a Port-Hamiltonian formulation of the Navier-Stokes equations restricted to the Lagrangian point of view we have to combine both proposed iPCHD system representations of Definition 3.4 and 3.8.

The Navier-Stokes Equations

In general, the Navier-Stokes equations characterise the flow of Newtonian fluids, where we consider the Cauchy stress form

$$\sigma = -\mathcal{P}g^{\alpha\beta}\partial_\alpha] \text{vol} \otimes \partial_\beta + \bar{\sigma}, \quad \bar{\sigma} = \mathcal{K}]d,$$

incorporating the viscous stress form $\bar{\sigma}$ which depend linearly on the components of the rate of deformation tensor with respect to (4.17) and (4.19). Since we restrict ourselves to the case of Cartesian coordinates the fourth-order tensor \mathcal{K} reads as

$$\mathcal{K} = (\lambda \delta^{\alpha\beta} \delta^{\gamma\delta} + \mu \delta^{\alpha\gamma} \delta^{\beta\delta} + \mu \delta^{\alpha\delta} \delta^{\beta\gamma}) \partial_\alpha] \text{vol} \otimes \partial_\beta \otimes \partial_\gamma \otimes \partial_\delta \quad (4.41)$$

and the rate of deformation tensor d simplifies to

$$d = \frac{1}{2} (\delta_{\varepsilon\delta} \partial_\gamma v^\varepsilon + \delta_{\gamma\varepsilon} \partial_\delta v^\varepsilon) dq^\gamma \otimes dq^\delta.$$

Therefore, (4.18) corresponds to

$$\bar{\sigma} = \mathcal{K}]d = (\lambda \delta^{\alpha\beta} \partial_\gamma v^\gamma + \mu (\delta^{\alpha\gamma} \partial_\gamma v^\beta + \delta^{\beta\delta} \partial_\delta v^\alpha)) \partial_\alpha] \text{vol} \otimes \partial_\beta$$

and, thus, for the considered case of a Newtonian fluid the Cauchy stress form takes the form of

$$\sigma = (-\mathcal{P}\delta^{\alpha\beta} + \lambda \delta^{\alpha\beta} \partial_\gamma v^\gamma + \mu (\delta^{\alpha\gamma} \partial_\gamma v^\beta + \delta^{\beta\delta} \partial_\delta v^\alpha)) \partial_\alpha] \text{vol} \otimes \partial_\beta.$$

Consequently, the covariant differential associated with Λ_c of the Cauchy stress form reads as

$$\begin{aligned} d_{\Lambda_c}(\sigma) \wedge dt^0 &= (-\partial_\alpha \mathcal{P} \delta^{\alpha\beta} + \lambda \partial_\alpha (\delta^{\alpha\beta} \partial_\gamma v^\gamma) + \mu \partial_\alpha (\delta^{\alpha\gamma} \partial_\gamma v^\beta) + \mu \partial_\alpha (\delta^{\beta\delta} \partial_\delta v^\alpha)) \text{vol} \otimes \partial_\beta \\ &= (-\partial_\alpha \mathcal{P} \delta^{\alpha\beta} + (\lambda + \mu) \partial_\alpha (\delta^{\alpha\beta} \partial_\gamma v^\gamma) + \mu \partial_\alpha (\delta^{\alpha\gamma} \partial_\gamma v^\beta)) \text{vol} \otimes \partial_\beta, \end{aligned}$$

and, thus, in the Eulerian setting the governing equations for a compressible Newtonian fluid (in Cartesian coordinates) are given by¹¹

$$\rho (\partial_0 v^\beta + v^\delta \partial_\delta v^\beta) = -\partial_\alpha \mathcal{P} \delta^{\alpha\beta} + (\lambda + \mu) \partial_\alpha (\delta^{\alpha\beta} \partial_\gamma v^\gamma) + \mu \partial_\alpha (\delta^{\alpha\gamma} \partial_\gamma v^\beta) \quad (4.42)$$

together with (4.12) representing the well-known Navier-Stokes equations (in Cartesian coordinates).

In order to obtain the Lagrangian counterpart of these equations we consider the first viscous Piola-Kichhoff stress form, see Appendix A.5,

$$\bar{P} = J \left(\hat{F}_\alpha^i \hat{F}_\gamma^k \circ \Phi \right) \left(\lambda \delta^{\alpha\beta} \partial_k V_0^\gamma + \mu \delta^{\alpha\gamma} \partial_k V_0^\beta + \mu \delta^{\beta\gamma} \partial_k V_0^\alpha \right) \partial_i] \text{VOL} \otimes \partial_\beta.$$

In consideration of (4.31) we finally obtain

$$\begin{aligned} d_{\Lambda_c}^\Phi(P) \wedge dt^0 &= J \left[-\delta^{\alpha\beta} \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi) + (\lambda + \mu) \delta^{\alpha\beta} \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_\gamma^k \circ \Phi \right) \partial_k V_0^\gamma \right) \right. \\ &\quad \left. + \mu \delta^{\alpha\gamma} \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_\gamma^k \circ \Phi \right) \partial_k V_0^\beta \right) \right] \text{VOL} \otimes \partial_\beta, \end{aligned}$$

¹¹Again, the body force density is neglected for simplicity.

where the exact computation can also be found in Appendix A.5. Consequently, the Lagrangian or material form of (4.42) reads as

$$\begin{aligned} (\rho \partial_0 V_0^\beta) \circ \Phi &= -\delta^{\alpha\beta} \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi) + (\lambda + \mu) \delta^{\alpha\beta} \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_\gamma^k \circ \Phi \right) \partial_k V_0^\gamma \right) \\ &\quad + \mu \delta^{\alpha\gamma} \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_\gamma^k \circ \Phi \right) \partial_k V_0^\beta \right) \end{aligned} \quad (4.43)$$

and together with (4.11) these equations represent the Navier-Stokes equations in the Lagrangian form (and in Cartesian coordinates).

Port-Hamiltonian Formulation of the Navier-Stokes Equations

As mentioned before, the main objective is to represent the governing equations (4.43) in the Port-Hamiltonian form

$$\dot{x} = (\mathcal{J} - \mathfrak{R}) (\delta (\mathcal{H} dX)) ,$$

where we extend the Hamiltonian formulation of the ideal fluid with respect to the viscous stress terms by means of an appropriate differential operator \mathfrak{R} . Accordingly, in the following part we will investigate the dissipation loss caused by the viscous stresses in more detail. Therefore, we consider the resulting viscous force density f^v which is defined by

$$f^v = (d_{\Lambda_c}^\Phi (\bar{P})] (g \circ \Phi)) \wedge dt^0 ,$$

cf. (A.5), and, hence, the dissipation loss concerning the viscous forces – abbreviated by $\mathcal{Q}_S^{v_1}$ – takes the form of

$$\mathcal{Q}_S^{v_1} = \int_S V] f^v = \int_S (d_{\Lambda_c}^\Phi (\bar{P})] V] (g \circ \Phi)) \wedge dt^0 \quad (4.44)$$

with respect to $V = (v \circ \Phi) \partial_\alpha = V_0^\alpha \partial_\alpha$ and $(g \circ \Phi) = (g_{\alpha\beta} \circ \Phi) dq^\alpha \otimes dq^\beta$. Since the components of the viscous stresses depend linearly on the components of the rate of deformation tensor and, thus, on the components of the (material) velocity we mark this relationship by $\bar{P}(\tilde{V})$ with respect to $\tilde{V} = \tilde{V}_0^\alpha \partial_\alpha$ for clarity (by a slight abuse of notation). According to [Schlacher et al., 2004, Schöberl, 2007], (4.44) can be rewritten as

$$\mathcal{Q}_S^{v_1} = - \int_S \hat{\otimes} \left(\bar{S}(\tilde{V})] D(V) \right) + \int_{\partial S} \bar{P}(\tilde{V})] V] (g \circ \Phi) = \mathcal{Q}_S^0 + \mathcal{Q}_\partial^{v_1} , \quad (4.45)$$

where $\hat{\otimes}(\cdot)$ denotes the replacement of \otimes by \wedge in the corresponding expression (\cdot) . Thus, we have

$$\mathcal{Q}_S^0 = - \int_S \hat{\otimes} \left(\bar{S}(\tilde{V})] D(V) \right) = - \int_S \bar{S}^{ij}(\tilde{V}) D_{ij}(V) \text{VOL} \quad (4.46)$$

as well as

$$\mathcal{Q}_\partial^{v_1} = \int_{\partial S} \bar{P}(\tilde{V})] V] (g \circ \Phi) . \quad (4.47)$$

In consideration of the components of the second viscous Piola-Kichhoff stress form (4.22) we obtain

$$\begin{aligned} \hat{\otimes} \left(\bar{S}(\tilde{V}) \rfloor D(V) \right) &= J D_{ij}(V) \left(\hat{F}_\alpha^i \hat{F}_\beta^j \mathcal{K}^{\alpha\beta\gamma\delta} \hat{F}_\gamma^k \hat{F}_\delta^l \circ \Phi \right) D_{kl}(\tilde{V}) \\ &= J D_{kl}(V) \left(\hat{F}_\gamma^k \hat{F}_\delta^l \mathcal{K}^{\gamma\delta\alpha\beta} \hat{F}_\alpha^i \hat{F}_\beta^j \circ \Phi \right) D_{ij}(\tilde{V}) \end{aligned} \quad (4.48)$$

by relabelling the indices and, furthermore, we conclude that

$$\mathcal{Q}_S^0 = - \hat{\otimes} \left(\bar{S}(V) \rfloor D(V) \right) \leq 0 \quad (4.49)$$

is met due to the form of (4.48) involving the components $\mathcal{K}^{\alpha\beta\gamma\delta} \geq 0$ of (4.19) and $J > 0$. With regard to the symmetry properties (4.20) we state the important result

$$\hat{\otimes} \left(\bar{S}(\tilde{V}) \rfloor D(V) \right) = \hat{\otimes} \left(\bar{S}(V) \rfloor D(\tilde{V}) \right)$$

which enables us to write

$$\begin{aligned} \mathcal{Q}_S^0 &= - \int_S \hat{\otimes} \left(\bar{S}(V) \rfloor D(\tilde{V}) \right) \\ &= \int_S \left(d_{\Lambda_c}^\Phi (\bar{P}(V)) \rfloor \tilde{V} \rfloor (g \circ \Phi) \right) \wedge dt^0 - \int_{\partial S} \bar{P}(V) \rfloor \tilde{V} \rfloor (g \circ \Phi) \\ &= \mathcal{Q}_S^{v_2} - \mathcal{Q}_\partial^{v_2}. \end{aligned} \quad (4.50)$$

Finally, by combining (4.45) and (4.50) we are able to end up with the result

$$\mathcal{Q}_S^{v_1} - \mathcal{Q}_S^{v_2} = \mathcal{Q}_\partial^{v_1} - \mathcal{Q}_\partial^{v_2}$$

which is equivalent to

$$\begin{aligned} \int_S \left[d_{\Lambda_c}^\Phi \left(\bar{P}(\tilde{V}) \right) \rfloor V \rfloor (g \circ \Phi) - d_{\Lambda_c}^\Phi (\bar{P}(V)) \rfloor \tilde{V} \rfloor (g \circ \Phi) \right] \wedge dt^0 \\ = \int_{\partial S} \left[\bar{P}(\tilde{V}) \rfloor V \rfloor (g \circ \Phi) - \bar{P}(V) \rfloor \tilde{V} \rfloor (g \circ \Phi) \right]. \end{aligned} \quad (4.51)$$

As already mentioned, this remarkable result enables us to extend the Hamiltonian formulation of Proposition 4.2 via an appropriate differential operator according to

$$f^v = (d_{\Lambda_c}^\Phi (\bar{P})) \rfloor (g \circ \Phi) \wedge dt^0 = - (j^3 \Phi)^* (d_i (\mathfrak{R}_{\alpha\beta}^{ij} d_j (\delta^\beta \mathcal{H}))) \text{VOL} \otimes dq^\alpha \quad (4.52)$$

with

$$\mathfrak{R}_{\alpha\beta}^{ij} = - \check{J} \hat{F}_\tau^i \delta_{\alpha\varepsilon} \mathcal{K}^{\tau\varepsilon\gamma\delta} \hat{F}_\gamma^j \delta_{\beta\delta} = \mathfrak{R}_{\beta\alpha}^{ji} \quad (4.53)$$

which corresponds to the operator introduced in (3.33), though, with respect to the choice of the dependent coordinates, cf. (A.5), too. In this context the Hamiltonian functional is again given by the sum of the kinetic and the potential energy of the fluid continuum (4.33), (4.34) respectively – restricted to the case of trivial metric coefficients – and reads as

$$\mathfrak{H}(\Phi) = \mathfrak{T}(\Phi) + \mathfrak{V}(\Phi) = \int_S (j^1 \Phi)^* (\mathcal{H} dX) \quad (4.54)$$

with respect to the first-order Hamiltonian density¹²

$$\mathcal{H} dX = \left(\frac{1}{2\rho_{\mathcal{R}}} \delta^{\alpha\beta} p_{\alpha} p_{\beta} + \rho_{\mathcal{R}} E_{st} \right) dX, \quad \mathcal{H} \in C^{\infty}(\mathcal{J}^1(\mathcal{X})). \quad (4.55)$$

Obviously, the differential operator of (4.52) fulfils the relation (4.51) and, therefore, it is a self-adjoint operator since in this context (4.51) (formally) corresponds to (3.34).

Proposition 4.3 *Consider the Hamiltonian functional (4.54) with the first-order Hamiltonian density (4.55). The Port-Hamiltonian formulation of the Navier-Stokes equations in a Lagrangian description (4.43) is given by*

$$\begin{aligned} \partial_0 \Phi^{\alpha} &= \delta^{\alpha} \mathcal{H} \circ j^1 \Phi = \frac{1}{\rho_{\mathcal{R}}} \delta^{\alpha\beta} P_{\beta}, \\ \partial_0 P_{\alpha} &= (-\delta_{\alpha} \mathcal{H} - d_i (\mathfrak{R}_{\alpha\beta}^{ij} d_j (\delta^{\beta} \mathcal{H}))) \circ j^3 \Phi \\ &= -J \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi) + J \left(\hat{F}_{\tau}^i \circ \Phi \right) \partial_i \left(\delta_{\alpha\epsilon} \mathcal{K}^{\tau\epsilon\gamma\delta} \left(\hat{F}_{\gamma}^j \circ \Phi \right) \delta_{\beta\delta} \partial_j \left(\frac{1}{\rho_{\mathcal{R}}} \delta^{\mu\beta} P_{\mu} \right) \right) \end{aligned}$$

and the formal change of (4.54) reads as¹³

$$v(\mathfrak{H}(\Phi)) = - \int_S (j^3 \Phi)^* (\delta^{\alpha} \mathcal{H} d_i (\mathfrak{R}_{\alpha\beta}^{ij} d_j (\delta^{\beta} \mathcal{H}))) \text{VOL} - \int_{\partial S} J \left(\frac{1}{\rho_{\mathcal{R}}} \mathcal{P} \delta^{\alpha\beta} \hat{F}_{\alpha}^i \circ \Phi \right) P_{\beta} \partial_i \rfloor \text{VOL}. \quad (4.56)$$

The equivalence of the equations of Proposition 4.3 and of (4.43) follows directly by substituting the first set into the second set of the equations by considering the components of (4.41), also see Appendix A.5. Furthermore, it is obvious that the equations of Proposition 4.3 are equivalent to

$$\dot{x} = \begin{bmatrix} \dot{q}^{\alpha} \\ \dot{p}_{\alpha} \end{bmatrix} = \left(\left[\begin{array}{cc} 0 & \delta_{\beta}^{\alpha} \\ -\delta_{\alpha}^{\beta} & 0 \end{array} \right] - \left[\begin{array}{cc} 0 & 0 \\ 0 & d_i (\mathfrak{R}_{\alpha\beta}^{ij} d_j (\cdot)) \end{array} \right] \right) \begin{bmatrix} \delta_{\beta} \mathcal{H} \\ \delta^{\beta} \mathcal{H} \end{bmatrix} = (\mathcal{J} - \mathfrak{R})(\delta(\mathcal{H} dX)) \quad (4.57)$$

with respect to the introduced self-adjoint operator (4.52) by suppressing the motion.

In the end we intend to analyse the formal change of the Hamiltonian functional. Therefore, the application of the adjoint operator leads to

$$\begin{aligned} v(\mathfrak{H}(\Phi)) &= \int_S (j^3 \Phi)^* (d_i (\delta^{\alpha} \mathcal{H}) \mathfrak{R}_{\alpha\beta}^{ij} d_j (\delta^{\beta} \mathcal{H})) \text{VOL} - \int_{\partial S} (j^2 \Phi)^* (\delta^{\alpha} \mathcal{H} \mathfrak{R}_{\alpha\beta}^{ij} d_j (\delta^{\beta} \mathcal{H}) \partial_i \rfloor \text{VOL}) \\ &\quad - \int_{\partial S} J \left(\frac{1}{\rho_{\mathcal{R}}} \mathcal{P} \delta^{\alpha\beta} \hat{F}_{\alpha}^i \circ \Phi \right) P_{\beta} \partial_i \rfloor \text{VOL}, \end{aligned}$$

where the term evaluated inside the domain

$$\int_S (j^3 \Phi)^* (d_i (\delta^{\alpha} \mathcal{H}) \mathfrak{R}_{\alpha\beta}^{ij} d_j (\delta^{\beta} \mathcal{H})) \text{VOL} \leq 0$$

¹²Note that in the case of Cartesian coordinates we clearly have $\text{VOL} = \sqrt{\det[\delta_{ij}]} dX = dX$.

¹³For readability purposes the inclusion mapping is omitted.

equals (4.46) and, thus, this expression is clearly non-positive due to (4.49), cf. (3.35). Thus, it is obvious that the operator of (4.52) is a non-negative self-adjoint differential operator. Combining the boundary terms we obtain

$$\int_{\partial\mathcal{S}} V_0^\alpha \left[-J \left(\mathcal{P}\hat{F}_\alpha^i \circ \Phi \right) + J \hat{F}_\tau^i \delta_{\alpha\varepsilon} \mathcal{K}^{\tau\varepsilon\gamma\delta} \hat{F}_\gamma^j \delta_{\beta\delta} \partial_j V_0^\beta \right] \partial_i] \text{VOL}$$

in terms of the material velocities and in consideration of (4.53). This expression is just equivalent to

$$\int_{\partial\mathcal{S}} P] V] (g \circ \Phi) = \int_{\Phi(t^0, \partial\mathcal{S})} \sigma] v] g$$

with respect to (A.5) as well as (4.13) and (4.14). Hence, this term completely reflects the influence of the boundary conditions; i.e., they determine the values of the velocity and the stresses on the boundary. For more detailed information and a general discussion concerning the physical interpretation of the boundary conditions for a viscous flow in a Lagrangian setting the interested reader is referred to [Bennett, 2006] and references therein.

4.3 Port-Hamiltonian Formulation of Magnetohydrodynamics

Magnetohydrodynamics (abbreviated MHD) is a well-established and mainly challenging discipline since it combines two main field theories in physics: These two main field theories are fluid mechanics, mostly represented by the Navier-Stokes equations, on the one hand and electrodynamics described by Maxwell's equations on the other hand, both linked together via Ohm's law and Lorentz forces. Roughly speaking, it deals with the interaction of free currents and electromagnetic fields with fluid matter (liquids and gases), usually equipped with a high electrical conductivity. Furthermore, MHD finds practical use in many areas of engineering and pure science; e.g., pumping and levitation of liquid metals in recasting and welding processes (as it is the case for remelting furnaces) or magnetohydrodynamic drive concepts such as the magnetoplasmadynamic thruster. For detailed information the interested reader is referred to [Davidson, 2001, Eringen and Maugin, 1990, Sutton and Sherman, 2006], for instance.

In this section we will investigate the governing equations on the basis of the obtained description of the last section, where it is worth noting that we restrict ourselves to the so-called inductionless MHD case – iMHD for short – meaning that the dynamic of the additionally induced electromagnetic parts can be neglected with respect to the external electromagnetic fields (at a low magnetic Reynold's number). Hence, the purpose of this section which is mainly based on [Siuka et al., 2010] is to extend the framework from the last section in order to also take electrically conducting fluids in the presence of external electromagnetic fields into account, where we are interested in deriving a Port-Hamiltonian representation of the governing iMHD equations based on the Lagrangian point of view. First of all, we have to introduce the main electromagnetic body forces which are important for the considered iMHD case.

4.3.1 Electromagnetic Body Forces

For the specification of the resulting electromagnetic body forces concerning a conducting fluid in the presence of external electromagnetic fields we intend to make heavy use of the classical MHD approximation, see, e.g., [Eringen and Maugin, 1990, Sutton and Sherman, 2006]. Before we will investigate this important result we intend to analyse the relevant electromagnetic objects and the constitutive relations in detail based on a purely geometric point of view.

The Electromagnetic Field Tensors

It is worth mentioning that the governing balance equations of electrodynamics are formulated in the configuration space. Therefore, let us introduce the electromagnetic field tensor $F : \mathcal{C} \rightarrow \bigwedge^2 T^*(\mathcal{C})$ corresponding to

$$F = E \wedge dt^0 + B$$

which meets $dF = 0$, see, e.g., [Burke, 1994, Frankel, 2004, Jadczyk et al., 1998], including the electric field strength $E : \mathcal{C} \rightarrow T^*(\mathcal{C})$ given by¹⁴

$$E = E_{0\alpha} dq^\alpha, \quad E_{0\alpha} \in C^\infty(\mathcal{C}), \quad (4.58)$$

and the magnetic flux density $B : \mathcal{C} \rightarrow \bigwedge^2 T^*(\mathcal{C})$ which reads as

$$B = \frac{1}{2} B_{\alpha\beta} dq^\alpha \wedge dq^\beta, \quad B_{\alpha\beta} = -B_{\beta\alpha} \in C^\infty(\mathcal{C}). \quad (4.59)$$

Remark 4.6 *It is worth noting that the magnetic flux density has been introduced as a pure two form for computational reasons. We may also write*

$$B = \frac{1}{2} B_{\alpha\beta} dq^\alpha \wedge dq^\beta = B^\gamma \partial_\gamma \rfloor \text{vol},$$

where the identification

$$B_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} B^\gamma \sqrt{\det [g_{\alpha\beta}]}, \quad B^\gamma \in C^\infty(\mathcal{C}), \quad (4.60)$$

is met¹⁵. Furthermore, it is worth noting that the electromagnetic field strength tensor meets the relation $dF = 0$ which is equivalent to the homogeneous Maxwell's equations since we obtain

$$dF = \frac{1}{2} (\partial_0 B_{\alpha\beta} + \partial_\alpha E_{0\beta} - \partial_\beta E_{0\alpha}) dq^\alpha \wedge dq^\beta \wedge dt^0 + \frac{1}{2} (\partial_\gamma B_{\alpha\beta}) dq^\alpha \wedge dq^\beta \wedge dq^\gamma = 0,$$

where the first term in brackets describes Faraday's law and the second term in brackets the Absence of Magnetic Charges. Especially, when we take the parameterisation (4.60) into account we are able to conclude

$$\frac{1}{2} (\partial_\gamma B_{\alpha\beta}) dq^\alpha \wedge dq^\beta \wedge dq^\gamma = \text{div}(B) \text{vol} = 0$$

¹⁴Throughout this section we consider the case $m_x = n_q = 3$, i.e., three-dimensional spatial domains.

¹⁵We have $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ with $\epsilon_{\alpha\beta\gamma} = -\epsilon_{\beta\alpha\gamma}$ and $\epsilon_{\alpha\beta\gamma} = 0$ for $\alpha = \beta$ or $\beta = \gamma$ or $\gamma = \alpha$.

with

$$\operatorname{div}(B) = \frac{1}{\sqrt{\det[g_{\alpha\beta}]}} \partial_\gamma \left(B^\gamma \sqrt{\det[g_{\alpha\beta}]} \right)$$

by considering the summation convention for the symbol $\epsilon_{\alpha\beta\gamma}$.

If the Lemma of Poincaré may be applied, see, e.g., [Burke, 1994, Frankel, 2004, Jadczyk et al., 1998], it is convenient to introduce the electromagnetic potential $A : \mathcal{C} \rightarrow \mathcal{T}^*(\mathcal{C})$ which reads as

$$A = A_0 dt^0 + A_\alpha dq^\alpha,$$

where $A_0 \in C^\infty(\mathcal{C})$ denotes the electrostatic potential and $A_\alpha dq^\alpha$ is the vector potential with components $A_\alpha \in C^\infty(\mathcal{C})$. The electromagnetic potential meets $F = dA$ leading to the parameterisation

$$E_{0\alpha} = \partial_\alpha A_0 - \partial_0 A_\alpha, \quad B_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

Furthermore, we introduce the field tensor $G : \mathcal{C} \rightarrow \bigwedge^2 \mathcal{T}^*(\mathcal{C})$ corresponding to

$$G = D - H \wedge dt^0,$$

see, e.g., [Burke, 1994, Frankel, 2004], which includes the magnetic field strength $H : \mathcal{C} \rightarrow \mathcal{T}^*(\mathcal{C})$ and the electric flux density $D : \mathcal{C} \rightarrow \bigwedge^2 \mathcal{T}^*(\mathcal{C})$ given by

$$H = H_{0\alpha} dq^\alpha, \quad D = \frac{1}{2} D_{\alpha\beta} dq^\alpha \wedge dq^\beta$$

with components $H_{0\alpha} \in C^\infty(\mathcal{C})$ as well as $D_{\alpha\beta} = -D_{\beta\alpha} \in C^\infty(\mathcal{C})$ and which meets

$$dG = \mu \operatorname{vol} - j \wedge dt^0, \quad (4.61)$$

see [Frankel, 2004], for instance, where we assume the existence of a (continuous) charge density $\mu \in C^\infty(\mathcal{C})$ and where we have introduced the current density $j = j^\gamma \partial_\gamma \rfloor \operatorname{vol}$ with components $j^\gamma \in C^\infty(\mathcal{C})$. Hence, (4.61) yields the continuity equation for conservation of charge according to

$$d(dG) = (\partial_0 \mu + \operatorname{div}(j)) dt^0 \wedge \operatorname{vol} = 0 \quad (4.62)$$

with

$$\operatorname{div}(j) = \frac{1}{\sqrt{\det[g_{\alpha\beta}]}} \partial_\gamma \left(j^\gamma \sqrt{\det[g_{\alpha\beta}]} \right).$$

Remark 4.7 It is worth mentioning that the inhomogeneous Maxwell's equations are equivalent to (4.61), where we have

$$dG = \frac{1}{2} (\partial_0 D_{\alpha\beta} - \partial_\alpha H_{0\beta} + \partial_\beta H_{0\alpha}) dq^\alpha \wedge dq^\beta \wedge dt^0 + \frac{1}{2} (\partial_\gamma D_{\alpha\beta}) dq^\alpha \wedge dq^\beta \wedge dq^\gamma,$$

and the comparison to (4.61) leads to the Ampère-Maxwell Law and to Gauss's Law; by taking the former introduced parameterisation applied to the electric flux density D into account, i.e.,

$$D = \frac{1}{2} D_{\alpha\beta} dq^\alpha \wedge dq^\beta = D^\gamma \partial_\gamma \rfloor \operatorname{vol}, \quad D_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} D^\gamma \sqrt{\det[g_{\alpha\beta}]},$$

Gauss's Law takes the form of

$$\frac{1}{2} (\partial_\gamma D_{\alpha\beta}) dq^\alpha \wedge dq^\beta \wedge dq^\gamma = \operatorname{div} (D) \operatorname{vol} = \mu \operatorname{vol}$$

with

$$\operatorname{div} (D) = \frac{1}{\sqrt{\det [g_{\alpha\beta}]}} \partial_\gamma \left(D^\gamma \sqrt{\det [g_{\alpha\beta}]} \right).$$

Let the charge $c(\mathcal{S}) \in \mathbb{R}$ of a continuum filled with fluid matter be defined as

$$c(\mathcal{S}) = \int_{\Phi(t^0, \mathcal{S})} \mu \operatorname{vol}, \quad (4.63)$$

where the integral has to be evaluated at a fixed time t^0 for a configuration $\Phi(t^0, \mathcal{S}) \subset \mathcal{Q}$. Furthermore, from the electromagnetic field tensor F we are able to derive

$$f = -v_\Phi \rfloor F = -v^\alpha E_{0\alpha} dt^0 + (E_{0\beta} - v^\alpha B_{\alpha\beta}) dq^\beta$$

with respect to the vector field $v_\Phi = \partial_0 + v^\alpha \partial_\alpha$ which we have already used before. This relation enables us to introduce the force density

$$f^L = \mu \operatorname{vol} \otimes \gamma_c \rfloor f = \mu (E_{0\beta} - v^\alpha B_{\alpha\beta}) \operatorname{vol} \otimes dq^\beta$$

in consideration of $\gamma_c = dq^\beta \otimes \partial_\beta$. It is worth mentioning that the force density f^L consists of the sum of the electrostatic force density f^{es} given by

$$f^{es} = \mu \operatorname{vol} \otimes E = \mu E_{0\beta} \operatorname{vol} \otimes dq^\beta \quad (4.64)$$

and the resulting force density caused by the convective transport of charge f^{co} which takes the form of

$$f^{co} = \mu \operatorname{vol} \otimes (-v \rfloor B) = -(\mu v^\alpha B_{\alpha\beta}) \operatorname{vol} \otimes dq^\beta. \quad (4.65)$$

In this context it is convenient to introduce the convective current density defined by

$$j = j^\alpha \partial_\alpha \rfloor \operatorname{vol} = \mu v^\alpha \partial_\alpha \rfloor \operatorname{vol} \quad (4.66)$$

with components $j^\alpha = \mu v^\alpha \in C^\infty(\mathcal{C})$, see [Frankel, 2004], for instance.

Remark 4.8 For the case of convective current densities only, we may introduce the charge density in the reference state in an analogous manner as we have done for the mass density. In fact, the pull-back of (4.63) by the motion leads to $\mu_{\mathcal{R}} = J(\mu \circ \Phi)$. Therefore, conservation of charge in the material picture reads as $\partial_0(\mu_{\mathcal{R}}) = 0$ and in the spatial picture we obtain $v_\Phi(\mu) + \mu \operatorname{div}(v) = 0$ which is equivalent to (4.62) evaluated for $j^\gamma = \mu v^\gamma$. It must be emphasised that these laws are the correct conservation laws in the case of convective current densities (4.66) only, see [Schöberl et al., 2010, Siuka et al., 2010].

The Convected Picture

Since in (i)MHD a conducting fluid in the presence of external electromagnetic fields is taken into account it is obvious that for the case of finite conductivity we have to consider an induced conductive current density which causes – in combination with the external electromagnetic fields – an electromagnetic force density. This force density may be interpreted as a body force which counteracts the motion of the fluid (according to Lenz's law) and, therefore, leads to dissipative effects. Before this force density will be introduced we intend to analyse in detail the constitutive relation for the conductive current density which is given by Ohm's law. There, special care must be taken since the constitutive relations for electrodynamics are only valid in the so-called fluid frame which may be interpreted as a frame *attached* to the considered (fluid) continuum, see, e.g., [Burke, 1994], for a more general discussion about this important topic.

In order to overcome this problem we intend to make heavy use in the sequel of the so-called convected picture, see [Aris, 1989, Marsden and Hughes, 1994, Simo et al., 1988], for instance, which allows us to introduce a frame whose coordinate lines are fixed to the deforming medium (the reference frame of the continuum). Before we proceed, we consider a bundle morphism (without time reparameterisation) from the configuration bundle $\pi_C : \mathcal{C} \rightarrow \mathcal{I}$ to $\bar{\pi}_C : \bar{\mathcal{C}} \rightarrow \bar{\mathcal{I}} = \mathcal{I}$ of the form

$$\begin{aligned} \bar{t}^0 &= \delta_0^0 t^0, & t^0 &= \delta_0^0 \bar{t}^0, \\ \bar{q}^{\bar{\alpha}} &= \varphi^{\bar{\alpha}}(t^0, q^\alpha), & q^\alpha &= \hat{\varphi}^\alpha(\bar{t}^0, \bar{q}^{\bar{\alpha}}) \end{aligned} \quad (4.67)$$

with respect to a diffeomorphism φ , where the inverse of φ is denoted by $\hat{\varphi}$. Furthermore, for the derivative coordinates we obtain

$$q_0^{\bar{\alpha}} = (\partial_0 \varphi^{\bar{\alpha}} + \partial_\alpha \varphi^{\bar{\alpha}} q_0^\alpha) \delta_0^0, \quad (4.68)$$

by considering the prolongation $j^1 \varphi$ of φ . Applying this bundle morphism the reference frame takes the form of

$$\Lambda = d\bar{t}^0 \otimes (\partial_0 + \delta_0^0 (\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}) \partial_{\bar{\alpha}}), \quad (4.69)$$

where in this context the components $\partial_0 \varphi^{\bar{\alpha}}$ represent the components of the velocity of an observer, see, e.g., [Giachetta et al., 1997, Schöberl, 2007] and references therein.

Remark 4.9 *It is worth noting that by applying this bundle morphism the velocity which is defined as the vector field $v : \mathcal{J}^1(\bar{\mathcal{C}}) \rightarrow (\bar{\pi}_{\bar{\mathcal{C}},0}^1)^*(\mathcal{V}(\bar{\mathcal{C}}))$ takes the form of*

$$v = (q_0^{\bar{\alpha}} - \delta_0^0 (\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi})) \partial_{\bar{\alpha}}$$

including the resulting transition functions of (4.69), see [Schöberl, 2007].

In order to obtain a reference frame that has coordinate lines fixed to the deforming medium leading to so-called convected coordinates, we consider the special bundle morphism

$$\bar{t}^0 = \delta_0^0 t^0, \quad \bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(t^0, q^\alpha) = \delta_i^{\bar{\alpha}} \hat{\Phi}^i(t^0, q^\alpha), \quad (4.70)$$

involving the inverse of the motion. In this context we are able to conclude¹⁶

$$\bar{q}^{\bar{\alpha}} \circ \Phi = \delta_i^{\bar{\alpha}} X^i,$$

where it is clear that

$$\partial_0 (\bar{q}^{\bar{\alpha}} \circ \Phi) = d_0 (\bar{q}^{\bar{\alpha}}) \circ j^1 \Phi = \bar{q}_0^{\bar{\alpha}} \circ j^1 \Phi = 0$$

is met. When this relation is applied to (4.68) we end up with

$$\bar{q}_0^{\bar{\alpha}} \circ j^1 \Phi = (\partial_0 \varphi^{\bar{\alpha}} \circ \Phi) + (\partial_\alpha \varphi^{\bar{\alpha}} \circ \Phi) V_0^\alpha = 0$$

which results in

$$\partial_0 \varphi^{\bar{\alpha}} = -\partial_\alpha \varphi^{\bar{\alpha}} \left(V_0^\alpha \circ \hat{\Phi} \right) = -\delta_i^{\bar{\alpha}} \hat{F}_\alpha^i v^\alpha \quad (4.71)$$

since with respect to (4.70) we have $\partial_\alpha \varphi^{\bar{\alpha}} = \delta_i^{\bar{\alpha}} \partial_\alpha \hat{\Phi}^i = \delta_i^{\bar{\alpha}} \hat{F}_\alpha^i$.

Remark 4.10 *With regard to Remark 4.9 the corresponding velocity field reads as*

$$v = -\delta_0^0 (\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}) \partial_{\bar{\alpha}} = \delta_0^0 \delta_i^{\bar{\alpha}} \left(\hat{F}_\alpha^i \circ \Phi \right) V_0^\alpha \partial_{\bar{\alpha}}$$

which is called the convective velocity, see [Simo et al., 1988].

Furthermore, with the help of

$$d\bar{q}^{\bar{\alpha}} = \partial_0 \varphi^{\bar{\alpha}} dt^0 + \partial_\alpha \varphi^{\bar{\alpha}} dq^\alpha,$$

obtained via (4.70), we are able to compute the expression of the metric g (4.7) in the convected picture due to $g = \varphi^* (\bar{g})$ resulting in¹⁷

$$\bar{g} = \bar{g}_{\bar{\alpha}\bar{\beta}} \left(d\bar{q}^{\bar{\alpha}} - (\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}) dt^0 \right) \otimes \left(d\bar{q}^{\bar{\beta}} - (\partial_0 \varphi^{\bar{\beta}} \circ \hat{\varphi}) dt^0 \right) \quad (4.72)$$

with respect to the components $\bar{g}_{\bar{\alpha}\bar{\beta}} = (g_{\alpha\beta} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^\alpha \partial_{\bar{\beta}} \hat{\varphi}^\beta$ with $\bar{g}_{\bar{\alpha}\bar{\beta}} \in C^\infty (\bar{\mathcal{C}})$. In fact, these components may be formally identified with the components of (4.16) in consideration of (4.70). The appropriate volume form can be derived similarly and reads as

$$\overline{\text{vol}} = \sqrt{\det [\bar{g}_{\bar{\alpha}\bar{\beta}}]} \left(d\bar{q}^1 - (\partial_0 \varphi^1 \circ \hat{\varphi}) dt^0 \right) \wedge \dots \wedge \left(d\bar{q}^{n_q} - (\partial_0 \varphi^{n_q} \circ \hat{\varphi}) dt^0 \right) \quad (4.73)$$

which meets $\text{vol} = \varphi^* (\overline{\text{vol}})$. It is worth mentioning that the metric (4.72) and the volume form (4.73) in the convected picture are explicitly time dependent in contrast to the spatial picture, where an inertial frame is used.

¹⁶Thus, the term *convected coordinates* should be obvious now.

¹⁷In fact, we consider a bundle morphism without time reparameterisation and, therefore, we do not distinguish between dt^0 and $d\bar{t}^0$ since we have $d\bar{t}^0 = \delta_0^0 dt^0$.

Ohm's Law and the Conductive Force Density

As mentioned before we intend to derive the constitutive relation for the conductive current density represented by Ohm's law. Therefore, we focus our interests on the derivation of the electromagnetic fields in the convected picture, then we use the classical relations and transform them back in order to obtain the correct relations in the spatial picture with respect to the inertial frame.

The electromagnetic field tensor in the convected picture is obtained by $F = \varphi^* (\bar{F})$ leading to

$$\begin{aligned} \bar{F} &= (E_{0\alpha} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} (d\bar{q}^{\bar{\alpha}} - (\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}) dt^0) \wedge dt^0 \\ &\quad + \frac{1}{2} (B_{\alpha\beta} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} (d\bar{q}^{\bar{\alpha}} - (\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}) dt^0) \wedge (d\bar{q}^{\bar{\beta}} - (\partial_0 \varphi^{\bar{\beta}} \circ \hat{\varphi}) dt^0) \end{aligned}$$

which results in

$$\begin{aligned} \bar{F} &= (E_{0\alpha} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} d\bar{q}^{\bar{\alpha}} \wedge dt^0 - \frac{1}{2} (B_{\alpha\beta} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} (\partial_0 \varphi^{\bar{\beta}} \circ \hat{\varphi}) d\bar{q}^{\bar{\alpha}} \wedge dt^0 \\ &\quad - \frac{1}{2} (B_{\alpha\beta} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} (\partial_0 \varphi^{\bar{\alpha}} \circ \hat{\varphi}) dt^0 \wedge d\bar{q}^{\bar{\beta}} + \frac{1}{2} (B_{\alpha\beta} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} d\bar{q}^{\bar{\alpha}} \wedge d\bar{q}^{\bar{\beta}}. \end{aligned}$$

By relabelling the indices and by considering the skew-symmetry condition for the components of the magnetic flux density we end up with

$$\bar{F} = \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \left(E_{0\alpha} - B_{\alpha\beta} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} \partial_0 \varphi^{\bar{\beta}} \right) \circ \hat{\varphi} d\bar{q}^{\bar{\alpha}} \wedge dt^0 + \frac{1}{2} (B_{\alpha\beta} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} d\bar{q}^{\bar{\alpha}} \wedge d\bar{q}^{\bar{\beta}}.$$

In consideration of (4.71) we are able to obtain

$$\begin{aligned} \bar{F} &= \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} (E_{0\alpha} - v^{\gamma} B_{\gamma\alpha}) \circ \hat{\varphi} d\bar{q}^{\bar{\alpha}} \wedge dt^0 + \frac{1}{2} (B_{\alpha\beta} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} d\bar{q}^{\bar{\alpha}} \wedge d\bar{q}^{\bar{\beta}} \\ &= \bar{E} \wedge dt^0 + \bar{B}, \end{aligned} \tag{4.74}$$

where in the convected picture the electric field strength reads as

$$\bar{E} = \bar{E}_{0\bar{\alpha}} d\bar{q}^{\bar{\alpha}} = \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} (E_{0\alpha} - v^{\gamma} B_{\gamma\alpha}) \circ \hat{\varphi} d\bar{q}^{\bar{\alpha}} \tag{4.75}$$

and the magnetic flux density takes the form of

$$\bar{B} = \frac{1}{2} \bar{B}_{\bar{\alpha}\bar{\beta}} d\bar{q}^{\bar{\alpha}} \wedge d\bar{q}^{\bar{\beta}} = \frac{1}{2} (B_{\alpha\beta} \circ \hat{\varphi}) \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} d\bar{q}^{\bar{\alpha}} \wedge d\bar{q}^{\bar{\beta}} \tag{4.76}$$

with respect to $\bar{E}_{0\bar{\alpha}} \in C^{\infty}(\bar{\mathcal{C}})$ as well as $\bar{B}_{\bar{\alpha}\bar{\beta}} = -\bar{B}_{\bar{\beta}\bar{\alpha}} \in C^{\infty}(\bar{\mathcal{C}})$. Therefore, we have derived the well-known result that the electric field strength seen by the (moving) fluid continuum – given by (4.75) – consists of the applied electric field strength (4.58) and a contribution of the applied magnetic flux density (4.59) in combination with the velocity of the continuum.

Furthermore, we introduce the electrical conductivity form as a vector valued form in the convected picture corresponding to

$$\bar{\kappa} = \bar{\kappa}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} \overline{\text{vol}} \otimes \partial_{\bar{\beta}}, \quad \bar{\kappa}^{\bar{\alpha}\bar{\beta}} \in C^{\infty}(\bar{\mathcal{C}}). \tag{4.77}$$

Thus, Ohm's law can be formulated in the convected picture according to

$$\bar{j} = -\bar{\kappa}] (\partial_0] \bar{F}) = \bar{\kappa}] \bar{E} = \bar{\kappa}^{\bar{\alpha}\bar{\beta}} \bar{E}_{0\bar{\beta}} \partial_{\bar{\alpha}}] \overline{\text{vol}},$$

where the conductive current density $\bar{j} = \bar{j}^{\bar{\alpha}} \partial_{\bar{\alpha}}] \overline{\text{vol}}$ results from the electric field which the continuum actually receives, i.e., this is the current density measured by an observer moving with the fluid continuum. For the equivalent expression in the spatial picture we have to evaluate $j = \varphi^* (\bar{j})$ resulting in

$$j = \left(\bar{\kappa}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} \right) \circ \varphi (E_{0\beta} - v^{\gamma} B_{\gamma\beta}) \partial_{\alpha}] \text{vol} = \kappa^{\alpha\beta} (E_{0\beta} - v^{\gamma} B_{\gamma\beta}) \partial_{\alpha}] \text{vol} \quad (4.78)$$

since from $\kappa = \varphi^* (\bar{\kappa})$ we obtain

$$\kappa = \left(\bar{\kappa}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \partial_{\bar{\beta}} \hat{\varphi}^{\beta} \right) \circ \varphi \partial_{\alpha}] \text{vol} \otimes \partial_{\beta} = \kappa^{\alpha\beta} \partial_{\alpha}] \text{vol} \otimes \partial_{\beta},$$

where the relation $\hat{\varphi}_* (\partial_{\bar{\alpha}}) = (\partial_{\bar{\alpha}} \hat{\varphi}^{\alpha} \circ \varphi) \partial_{\alpha}$ has been used with respect to the push-forward of $\hat{\varphi}$ denoted by $\hat{\varphi}_*$.

It is worth noting that (4.78) represents the simplest case of Ohm's law in MHD, where thermoelectric effects as well as the Hall current (reasonable approximation for conducting liquids) are neglected. For more detailed information see [Eringen and Maugin, 1990, Sutton and Sherman, 2006], for instance.

Remark 4.11 For the isotropic case the conductivity form in the convected picture may take the form of

$$\bar{\kappa} = \eta \bar{g}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}}] \overline{\text{vol}} \otimes \partial_{\bar{\beta}}, \quad \eta \in \mathbb{R}^+,$$

and, consequently, in the spatial picture we obtain

$$\bar{\kappa} = \eta g^{\alpha\beta} \partial_{\alpha}] \text{vol} \otimes \partial_{\beta}$$

in consideration of the components of (4.72).

Finally, the electromagnetic force density caused by the conductive current density takes the form of

$$\begin{aligned} f^D &= -j] B = -(j^{\alpha} B_{\alpha\beta}) \text{vol} \otimes dq^{\beta} \\ &= -\kappa^{\alpha\delta} (E_{0\delta} - v^{\gamma} B_{\gamma\delta}) B_{\alpha\beta} \text{vol} \otimes dq^{\beta} \end{aligned} \quad (4.79)$$

since the conductive current density is represented by a vector valued form which is isomorphic to $\text{vol} \otimes j^{\alpha} \partial_{\alpha}$. Therefore, with respect to the motion we are able to derive

$$F^D = -J (\kappa^{\alpha\delta} E_{0\delta} B_{\alpha\beta}) \circ \Phi \text{VOL} \otimes dq^{\beta} - J V_0^{\gamma} (\kappa^{\alpha\delta} B_{\beta\alpha} B_{\gamma\delta}) \circ \Phi \text{VOL} \otimes dq^{\beta} \quad (4.80)$$

which represents the corresponding force density in the material picture.

Remark 4.12 As stated earlier, the former introduced Cauchy stress form (4.13) – including the hydrostatic pressure and the viscous stress form (4.18) – is still valid if the corresponding constitutive relations would be introduced in the convected picture and, afterwards, transformed back to obtain the equivalent expression in the spatial picture. The interested reader is referred to [Aris, 1989] (and references therein) for a profound discussion about this topic.

MHD Approximation

Now, we are able to introduce the so-called MHD approximation. First of all, the MHD approximation states that all electro- and magnetostrictive effects are negligible with respect to conductive force densities and viscous stresses as well as the hydrostatic pressure. Thus, it is convenient to consider the Cauchy stress form as we have already introduced in (4.13). Furthermore, by neglecting polarisation and magnetisation effects the relevant electromagnetic body forces for the iMHD case – where the additionally induced electromagnetic parts can be neglected with respect to the external electromagnetic fields (at a low magnetic Reynold's number) – are given by the electrostatic body force (4.64), the body force caused by the convective transport of charge (4.65) and the conductive force density (4.79) which results from the induced conductive current in combination with the external electromagnetic fields (since we consider a conducting fluid with finite electrical conductivity). It is worth mentioning that the classical MHD approximation further states that the electrostatic force density as well as the force caused by the convective transport of charge are negligible in comparison with the conductive force density, i.e.,

$$f^L = f^{es} + f^{co} \ll f^D$$

is met¹⁸.

4.3.2 Port-Hamiltonian Formulation of inductionless Magnetohydrodynamics

This subsection mainly focuses on the Hamiltonian formulation of the governing equations of the iMHD case in a Lagrangian setting, where the objective is dedicated to the derivation of a formal Port-Hamiltonian representation of these equations. Roughly speaking, we intend to derive a Port-Hamiltonian formulation of an electrically conducting fluid continuum in the presence of external electromagnetic fields with respect to the considered iMHD case together with the MHD approximation. First of all, we analyse the governing equations in the Eulerian and Lagrangian picture, where for simplicity we neglect the viscous stresses since they can be incorporated in the presented framework in an analogous manner as before. In fact, we are interested in representing the governing equations in the form

$$\begin{aligned} \dot{x} &= (\mathcal{J} - \mathcal{R})(\delta(\mathcal{H} dX)) + \mathfrak{G}(u), \\ y &= \mathfrak{G}^*(\delta(\mathcal{H} dX)), \end{aligned}$$

where we again extend the Hamiltonian formulation of the ideal fluid continuum with an appropriate choice for the dissipation map \mathcal{R} – represented by a multilinear map according to (3.19) – and we take an appropriate input operator \mathfrak{G} as well as its adjoint operator \mathfrak{G}^* into account which corresponds to the operator introduced in (3.36), (3.37) respectively. As we will see later on, as distributed system input we choose the electrostatic

¹⁸In classical MHD this relation is equivalent to the assumption $\mu \rightarrow 0$, i.e., the charge density is set to zero leading to a vanishing convective current density.

potential. Thus, in order to obtain a Port-Hamiltonian formulation of the governing equations concerning the iMHD case in a Lagrangian setting we have to combine both proposed iPCHD system representations.

The iMHD Case

In consideration of the former introduced MHD approximation and of (4.79) the governing equations of the iMHD case in the Eulerian or spatial picture take the form of

$$\rho \left(\partial_0 v^\beta + v^\delta \partial_\delta v^\beta + \gamma_{\gamma\delta}^\beta v^\gamma v^\delta \right) = - (\partial_\alpha \mathcal{P}) g^{\alpha\beta} + \kappa^{\alpha\delta} (E_{0\delta} - v^\gamma B_{\gamma\delta}) B_{\varepsilon\alpha} g^{\varepsilon\beta} \quad (4.81)$$

together with (4.12), where it is worth mentioning that we neglect the viscous stresses in order to keep the forthcoming calculations short and readable. Nevertheless, they can be treated in an analogous manner as in section 4.2.6. The corresponding equations in the Lagrangian or material picture read as

$$\begin{aligned} \left(\rho \partial_0 V_0^\beta + \rho \gamma_{\gamma\delta}^\beta V_0^\gamma V_0^\delta \right) \circ \Phi = & - \left(\hat{F}_\alpha^i g^{\alpha\beta} \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi) - V_0^\gamma \left(\kappa^{\alpha\delta} B_{\varepsilon\alpha} B_{\gamma\delta} g^{\varepsilon\beta} \right) \circ \Phi \\ & + \left(\kappa^{\alpha\delta} E_{0\delta} B_{\varepsilon\alpha} g^{\varepsilon\beta} \right) \circ \Phi \end{aligned} \quad (4.82)$$

together with (4.11) and characterise a set of PDEs for the motion as illustrated before.

Port-Hamiltonian Representation of iMHD

Particularly with regard to a Port-Hamiltonian representation of (4.82) we intend to extend the Hamiltonian formulation of the ideal fluid of Proposition 4.2. Therefore, in consideration of (4.30) as well as (4.81) it is clear that we have to analyse the conductive force density (4.80) in detail. Since in the iMHD case the induced electromagnetic parts are negligible compared with the external electromagnetic fields we assume in the sequel a (quasi-)stationary external magnetic field, i.e., $\partial_0 B_{\alpha\beta} = 0$ and, therefore, we have $B_{\alpha\beta} \in C^\infty(\mathcal{Q})$ as well as $A_\alpha \in C^\infty(\mathcal{Q})$. In this case it is convenient to consider the electrostatic potential as the system input. Thus, we set $u = A_0$ with $A_0 \in C^\infty(\mathcal{Q})$. By taking these considerations into account the conductive force density in the material picture can be rewritten as

$$F^D = -JV_0^\gamma \left(\kappa^{\alpha\delta} B_{\beta\alpha} B_{\gamma\delta} \right) \circ \Phi \text{VOL} \otimes dq^\beta + J \left(\kappa^{\alpha\delta} B_{\beta\alpha} \partial_\delta A_0 \right) \circ \Phi \text{VOL} \otimes dq^\beta, \quad (4.83)$$

where it is obvious that this force density splits into two parts; the first part consists of a quadratic term with respect to the magnetic flux density and the second part contains the chosen system input which acts on the domain. First, we analyse the quadratic term. With regard to the consideration of this expression in the Port-Hamiltonian context we have to demand on the one hand that the components of the conductivity form satisfy $\kappa^{\alpha\beta} = \kappa^{\beta\alpha} \in C^\infty(\mathcal{Q})$ and on the other hand we assume that the conductivity form serves as a positive definite map in order that the matrix representation $[\kappa^{\alpha\delta} B_{\beta\alpha} B_{\gamma\delta}]$ is positive semidefinite. The exact computation can be found in Appendix A.6.

Remark 4.13 Obviously, for the isotropic case, where we have $\kappa^{\alpha\beta} = \eta g^{\alpha\beta}$ with $\eta \in \mathbb{R}^+$, it is guaranteed that the conductivity form serves as a symmetric and positive definite map with components $\kappa^{\alpha\beta} = \eta g^{\alpha\beta} \in C^\infty(\mathcal{Q})$.

Next we analyse the second part of the conductive force density (4.83) which contains the chosen system input. It is worth noting that the components of this part can be rewritten in the form

$$J(\kappa^{\alpha\delta} B_{\beta\alpha} \partial_\delta A_0) \circ \Phi \text{VOL} \otimes dq^\beta = J(\kappa^{\alpha\delta} B_{\beta\alpha} \hat{F}_\delta^i d_i A_0) \circ j^1\Phi \text{VOL} \otimes dq^\beta$$

since $(\hat{F}_\delta^i d_i A_0) \circ j^1\Phi = \partial_\delta A_0 \circ \Phi$. Thus, it is clear that for the Port-Hamiltonian representation of (4.82) we have to consider an appropriate input differential operator.

Proposition 4.4 Consider the Hamiltonian functional (4.36) with the first-order Hamiltonian density (4.37). The Port-Hamiltonian representation of the governing equations of the iMHD case in a Lagrangian description (4.82) with the electrostatic potential as the system input is given by

$$\begin{aligned} \partial_0 \Phi^\alpha &= \delta^\alpha \mathcal{H} \circ j^1\Phi, \\ \partial_0 P_\alpha &= -\delta_\alpha \mathcal{H} \circ j^2\Phi - (\mathcal{R}_{\alpha\beta} \delta^\beta \mathcal{H}) \circ j^1\Phi + \mathfrak{G}_\alpha^i d_i(A_0) \circ j^1\Phi \end{aligned}$$

with

$$\mathcal{R}_{\alpha\beta} = \check{J} \sqrt{\det[G_{ij}] \kappa^{\gamma\delta} B_{\alpha\gamma} B_{\beta\delta}} = \mathcal{R}_{\beta\alpha} \in C^\infty(\mathcal{J}^1(\mathcal{X})), \quad [\mathcal{R}_{\alpha\beta}] \geq 0,$$

as well as

$$\mathfrak{G}_\alpha^i d_i(A_0) = \check{J} \sqrt{\det[G_{ij}] \kappa^{\beta\gamma} B_{\alpha\beta} \hat{F}_\gamma^i d_i(A_0)}, \quad (4.84)$$

and the formal change of (4.36) reads as¹⁹

$$\begin{aligned} v(\mathfrak{H}(\Phi)) &= - \int_S (j^1\Phi)^* (\delta^\alpha \mathcal{H} \mathcal{R}_{\alpha\beta} \delta^\beta \mathcal{H} dX) + \int_S (j^1\Phi)^* (\delta^\alpha \mathcal{H} \mathfrak{G}_\alpha^i d_i(A_0) dX) \\ &\quad - \int_{\partial S} J \left(\frac{\mathcal{P} g^{\alpha\beta} \hat{F}_\alpha^i}{\rho_{\mathcal{R}} \sqrt{\det[G_{ij}]}} \circ \Phi \right) P_\beta \partial_i \text{VOL}. \end{aligned} \quad (4.85)$$

The equivalence of the equations of Proposition 4.4 and of (4.82) follows directly since we have only extended the formulation of Proposition 4.2 with respect to the conductive force density (4.83). In fact, the introduced input operator (4.84) corresponds to the operator introduced in (3.37), though, with respect to the choice of the dependent coordinates. Therefore, it can be easily verified that the equations of Proposition 4.4 are equivalent to

$$\begin{aligned} \begin{bmatrix} \dot{q}^\alpha \\ \dot{p}_\alpha \end{bmatrix} &= \left(\begin{bmatrix} 0 & \delta_\beta^\alpha \\ -\delta_\alpha^\beta & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{R}_{\alpha\beta} \end{bmatrix} \right) \begin{bmatrix} \delta_\beta \mathcal{H} \\ \delta^\beta \mathcal{H} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathfrak{G}_\alpha^i d_i(u) \end{bmatrix} = (\mathcal{J} - \mathcal{R})(\delta(\mathcal{H} dX)) + \mathfrak{G}(u), \\ y &= \mathfrak{G}^*(\delta(\mathcal{H} dX)), \end{aligned} \quad (4.86)$$

by suppressing the motion, where the physical interpretation of the distributed collocated output follows by the analysis of the formal change of the Hamiltonian functional.

¹⁹In order to enhance the readability the inclusion mapping is omitted.

Remark 4.14 *It is worth mentioning that this system representation can be extended directly with respect to the consideration of the viscous stresses by combining this representation with the one of Proposition 4.3 (restricted to the case of a trivial metric).*

Concerning the formal change of the Hamiltonian functional given by (4.85) we investigate the term involving the input operator first. This term takes the form of

$$\int_S (j^1 \Phi)^* (\delta^\alpha \mathcal{H} \mathfrak{G}_\alpha^i d_i (A_0) dX) = - \int_S (j^2 \Phi)^* (A_0 d_i (\mathfrak{G}_\alpha^i \delta^\alpha \mathcal{H}) dX) + \int_{\partial S} (j^1 \Phi)^* (\delta^\alpha \mathcal{H} \mathfrak{G}_\alpha^i A_0 \partial_i] dX) \quad (4.87)$$

by applying the adjoint operator, cf (3.38). Hence, the term including the input operator splits into two parts; the first part is again a term acting inside the domain containing the adjoint operator – leading to the formal definition of the distributed collocated output – and the second part degenerates to a term on the boundary. The first part may be rewritten as

$$- \int_S (j^2 \Phi)^* (A_0 d_i (\mathfrak{G}_\alpha^i \delta^\alpha \mathcal{H}) dX) = \int_S (A_0 \circ \Phi) \text{DIV} (S) \text{VOL}$$

with respect to

$$S = S^i \partial_i = -J \left(\hat{F}_\gamma^i \kappa^{\beta\gamma} B_{\alpha\beta} \circ \Phi \right) V_0^\alpha \partial_i, \quad \text{DIV} (S) = \frac{1}{\sqrt{\det [G_{ij}]}} \partial_i \left(S^i \sqrt{\det [G_{ij}]} \right),$$

since $V_0^\alpha = \delta^\alpha \mathcal{H} \circ j^1 \Phi$. Furthermore, by introducing

$$s = s^\gamma \partial_\gamma = -\kappa^{\beta\gamma} v^\alpha B_{\alpha\beta} \partial_\gamma \quad (4.88)$$

it is clear that $S^i = J \left(\hat{F}_\gamma^i s^\gamma \circ \Phi \right)$ is met and, therefore, it can be shown that the relation

$$\text{DIV} (S) = J (\text{div} (s)) \circ \Phi$$

– which corresponds to the formal definition of the distributed collocated output – is fulfilled with²⁰

$$\text{div} (s) = \frac{1}{\sqrt{\det [g_{\alpha\beta}]}} \partial_\alpha \left(s^\alpha \sqrt{\det [g_{\alpha\beta}]} \right).$$

It is worth noting that the components of s equal the components of the conductive current density caused by the motion only, cf. (4.78). Consequently, we are able to conclude

$$\int_S (A_0 \circ \Phi) \text{DIV} (S) \text{VOL} = \int_{\Phi(t^0, S)} A_0 \text{div} (s) \text{vol}. \quad (4.89)$$

Finally, we analyse the second part of (4.87) which takes the form of

$$\int_{\partial S} (j^1 \Phi)^* (\delta^\alpha \mathcal{H} \mathfrak{G}_\alpha^i A_0 \partial_i] dX) = - \int_{\partial S} (A_0 \circ \Phi) S^i \partial_i] \text{VOL} = - \int_{\Phi(t^0, \partial S)} A_0 s^\gamma \partial_\gamma] \text{vol}, \quad (4.90)$$

²⁰More precisely, it can be shown that S corresponds to the Piola transform of s . For detailed information the interested reader is referred to [Marsden and Hughes, 1994], for instance.

including the components of (4.88), too.

The remaining terms of (4.85) are given by the boundary term including the hydrostatic pressure which is equivalent to (4.40) and by the term

$$-\int_S (j^1 \Phi)^* (\delta^\alpha \mathcal{H} \mathcal{R}_{\alpha\beta} \delta^\beta \mathcal{H} dX) = -\int_S J(\kappa_{\gamma\varepsilon} s^\varepsilon s^\gamma) \circ \Phi \text{VOL} = -\int_{\Phi(t^0, S)} \kappa_{\gamma\varepsilon} s^\varepsilon s^\gamma \text{vol}, \quad (4.91)$$

where we have $\kappa_{\gamma\varepsilon} \kappa^{\varepsilon\delta} = \delta_\gamma^\delta$ since $[\kappa^{\varepsilon\delta}]$ is invertible due to the assumption of the positive definiteness of the conductivity form. Therefore, this term can be interpreted such that it characterises the dissipation loss concerning the part of the conductive current density caused only by the motion.

Remark 4.15 *It must be emphasised that in the expressions of (4.89), (4.90) and (4.91) no electrostatic dissipation loss is contained directly. In fact, we have not yet considered the continuity equation for the conductive current density. However, by a rearrangement of the terms in (4.85) it is possible to show that the formal change of the Hamiltonian functional consists of a term describing the full Ohmic power loss inside the domain inclusive the electrostatic loss and – besides the boundary term including the hydrostatic pressure – a boundary term which contains the product of the electrostatic potential and the conductive current density restricted to the boundary, provided that the continuity equation for the conductive current density may be taken into account, i.e., we have not considered (4.62) restricted to the iMHD case ($\mu \rightarrow 0$) and the components of (4.78). This fact is completely omitted in the representation of Proposition 4.4.*

Remark 4.16 *It is worth noting that only for the case of convective currents of the form (4.66) – i.e., we neglect the conductivity of the fluid (as well as the MHD approximation) – we have to consider the electrostatic body force (4.64) and the force density caused by the convective transport of charge (4.65). Therefore, we may extend the Hamiltonian functional – which has been corresponded so far to the sum of the kinetic and potential energy of the continuum with fluid matter – by an electromagnetic energy density; this fact leads to a redefinition of the temporal momenta which additionally consist of electromagnetic parts in contrast to the pure mechanical momenta used so far in order to obtain a Hamiltonian representation of the governing equations (in the material picture) for this setting. This case is omitted in this thesis. For detailed information about this topic the interested reader is referred to [Siuka et al., 2010].*

Control of infinite dimensional Port-Hamiltonian Systems

Due to the fact that in many applications the physics behind the governing equations becomes apparent by the Port-Hamiltonian framework it is now obvious to take advantage of the system representation with respect to control purposes. In the finite dimensional case a key benefit of the Port-Hamiltonian system class lies in the possibility of coupling PCHD systems via their (energy) ports which can be exploited not only for the modelling of networks but also for the well-known control by interconnection methodology, see [Ortega et al., 2001, van der Schaft, 2000], for instance. In particular, for the *control via structural invariants* approach which is based on the control by interconnection concept the structural invariants of such coupled PCHD systems – interconnected by their ports – play a crucial role. Of course, this control concept is not limited to the finite dimensional case and concerning infinite dimensional systems an approach based on Stokes-Dirac structures was proposed, see, e.g., [Macchelli and Melchiorri, 2004a,b, Macchelli et al., 2004d, Macchelli and Melchiorri, 2005, Rodriguez et al., 2001] and references therein. In particular, for this approach, where so-called energy variables are used – for the Timoshenko beam the strain is used instead of the displacements, cf. section 4.1 –, the PDEs are considered as a kind of transmission systems between two finite dimensional ones, i.e., the finite dimensional controller system is interconnected to the finite dimensional plant via infinite dimensional transmission systems. However, by taking the presented Port-Hamiltonian framework into account we directly consider the interconnection of a finite dimensional system with an infinite dimensional one, i.e., the finite dimensional controller is interconnected to the infinite dimensional plant. In fact, for infinite dimensional mechanical systems this point of view seems to be advantageously whenever position control is the objective.

Due to the direct analogies of the iPCHD system class to finite dimensional PCHD systems we intend to generalise the *control via structural invariants* method to the proposed Port-Hamiltonian framework for distributed-parameter systems, where we confine ourselves to the non-differential operator case only, cf. Definition 3.4, in order to obtain a systematic and a most general approach. More precisely, the objective of this chapter is to directly adapt the approach from the finite dimensional case based on [Ortega et al., 2001, van der Schaft, 2000]; we mainly focus our interests on a systematic derivation of the ne-

cessary conditions for structural invariants of the considered closed-loop system which consists of the interconnection of the finite dimensional controller system and the infinite dimensional plant represented as an iPCHD system according to Definition 3.4, where we restrict our considerations to so-called *Hamiltonian boundary control systems* (we consider boundary ports only) with one-dimensional spatial domains ($\dim(\mathcal{D}) = 1$), i.e., the finite dimensional controller is interconnected to the infinite dimensional plant by means of the ports of the controller system and of the boundary ports of the plant. In fact, it will be shown that in the infinite dimensional scenario we will obtain analogous conditions for the structural invariants of the closed-loop system as in the lumped-parameter case depending on the considered case of the boundary ports, i.e., we will take the parameterisations of (3.23) as well as (3.24) into account for a systematic derivation of these conditions.

This chapter which is mainly based on [Siuka et al., 2011] is organised as follows; in section 5.1 we recapitulate the well-known control via structural invariants method in the finite dimensional case, where we intend to focus on the key ideas of this approach which will play a crucial role for the generalisation to the infinite dimensional scenario. Section 5.2 deals with the adaption of the method to the infinite dimensional case, where the coupling of a finite dimensional PCHD system (the controller) and an iPCHD system (the plant regarded as a Hamiltonian boundary control system) will be performed and analysed in detail in order to systematically derive conditions for the structural invariants of the coupled system (the closed-loop system) depending on the interconnection of the two (sub)systems. In section 5.3 the usability and the efficiency of the proposed approach is demonstrated, where the control concept is applied to the boundary control of the Timoshenko beam.

Finally, it is worth noting that other (energy based) control approaches dealing with the boundary control of the Timoshenko beam can be found in, e.g., [Kim and Renardy, 1987, Luo et al., 1999] for the case of pure damping injection control laws and/or (finite dimensional) dynamic boundary controllers, in, e.g., [Zhang, 2007] for the case of a standard PD control law or in, e.g., [Macchelli and Melchiorri, 2004a,b] as already mentioned, where the authors apply the *control via structural invariants* methodology to the infinite dimensional Port-Hamiltonian representation of the Timoshenko beam based on the Stokes-Dirac structures; more precisely, they focus on the interconnection of a (finite dimensional) boundary controller with the Timoshenko beam with an end mass, where the main objective is dedicated to the position control of the end mass, i.e., the partial differential equations of the Timoshenko beam characterise a transmission system as described before.

5.1 Control of finite dimensional Port-Hamiltonian Systems based on Structural Invariants

The main purpose of this section is to recapitulate the well-known *control via structural invariants* method based on the *control by interconnection* concept for finite dimensional PCHD systems, see [Ortega et al., 2001, van der Schaft, 2000]. We intend to present the basic ideas of this approach insofar as that we are able to directly adapt this method to the infinite dimensional case in the next section.

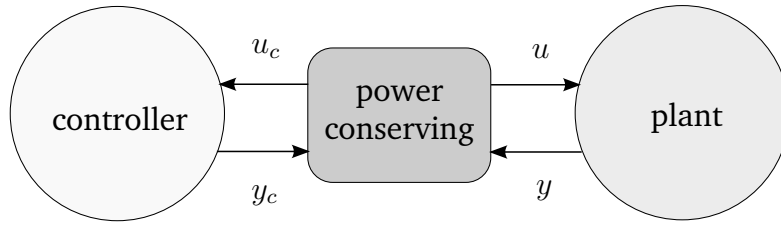


Figure 5.1: Power conserving interconnection of finite dimensional PCHD systems (the controller and the plant).

In the sequel we investigate the system interconnection depicted in Figure. 5.1, where the plant – represented by a PCHD system of the form

$$\begin{aligned} \dot{x} = v &= (J - R) \rfloor dH + u \rfloor G & \dot{x}^\alpha = v^\alpha &= (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta H + G_\xi^\alpha u^\xi \\ y &= G^* \rfloor dH & y_\xi &= G_\xi^\alpha \partial_\alpha H \end{aligned}, \quad (5.1)$$

according to Definition 3.1 with the Hamiltonian $H \in C^\infty(\mathcal{M})$, $\alpha, \beta = 1, \dots, n$ and $\xi = 1, \dots, m$ as well as $\dim(\mathcal{M}) = n$ – is interconnected in a power conserving manner to the (dynamical) controller which is also represented by a PCHD system of the form

$$\begin{aligned} \dot{x}_c = v_c &= (J_c - R_c) \rfloor dH_c + u_c \rfloor G_c & \dot{x}_c^{\alpha_c} = v_c^{\alpha_c} &= (J_c^{\alpha_c \beta_c} - R_c^{\alpha_c \beta_c}) \partial_{\beta_c} H_c + G_{c,\xi}^{\alpha_c} u_c^\xi \\ y_c &= G_c^* \rfloor dH_c & y_{c,\xi} &= G_{c,\xi}^{\alpha_c} \partial_{\alpha_c} H_c \end{aligned}, \quad (5.2)$$

with the controller Hamiltonian $H_c \in C^\infty(\mathcal{M}_c)$ and $\alpha_c, \beta_c = 1, \dots, n_c$. The controller state manifold is denoted by \mathcal{M}_c with $\dim(\mathcal{M}_c) = n_c$ and is equipped with coordinates $(x_c^{\alpha_c})$. Furthermore, we introduce the input vector bundle of the controller $v_c : \mathcal{U}_c \rightarrow \mathcal{M}_c$ which possesses the coordinates $(x_c^{\alpha_c}, u_c^\xi)$, $\xi = 1, \dots, m$, with respect to the holonomic basis $\{e_{c,\xi}\}$ as well as the output vector bundle of the controller $v_c^* : \mathcal{Y}_c = \mathcal{U}_c^* \rightarrow \mathcal{M}_c$ equipped with coordinates $(x_c^{\alpha_c}, y_{c,\xi})$ and the basis $\{e_c^\xi\}$ for the fibres. The interconnection map J_c , the dissipation map R_c and the input map G_c (as well as its dual map G_c^*) of the controller are introduced in a standard manner. The plant (5.1) and the controller (5.2) are interconnected by the ports in a power conserving way according to

$$u \rfloor y + u_c \rfloor y_c = 0. \quad (5.3)$$

In fact, with regard to control purposes we are interested in a power conserving *feedback* interconnection.

Proposition 5.1 *In general, a power conserving feedback interconnection of the plant (5.1) and the controller (5.2) takes the form of*

$$u_c = K \rfloor y, \quad u = -K^* \rfloor y_c \quad (5.4)$$

with respect to the map $K : \mathcal{Y} \rightarrow \mathcal{U}_c$ as well as its adjoint map $K^* : \mathcal{U}_c^* = \mathcal{Y}_c \rightarrow \mathcal{U} = \mathcal{Y}^*$. These maps are represented by the tensor

$$K = K^{\xi\eta} e_{c,\xi} \otimes e_\eta, \quad K^{\xi\eta} \in C^\infty(\mathcal{M} \times \mathcal{M}_c), \quad \xi, \eta = 1, \dots, m,$$

leading in local coordinates to

$$u_c^\xi = K^{\xi\eta} y_\eta, \quad u^\xi = -K^{\eta\xi} y_{c,\eta}. \quad (5.5)$$

This result is easily verified by direct evaluation of (5.3) with respect to (5.4). Therefore, in consideration of (5.5) the closed-loop system serves as an overall PCHD system which takes in local coordinates the form of (in matrix representation)

$$\begin{bmatrix} \dot{x}^\alpha \\ \dot{x}^{\alpha_c} \end{bmatrix} = \left(\begin{bmatrix} J^{\alpha\beta} & -G_\xi^\alpha K^{\eta\xi} G_{c,\eta}^{\beta_c} \\ G_{c,\xi}^{\alpha_c} K^{\xi\eta} G_\eta^\beta & J_c^{\alpha_c\beta_c} \end{bmatrix} - \begin{bmatrix} R^{\alpha\beta} & 0 \\ 0 & R_c^{\alpha_c\beta_c} \end{bmatrix} \right) \begin{bmatrix} \partial_\beta H \\ \partial_{\beta_c} H_c \end{bmatrix} \quad (5.6)$$

with the closed-loop Hamiltonian

$$H_d = H + H_c \in C^\infty(\mathcal{M} \times \mathcal{M}_c), \quad (5.7)$$

whose total time change along the trajectories of the closed-loop system (5.6) results in

$$v_d(H_d) = -(\partial_\alpha H) R^{\alpha\beta} (\partial_\beta H) - (\partial_{\alpha_c} H_c) R_c^{\alpha_c\beta_c} (\partial_{\beta_c} H_c) \leq 0 \quad (5.8)$$

with respect to the corresponding vector field of the closed-loop system $v_d : \mathcal{M} \times \mathcal{M}_c \rightarrow \mathcal{T}(\mathcal{M} \times \mathcal{M}_c)$ locally given by

$$v_d = v^\alpha(x^\alpha, x_c^{\alpha_c}) \partial_\alpha + v_c^{\alpha_c}(x^\alpha, x_c^{\alpha_c}) \partial_{\alpha_c}.$$

Due to the power-conserving interconnection it is clear that the total time change of H_d is only characterised by the dissipative parts of the plant and of the controller and, hence, it is negative semidefinite. Consequently, with regard to control purposes the objective may be formulated as follows; choose the controller Hamiltonian H_c such that the closed-loop Hamiltonian H_d possesses a minimum at the desired equilibrium – so-called *energy shaping* – and, moreover, if the closed-loop Hamiltonian H_d is positive definite it can serve as a Lyapunov function candidate for the investigation of the stability of the desired equilibrium in the sense of Lyapunov.

In order to fulfil these requirements it is of interest to analyse the relation of the plant and the controller coordinates in detail; this relation exists due to the power-conserving interconnection and, therefore, the closed-loop dynamics is, in fact, restricted to a submanifold of $\mathcal{M} \times \mathcal{M}_c$. In particular, one possibility for the characterisation of this relation is the investigation of (non-trivial) structural invariants $C_d \in C^\infty(\mathcal{M} \times \mathcal{M}_c)$ of the closed-loop system. Since the closed-loop system possesses no external inputs it is clear that the structural invariants must serve as conserved quantities for the closed-loop system independently of the plant Hamiltonian H and the controller Hamiltonian H_c , cf. Definition 3.2. Obviously, these requirements lead to the set of PDEs

$$\begin{bmatrix} \partial_\alpha C_d & \partial_{\alpha_c} C_d \end{bmatrix} \left(\begin{bmatrix} J^{\alpha\beta} & -G_\xi^\alpha K^{\eta\xi} G_{c,\eta}^{\beta_c} \\ G_{c,\xi}^{\alpha_c} K^{\xi\eta} G_\eta^\beta & J_c^{\alpha_c\beta_c} \end{bmatrix} - \begin{bmatrix} R^{\alpha\beta} & 0 \\ 0 & R_c^{\alpha_c\beta_c} \end{bmatrix} \right) = 0, \quad (5.9)$$

where the structural invariants are clearly determined by the interconnection and the dissipation map of the closed-loop system. In order to simplify these conditions we confine ourselves to specific structural invariants which restrict the closed-loop dynamics to the submanifold given by

$$\{(x^\alpha, x_c^{\alpha_c}) \in \mathcal{M} \times \mathcal{M}_c \mid x_c^\lambda = -C^\lambda + \kappa^\lambda, C^\lambda \in C^\infty(\mathcal{M}), \kappa^\lambda \in \mathbb{R}, \lambda = 1, \dots, \bar{n} \leq n_c\},$$

see [Ortega et al., 2001, van der Schaft, 2000], where the constants κ^λ depend on the initial conditions of the plant and the controller to the initial point of time $t = t_0 \in \mathbb{R}_0^+$. In particular, we are interested in \bar{n} (non-trivial) structural invariants of the closed-loop system of the specific form

$$C_d^\lambda = x_c^\lambda + C^\lambda, \quad C^\lambda \in C^\infty(\mathcal{M}), \quad (5.10)$$

which must fulfil the set of PDEs (5.9). Thus, provided that \bar{n} such structural invariants exist it is ensured that \bar{n} coordinates of the controller are related to the plant coordinates via (by a slight abuse of notation)

$$x_c^\lambda = -C^\lambda + \kappa^\lambda, \quad \kappa^\lambda = C_d^\lambda|_{t=t_0}, \quad (5.11)$$

since the structural invariants (5.10) serve as conserved quantities for the closed-loop system and, therefore, they are constant along the trajectories of (5.6). Particularly, for structural invariants of the specific form (5.10) it is possible to simplify the conditions of (5.9).

Proposition 5.2 *The functions (5.10) are structural invariants of the closed-loop system (5.6) if and only if the conditions*

$$(\partial_\alpha C^\lambda) J^{\alpha\beta} (\partial_\beta C^\rho) = J_c^{\lambda\rho} \quad (5.12)$$

$$R^{\alpha\beta} (\partial_\beta C^\lambda) = 0 \quad (5.13)$$

$$R_c^{\lambda\rho} = 0 \quad (5.14)$$

$$(\partial_\alpha C^\lambda) J^{\alpha\beta} = -G_{c,\xi}^\lambda K^{\xi\eta} G_\eta^\beta \quad (5.15)$$

with $\lambda, \rho = 1, \dots, \bar{n} \leq n_c$ are fulfilled.

The exact computation can be found in Appendix A.7. For more detailed information as well as modifications of these conditions concerning the control methodology the interested reader is referred to [Ortega et al., 2001, van der Schaft, 2000].

It is worth noting that the order of the controller has not yet been determined and, thus, the order may be considered as an additional degree of freedom for the proposed approach. Especially, for the case $\bar{n} < n_c$ – i.e., not all controller coordinates are related to the plant coordinates by (5.10), (5.11) respectively – it is obvious that certain components of J_c , R_c and G_c are not determined by the conditions of Proposition 5.2; hence, they can also be considered as additional (free) design parameters in order to, e.g., (systematically) introduce additional damping in the closed-loop system by the controller. Therefore, in the case of $\bar{n} < n_c$ the closed-loop dynamics is restricted to the submanifold

$$\{(x^\alpha, x_c^{\alpha c}) \in \mathcal{M} \times \mathcal{M}_c \mid (x_c^{\alpha c}) = (-C^\lambda + \kappa^\lambda, x_c^\mu), \lambda = 1, \dots, \bar{n}, \mu = \bar{n} + 1, \dots, n_c\},$$

where for (x_c^λ) the relation (5.11) is met and (x_c^μ) denote those controller coordinates which are not related to the plant coordinates by (5.10), (5.11) respectively. Hence, by an appropriate choice of the controller Hamiltonian H_c – which has not been determined so far – the closed-loop Hamiltonian (5.7) may serve as an appropriate Lyapunov function candidate which can be used for the stability analysis of the desired equilibrium (of the

closed-loop system) in the sense of Lyapunov. Provided that the closed-loop Hamiltonian H_d is positive definite and that it serves as an appropriate Lyapunov function the stability of the desired equilibrium in the sense of Lyapunov is clearly ensured because of (5.8). In this case, it is worth noting that for many applications the asymptotic stability of the desired equilibrium can be mostly shown by applying LaSalle's invariance principle, see [Khalil, 2002], for instance.

Remark 5.1 *Let us consider the special case $\bar{n} = n_c$, i.e., all controller coordinates are related to the plant coordinates by (5.10), (5.11) respectively. Then, the controlled plant which reads as*

$$\dot{x}^\alpha = (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta H - G_\xi^\alpha K^{\eta\xi} G_{c,\eta}^{\beta c} (\partial_{\beta c} H_c) ,$$

cf. (5.6), can be rewritten as

$$\dot{x}^\alpha = (J^{\alpha\beta} - R^{\alpha\beta}) (\partial_\beta H - \partial_\beta C^{\beta c} \partial_{\beta c} H_c)$$

in consideration of (5.14) as well as (5.15) since

$$-G_\xi^\alpha K^{\eta\xi} G_{c,\eta}^{\beta c} = (\partial_\beta C^{\beta c}) J^{\beta\alpha} = -\partial_\beta C^{\beta c} (J^{\alpha\beta} - R^{\alpha\beta}) .$$

Due to the fact that all controller coordinates are related to the plant coordinates we clearly have $(x_c^{\beta c}) = (-C^{\beta c} + \kappa^{\beta c})$ with $C^{\beta c} \in C^\infty(\mathcal{M})$ and, therefore, by applying the chain rule we deduce $\partial_\beta H_c = -(\partial_{\beta c} H_c)(\partial_\beta C^{\beta c})$. Finally, this result enables us to state

$$\dot{x}^\alpha = (J^{\alpha\beta} - R^{\alpha\beta}) (\partial_\beta H + \partial_\beta H_c) = (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta H_d ,$$

i.e., in the case of $\bar{n} = n_c$ the controlled plant possesses the same interconnection and dissipation maps as in the uncontrolled case but a shaped Hamiltonian – namely the desired Hamiltonian H_d . For more detailed information see [van der Schaft, 2000], for instance.

5.2 Boundary Control of infinite dimensional Port-Hamiltonian Systems based on Structural Invariants

This section is dedicated to the extension of the former illustrated *control via structural invariants* method to the infinite dimensional case. As mentioned before we consider iPCHD systems with a one-dimensional spatial domain – $\dim(\mathcal{D}) = 1$ with $m = 1$ – and, in addition, we do not assume the existence of a distributed port, i.e., we confine ourselves to so-called *Hamiltonian boundary control systems*. In fact, we are interested in a direct extension of the *control via structural invariants* method based on the finite dimensional case and, therefore, with respect to the derivation of a systematic approach in the infinite dimensional scenario we treat iPCHD systems only, where the interconnection and the dissipation map are no differential operators; i.e., we only take iPCHD systems according to Definition 3.4 into account.

In the sequel we investigate the system interconnection depicted in Figure 5.2, where the plant is given by an iPCHD system of the form

$$\dot{x} = v = (\mathcal{J} - \mathcal{R})(\delta(\mathcal{H} dX)) , \quad \dot{x}^\alpha = v^\alpha = (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) \delta_\beta \mathcal{H} , \quad (5.16)$$

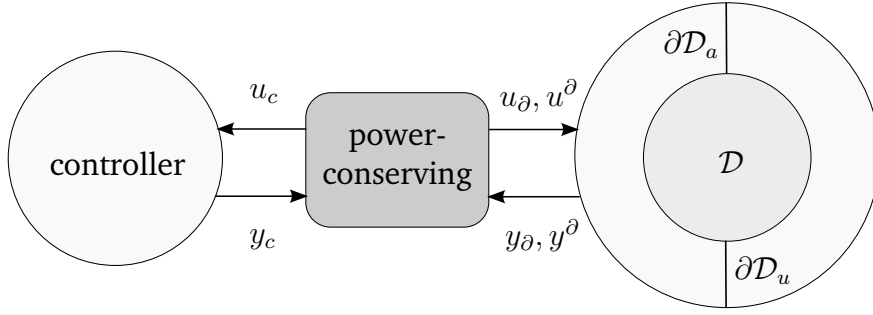


Figure 5.2: Power conserving interconnection of a finite dimensional PCHD system (the controller) and an iPCHD system (the plant) depending on the boundary port parameterisation.

which possesses the Hamiltonian functional

$$\mathfrak{H}(\Phi) = \int_{\mathcal{D}} (j^1\Phi)^* (\mathcal{H} dX), \quad \mathcal{H} \in C^\infty(\mathcal{J}^1(\mathcal{X})), \quad dX = dX^1.$$

Furthermore, we assume that the boundary of the plant is parameterised as¹ $\partial\mathcal{D} = \partial\mathcal{D}_a \cup \partial\mathcal{D}_u$, where $\partial\mathcal{D}_a$ denotes the actuated boundary with the boundary ports

$$\iota_a^*(v) \rfloor \iota_a^* (\delta^\partial (\mathcal{H} dX)) = u_\partial \rfloor y_\partial = y^\partial \rfloor u^\partial \quad (5.17)$$

and $\partial\mathcal{D}_u$ characterises an unactuated boundary by means of

$$\iota_u^*(v) \rfloor \iota_u^* (\delta^\partial (\mathcal{H} dX)) = 0 \quad (5.18)$$

with respect to the inclusion mappings $\iota_a : \partial\mathcal{D}_a \rightarrow \mathcal{D}$ and $\iota_u : \partial\mathcal{D}_u \rightarrow \mathcal{D}$. Analogous to section 5.3 the controller serves as a finite dimensional PCHD system of the form (5.2), where according to Figure 5.2 the plant (5.16) and the controller (5.2) are interconnected at $\partial\mathcal{D}_a$ by means of a power conserving feedback interconnection. As indicated in (5.17) we take in the sequel for the parameterisation of the boundary ports the two cases (3.23) as well as (3.24) into account.

Structural Invariants of the closed-loop System I

First, we consider the parameterisation of the boundary ports of the form

$$\begin{aligned} \iota_a^*(v) &= u_\partial \rfloor \mathcal{G}_\partial & \dot{x}^\alpha \circ \iota_a &= v^\alpha \circ \iota_a &= \mathcal{G}_{\partial,\xi}^\alpha u_\partial^\xi \\ y_\partial &= \mathcal{G}_\partial^* \rfloor \iota_a^* (\delta^\partial (\mathcal{H} dX)) & y_{\partial,\xi} &= \mathcal{G}_{\partial,\xi}^\alpha (\partial_\alpha^1 \mathcal{H} \circ \iota_a) \end{aligned}, \quad (5.19)$$

with $\xi = 1, \dots, m$. Therefore, we introduce the boundary bundles $\nu_\partial : \mathcal{U}_\partial \rightarrow \iota_a^*(\mathcal{J}^2(\mathcal{X}))$ – equipped with local coordinates $(X^1 \circ \iota_a, x^\alpha, x_J^\alpha, u_\partial^\xi)$ with $1 \leq \#J \leq 2$ and the holonomic

¹Since we consider an iPCHD system with a one-dimensional spatial domain the boundary $\partial\mathcal{D}$ consists of two points only; these are represented by $\partial\mathcal{D}_a$ as well as $\partial\mathcal{D}_u$.

basis $\{e_{\partial,\xi}\}$ – as well as $\nu_{\partial}^* : \mathcal{Y}_{\partial} = \mathcal{U}_{\partial}^* \rightarrow \iota_a^*(\mathcal{J}^2(\mathcal{X}))$ which possesses the local coordinates $(X^1 \circ \iota_a, x^\alpha, x_j^\alpha, y_{\partial,\xi})$ and the basis $\{e_{\partial}^\xi\}$ for the fibres².

In this case, the plant (5.16) and the controller (5.2) are interconnected by the ports at $\partial\mathcal{D}_a$ in a power-conserving way according to

$$u_c \rfloor y_c + (\Phi_{\partial_a}^2)^* (u_{\partial} \rfloor y_{\partial}) = 0, \quad \Phi_{\partial_a}^2 = j^2\Phi \circ \iota_a, \quad (5.20)$$

where we are interested in a power conserving feedback interconnection for this setting.

Proposition 5.3 *In general, a power conserving feedback interconnection of the plant (5.16) and the controller (5.2) takes the form of*

$$u_c = (\mathcal{K}_{\partial} \rfloor y_{\partial}) \circ \Phi_{\partial_a}^2, \quad (u_{\partial} \circ \Phi_{\partial_a}^2) = -(\mathcal{K}_{\partial}^* \circ \Phi_{\partial_a}^2) \rfloor y_c,$$

for the parameterisation of the boundary ports (5.19) with respect to the map $\mathcal{K}_{\partial} : \mathcal{Y}_{\partial} \rightarrow \mathcal{U}_c$ as well as its adjoint map $\mathcal{K}_{\partial}^* : \mathcal{U}_c^* = \mathcal{Y}_c \rightarrow \mathcal{U}_{\partial} = \mathcal{Y}_{\partial}^*$. These maps are represented by the tensor

$$\mathcal{K}_{\partial} = \mathcal{K}_{\partial}^{\xi\eta} e_{c,\xi} \otimes e_{\partial,\eta}, \quad \mathcal{K}_{\partial}^{\xi\eta} \in C^\infty(\mathcal{M}_c \times \iota_a^*(\mathcal{J}^2(\mathcal{X}))), \quad \xi, \eta = 1, \dots, m,$$

leading in local coordinates to

$$u_c^\xi = \left(\mathcal{K}_{\partial}^{\xi\eta} y_{\partial,\eta} \right) \circ \Phi_{\partial_a}^2, \quad \left(u_{\partial}^\xi \circ \Phi_{\partial_a}^2 \right) = - \left(\mathcal{K}_{\partial}^{\eta\xi} \circ \Phi_{\partial_a}^2 \right) y_{c,\eta}. \quad (5.21)$$

This result can directly be verified by direct computation. Therefore, the closed-loop system serves as an overall mixed-dimensional PCHD system – which consists of the (power-conserving) interconnection of a finite and an infinite dimensional Port-Hamiltonian (sub) system – of the form

$$\begin{aligned} \dot{x}^\alpha &= v^\alpha &= (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) \delta_\beta \mathcal{H} \\ \dot{x}_c^{\alpha_c} &= v_c^{\alpha_c} &= (J_c^{\alpha_c \beta_c} - R_c^{\alpha_c \beta_c}) \partial_{\beta_c} H_c + G_{c,\xi}^{\alpha_c} \left(\mathcal{K}_{\partial}^{\xi\eta} y_{\partial,\eta} \circ \Phi_{\partial_a}^2 \right) \end{aligned} \quad (5.22)$$

with respect to the boundary ports/terms of the infinite dimensional subsystem

$$\begin{aligned} \iota_a^*(v) \rfloor \iota_a^*(\delta^\partial(\mathcal{H} dX)) &= y_{\partial,\xi} u_{\partial}^\xi = -y_{\partial,\xi} \mathcal{K}_{\partial}^{\eta\xi} y_{c,\eta} = -y_{\partial,\xi} \mathcal{K}_{\partial}^{\eta\xi} G_{c,\eta}^{\alpha_c} (\partial_{\alpha_c} H_c), \\ \iota_u^*(v) \rfloor \iota_u^*(\delta^\partial(\mathcal{H} dX)) &= 0. \end{aligned} \quad (5.23)$$

The Hamiltonian functional of the closed-loop system takes the form of

$$\mathfrak{H}_d = \mathfrak{H}(\Phi) + H_c = \int_{\mathcal{D}} (j^1\Phi)^* (\mathcal{H} dX) + H_c, \quad (5.24)$$

whose formal change results in

$$v_d(\mathfrak{H}_d) = - \int_{\mathcal{D}} (j^2\Phi)^* ((\delta_\alpha \mathcal{H}) \mathcal{R}^{\alpha\beta} (\delta_\beta \mathcal{H}) dX) - (\partial_{\alpha_c} H_c) R_c^{\alpha_c \beta_c} (\partial_{\beta_c} H_c) \leq 0 \quad (5.25)$$

²In fact, we have $X^1 \circ \iota_a = \text{const.}$ and $dX_{\partial} = \partial_1 \rfloor dX = 1$ with respect to the considered one-dimensional spatial domains.

with respect to the corresponding vector field of the closed-loop system $v_d = v^\alpha \partial_\alpha + v^{\alpha c} \partial_{\alpha c}$. Due to the power conserving interconnection it is obvious that the formal change of \mathfrak{H}_d is only characterised by the dissipative parts of the plant (inside the domain) and of the controller. With regard to control purposes the objective is – analogous to the finite dimensional case – the stabilisation of a desired equilibrium of the closed-loop system (5.22), (5.23) respectively. Therefore, the controller Hamiltonian must be chosen such that the Hamiltonian functional of the closed-loop system possesses a minimum at the desired equilibrium and that it is positive definite due to (5.25) in order that \mathfrak{H}_d may serve as a Lyapunov function candidate for the investigation of the stability of the desired equilibrium in the sense of Lyapunov.

Consequently, it is again of interest to analyse the relation of the plant and the controller coordinates in detail which exists due to the power conserving interconnection of the controller and of the plant at $\partial\mathcal{D}_a$. In particular, we intend to investigate the structural invariants of the closed-loop system (5.22), (5.23) respectively. By analogy with the lumped-parameter case we are interested in \bar{n} structural invariants of the specific form

$$\mathfrak{C}^\lambda = x_c^\lambda + \int_{\mathcal{D}} (j^1\Phi)^* (\mathfrak{C}^\lambda dX) , \quad \mathfrak{C}^\lambda \in C^\infty (\mathcal{J}^1(\mathcal{X})) , \quad \lambda = 1, \dots, \bar{n} \leq n_c, \quad (5.26)$$

where – since the closed-loop system possesses neither external distributed inputs nor external boundary inputs – it is clear that the structural invariants must serve as conserved quantities for the closed-loop system independently of the closed-loop Hamiltonian functional \mathfrak{H}_d (i.e., independent of \mathfrak{H} and H_c), cf. Definition 3.5.

Proposition 5.4 *The functionals (5.26) are structural invariants of the closed-loop system (5.22), (5.23) respectively, with respect to the parameterisation of the boundary ports (5.19) if and only if the conditions*

$$\delta_\alpha \mathfrak{C}^\lambda (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0 \quad (5.27)$$

$$J_c^{\lambda\beta c} - R_c^{\lambda\beta c} - \mathcal{G}_{\partial, \xi}^\alpha (\partial_\alpha^1 \mathfrak{C}^\lambda \circ \iota_a) \mathcal{K}_\partial^{\eta\xi} G_{c, \eta}^{\beta c} = 0 \quad (5.28)$$

$$(\dot{x}^\alpha \partial_\alpha^1 \mathfrak{C}^\lambda) \circ \iota_u = 0 \quad (5.29)$$

$$G_{c, \xi}^\lambda \mathcal{K}_\partial^{\xi\eta} = 0 \quad (5.30)$$

with $\lambda = 1, \dots, \bar{n} \leq n_c$ are fulfilled.

In order to prove this proposition let us compute the formal change of structural invariants of the form (5.26) which reads as

$$v_d(\mathfrak{C}^\lambda) = \dot{x}_c^\lambda + \int_{\mathcal{D}} (j^2\Phi)^* (\delta_\alpha \mathfrak{C}^\lambda (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) \delta_\beta \mathcal{H} dX) + (\Phi_\partial^2)^* ((\dot{x}^\alpha \partial_\alpha^1 \mathfrak{C}^\lambda) \circ \iota) ,$$

with $\Phi_\partial^2 = j^2\Phi \circ \iota$, $\iota : \partial\mathcal{D}_a \cup \partial\mathcal{D}_u = \partial\mathcal{D} \rightarrow \mathcal{D}$, where due to the requirement $v_d(\mathfrak{C}^\lambda) = 0$ and due to the fact that \mathfrak{C}^λ must be defined independently of the Hamiltonian functional \mathfrak{H} of the plant and the controller Hamiltonian H_c we are able to conclude that

$$\delta_\alpha \mathfrak{C}^\lambda (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0$$

must be met which corresponds to (5.27). Hence, the remaining expression in consideration of (5.19) as well as (5.22) reads as

$$v_d(\mathfrak{e}^\lambda) = (J_c^{\lambda\beta_c} - R_c^{\lambda\beta_c}) \partial_{\beta_c} H_c + G_{c,\xi}^\lambda \left(\mathcal{K}_\partial^{\xi\eta} y_{\partial,\eta} \circ \Phi_{\partial_a}^2 \right) + (\Phi_{\partial_u}^2)^* \left((\dot{x}^\alpha \partial_\alpha^1 \mathcal{C}^\lambda) \circ \iota_u \right) \\ + (\Phi_{\partial_a}^2)^* \left(\mathcal{G}_{\partial,\xi}^\alpha u_\partial^\xi (\partial_\alpha^1 \mathcal{C}^\lambda \circ \iota_a) \right) = 0$$

– with $\Phi_{\partial_u}^2 = j^2 \Phi \circ \iota_u$ as well as $\Phi_{\partial_a}^2 = j^2 \Phi \circ \iota_a$ – and, consequently, we obtain

$$v_d(\mathfrak{e}^\lambda) = (\Phi_{\partial_a}^2)^* \left(G_{c,\xi}^\lambda \mathcal{K}_\partial^{\xi\eta} \right) (y_{\partial,\eta} \circ \Phi_{\partial_a}^2) + (\Phi_{\partial_u}^2)^* \left((\dot{x}^\alpha \partial_\alpha^1 \mathcal{C}^\lambda) \circ \iota_u \right) \\ + (\Phi_{\partial_a}^2)^* \left(J_c^{\lambda\beta_c} - R_c^{\lambda\beta_c} - \mathcal{G}_{\partial,\xi}^\alpha (\partial_\alpha^1 \mathcal{C}^\lambda \circ \iota_a) \mathcal{K}_\partial^{\eta\xi} G_{c,\eta}^{\beta_c} \right) \partial_{\beta_c} H_c = 0,$$

with respect to (5.23), from which the remaining conditions of Proposition 5.4 directly follow.

Structural Invariants of the closed-loop System II

Now, we take the parameterisation of the boundary ports of the form

$$\begin{aligned} \iota_a^* (\delta^\partial (\mathcal{H} \, dX)) &= \mathcal{G}^\partial \rfloor u^\partial & \partial_\alpha^1 \mathcal{H} \circ \iota_a &= \mathcal{G}_\alpha^{\partial,\xi} u_\xi^\partial \\ y^\partial &= \iota_a^* (v) \rfloor \mathcal{G}^{\partial,*} & y^{\partial,\xi} &= \mathcal{G}_\alpha^{\partial,\xi} (\dot{x}^\alpha \circ \iota_a) \end{aligned} \quad (5.31)$$

into account with $\xi = 1, \dots, m$. Accordingly, for this case we consider the boundary bundles $\nu^\partial : \mathcal{U}^\partial \rightarrow \iota_a^*(\mathcal{J}^2(\mathcal{X}))$ equipped with local coordinates $(X^1 \circ \iota_a, x^\alpha, x_j^\alpha, u_\xi^\partial)$, with $1 \leq \#J \leq 2$, and the holonomic basis $\{e^{\partial,\xi}\}$ as well as $\nu^{\partial,*} : \mathcal{Y}^\partial = \mathcal{U}^{\partial,*} \rightarrow \iota_a^*(\mathcal{J}^2(\mathcal{X}))$ which possesses the local coordinates $(X^1 \circ \iota_a, x^\alpha, x_j^\alpha, y^{\partial,\xi})$ and the basis $\{e_\xi^\partial\}$.

In this case, the plant (5.16) and the controller (5.2) are interconnected by the ports at $\partial\mathcal{D}_a$ in a power conserving manner according to

$$u_c \rfloor y_c + (\Phi_{\partial_a}^2)^* (y^\partial \rfloor u^\partial) = 0, \quad \Phi_{\partial_a}^2 = j^2 \Phi \circ \iota_a.$$

Again, we are interested in a power conserving feedback interconnection for this case.

Proposition 5.5 *In general, a power conserving feedback interconnection of the plant (5.16) and the controller (5.2) takes the form of*

$$u_c = (y^\partial \rfloor \mathcal{K}^\partial) \circ \Phi_{\partial_a}^2, \quad (u^\partial \circ \Phi_{\partial_a}^2) = -(\mathcal{K}^{\partial,*} \circ \Phi_{\partial_a}^2) \rfloor y_c,$$

for the parameterisation of the boundary ports (5.31) with respect to the map $\mathcal{K}^\partial : \mathcal{Y}^\partial \rightarrow \mathcal{U}_c$ as well as its adjoint map $\mathcal{K}^{\partial,*} : \mathcal{U}_c^* = \mathcal{Y}_c \rightarrow \mathcal{U}^\partial = \mathcal{Y}^{\partial,*}$. These maps are represented by the tensor

$$\mathcal{K}^\partial = \mathcal{K}_\eta^{\partial,\xi} e^{\partial,\eta} \otimes e_{c,\xi}, \quad \mathcal{K}_\eta^{\partial,\xi} \in C^\infty(\mathcal{M}_c \times \iota_a^*(\mathcal{J}^2(\mathcal{X}))), \quad \xi, \eta = 1, \dots, m,$$

leading in local coordinates to

$$u_c^\xi = (\mathcal{K}_\eta^{\partial,\xi} y^{\partial,\eta}) \circ \Phi_{\partial_a}^2, \quad (u_\xi^\partial \circ \Phi_{\partial_a}^2) = -(\mathcal{K}_\xi^{\partial,\eta} \circ \Phi_{\partial_a}^2) y_{c,\eta}. \quad (5.32)$$

This result can easily be verified by direct computation. The closed-loop system serves as an overall mixed-dimensional PCHD system of the form

$$\begin{aligned}\dot{x}^\alpha &= v^\alpha &= (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) \delta_\beta \mathcal{H} \\ \dot{x}_c^{\alpha c} &= v_c^{\alpha c} &= (J_c^{\alpha c \beta c} - R_c^{\alpha c \beta c}) \partial_{\beta c} H_c + G_{c,\xi}^{\alpha c} (\mathcal{K}_\eta^{\partial,\xi} y^{\partial,\eta} \circ \Phi_{\partial_a}^2)\end{aligned}\quad (5.33)$$

with respect to the boundary ports/terms of the infinite dimensional subsystem

$$\begin{aligned}\iota_a^*(v) \rfloor \iota_a^* (\delta^\partial (\mathcal{H} dX)) &= u_\xi^\partial y^{\partial,\xi} = -y_{c,\eta} \mathcal{K}_\xi^{\partial,\eta} y^{\partial,\xi} = -(\partial_{\alpha c} H_c) G_{c,\eta}^{\alpha c} \mathcal{K}_\xi^{\partial,\eta} y^{\partial,\xi}, \\ \iota_u^*(v) \rfloor \iota_u^* (\delta^\partial (\mathcal{H} dX)) &= 0.\end{aligned}\quad (5.34)$$

The Hamiltonian functional of the closed-loop system is again of the form (5.24) and its formal change corresponds to (5.25), of course. Therefore, we are again interested in \bar{n} structural invariants of the specific form (5.26) of the closed-loop system (5.33), (5.34) respectively. Since the closed-loop system possesses neither external distributed nor external boundary inputs the structural invariants must serve as conserved quantities for the closed-loop system independently of the Hamiltonian functional \mathfrak{H} of the plant and the controller Hamiltonian H_c , cf. Definition 3.5.

Proposition 5.6 *The functionals (5.26) are structural invariants of the closed-loop system (5.33), (5.34) respectively, with respect to the parameterisation of the boundary ports (5.31) if and only if the conditions*

$$\delta_\alpha \mathcal{C}^\lambda (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0 \quad (5.35)$$

$$G_{c,\xi}^\lambda \mathcal{K}_\eta^{\partial,\xi} \mathcal{G}_\alpha^{\partial,\eta} + (\partial_\alpha^1 \mathcal{C}^\lambda \circ \iota_a) = 0 \quad (5.36)$$

$$(\dot{x}^\alpha \partial_\alpha^1 \mathcal{C}^\lambda) \circ \iota_u = 0 \quad (5.37)$$

$$J_c^{\lambda\beta c} - R_c^{\lambda\beta c} = 0 \quad (5.38)$$

with $\lambda = 1, \dots, \bar{n} \leq n_c$ are fulfilled.

In order to prove this proposition we compute the formal change of structural invariants of the form (5.26) which takes the form of

$$v_d(\mathfrak{C}^\lambda) = \dot{x}_c^\lambda + \int_{\mathcal{D}} (j^2 \Phi)^* (\delta_\alpha \mathcal{C}^\lambda (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) \delta_\beta \mathcal{H} dX) + (\Phi_\partial^2)^* ((\dot{x}^\alpha \partial_\alpha^1 \mathcal{C}^\lambda) \circ \iota),$$

with $\Phi_\partial^2 = j^2 \Phi \circ \iota$, where due to the fact $v_d(\mathfrak{C}^\lambda) = 0$ and that \mathfrak{C}^λ must be independently defined of the Hamiltonian functional of the plant and the controller Hamiltonian we deduce

$$\delta_\alpha \mathcal{C}^\lambda (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) = 0$$

which equals (5.35). Furthermore, the remaining expression is given by

$$\begin{aligned}v_d(\mathfrak{C}^\lambda) &= (J_c^{\lambda\beta c} - R_c^{\lambda\beta c}) \partial_{\beta c} H_c + G_{c,\xi}^\lambda (\mathcal{K}_\eta^{\partial,\xi} y^{\partial,\eta} \circ \Phi_{\partial_a}^2) + (\Phi_{\partial_u}^2)^* ((\dot{x}^\alpha \partial_\alpha^1 \mathcal{C}^\lambda) \circ \iota_u) \\ &\quad + (\Phi_{\partial_a}^2)^* ((\dot{x}^\alpha \partial_\alpha^1 \mathcal{C}^\lambda) \circ \iota_a) = 0\end{aligned}$$

– with respect to $\Phi_{\partial_u}^2 = j^2\Phi \circ \iota_u$ as well as $\Phi_{\partial_a}^2 = j^2\Phi \circ \iota_a$ – which is equivalent to

$$v_d(\mathfrak{C}^\lambda) = (J_c^{\lambda\beta_c} - R_c^{\lambda\beta_c}) \partial_{\beta_c} H_c + (\Phi_{\partial_a}^2)^* ((G_{c,\xi}^\lambda \mathcal{K}_\eta^{\partial,\xi} \mathcal{G}_\alpha^{\partial,\eta} + (\partial_\alpha^1 \mathcal{C}^\lambda \circ \iota_a)) (\dot{x}^\alpha \circ \iota_a)) \\ + (\Phi_{\partial_u}^2)^* ((\dot{x}^\alpha \partial_\alpha^1 \mathcal{C}^\lambda) \circ \iota_u) = 0,$$

in consideration of (5.31), from which the remaining conditions of Proposition 5.6 directly follow.

Boundary Control of iPCHD Systems based on Structural Invariants

By analogy with the finite dimensional case the order of the controller has not been determined yet and, therefore, the order may be seen again as an additional degree of freedom for the proposed approach. In particular, for the case $\bar{n} < n_c$ – i.e., not all controller coordinates are considered by the relation (5.26) – it is clear that certain components of J_c , R_c and G_c are not determined by the conditions of the Propositions 5.4, 5.6 respectively; thus, they can also be considered as additional free design parameters in order to, e.g., add additional damping in the closed-loop system by the controller which acts through the boundary port on $\partial\mathcal{D}_a$.

In order to find solutions of the set of the conditions (5.27) – (5.30) as well as (5.35) – (5.38) the following strategy may be performed:

1. For the parameterisation (5.19) choose G_c and \mathcal{K}_∂ in order to satisfy (5.30) or for (5.31) choose J_c and R_c in order to satisfy (5.38).
2. Solve (5.27) which equals (5.35) concerning the boundary conditions which are given for the parameterisation (5.19) by (5.28), (5.29) with respect to the design parameters J_c and R_c or for (5.31) by (5.36), (5.37) with regard to the design parameters G_c and \mathcal{K}^∂ .

Analogous to the finite dimensional case – provided that \bar{n} structural invariants exist – the key feature of this approach is that \bar{n} controller coordinates are related to the plant coordinates via

$$x_c^\lambda = - \int_{\mathcal{D}} (j^1\Phi)^* (\mathcal{C}^\lambda dX) + \kappa^\lambda, \quad \mathcal{C}^\lambda \in C^\infty(\mathcal{J}^1(\mathcal{X})), \quad \kappa^\lambda \in \mathbb{R}, \quad (5.39)$$

with $\lambda = 1, \dots, \bar{n}$ in consideration of (5.26) since the structural invariants serve as conserved quantities for the closed-loop system. The constants $\kappa^\lambda \in \mathbb{R}$ depend on the initial conditions of the plant and the controller to the initial point of time $t = t_0 \in \mathbb{R}_0^+$ and are given by $\kappa^\lambda = \mathfrak{C}^\lambda|_{t=t_0}$ by a slight abuse of notation. Therefore, it is clear that – in the case of $\bar{n} < n_c$ – the controller coordinates are split into

$$(x_c^{\alpha_c}) = (x_c^\lambda, x_c^\mu) = \left(- \int_{\mathcal{D}} (j^1\Phi)^* (\mathcal{C}^\lambda dX) + \kappa^\lambda, x_c^\mu \right)$$

with $\lambda = 1, \dots, \bar{n}$ and $\mu = \bar{n} + 1, \dots, n_c$, where for (x_c^λ) the relation (5.39) is met and (x_c^μ) denote those coordinates which are not related to the plant coordinates by (5.26), (5.39)

respectively. Consequently, if it is possible to choose the controller Hamiltonian H_c – which has not been yet determined – such that the closed-loop Hamiltonian functional \mathfrak{H}_d of (5.24) is positive definite, then it may serve as an appropriate Lyapunov function candidate for the investigation of the stability of the desired equilibrium of the closed-loop system in the sense of Lyapunov. It is worth noting that the negative semidefiniteness of the formal change of the closed-loop Hamiltonian functional (5.25) is only a necessary condition for the stability of the desired equilibrium in the infinite dimensional case. In general, the proof of the stability in the infinite dimensional scenario is more sophisticated than in the finite dimensional case, where advanced functional analytic investigations which can be found for the case of linear PDEs (with a one-dimensional spatial domain) in [Liu and Zheng, 1999, Luo et al., 1999, Michel et al., 2007], for instance, must be usually accomplished. Therefore, the proof of the stability must be investigated for each particular application.

5.3 Boundary Control of the Timoshenko Beam based on Structural Invariants

In order to emphasise the results of the last section we intend to apply the proposed approach to the (energy based) boundary control of the Timoshenko beam.

Boundary Control via Structural Invariants

Let us consider a beam modelled according to the Timoshenko theory with the domain $\mathcal{D} = [0, L]$ and the spatial coordinate $X^1 \in [0, L]$, where for simplicity we neglect the gravitational potential and all beam parameters are assumed to be constant (and positive). According to section 4.1 the Hamiltonian functional – in this case – is given by

$$\mathfrak{H}(\Phi) = \frac{1}{2} \int_0^L (j^1\Phi)^* \left(\frac{1}{\rho} (p_w)^2 + \frac{1}{I_m} (p_\psi)^2 + EI_a (\psi_1)^2 + kGA (w_1 - \psi)^2 \right) dX \quad (5.40)$$

with the first-order Hamiltonian density

$$\mathcal{H} dX = \frac{1}{2} \left(\frac{1}{\rho} (p_w)^2 + \frac{1}{I_m} (p_\psi)^2 + EI_a (\psi_1)^2 + kGA (w_1 - \psi)^2 \right) dX$$

and the iPCH system representation reads as

$$\dot{x} = v = \begin{bmatrix} \dot{w} \\ \dot{\psi} \\ \dot{p}_w \\ \dot{p}_\psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_w \mathcal{H} \\ \delta_\psi \mathcal{H} \\ \delta_{p_w} \mathcal{H} \\ \delta_{p_\psi} \mathcal{H} \end{bmatrix} = \mathcal{J}(\delta(\mathcal{H} dX)) .$$

Furthermore, at $X^1 = 0$ we consider a free end and at $X^1 = L$ the beam is actuated via a (shearing) force and a (bending) moment, i.e., the boundary conditions are given by

$$\begin{aligned} u_1^\partial &= (kGA (w_1 - \psi)) \circ \iota_L & (kGA (w_1 - \psi)) \circ \iota_0 &= 0 \\ u_2^\partial &= (EI_a \psi_1) \circ \iota_L & (EI_a \psi_1) \circ \iota_0 &= 0 \end{aligned} ,$$

cf. Examples 4.1, 4.2 respectively. For this configuration it is obvious that $X^1 = 0$ characterises an unactuated boundary $\partial\mathcal{D}_u$ and $X^1 = L$ represents an actuated boundary $\partial\mathcal{D}_a$ with the boundary map

$$[\mathcal{G}_\alpha^{\partial,\xi}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{G}_\alpha^{\partial,\xi} = \delta_\alpha^\xi, \quad \xi = 1, 2,$$

and the corresponding collocated boundary outputs

$$y^{\partial,1} = \left(\frac{1}{\rho}p_w\right) \circ \iota_L, \quad y^{\partial,2} = \left(\frac{1}{I_m}p_\psi\right) \circ \iota_L, \quad (5.41)$$

cf. Example 4.2. Furthermore, it is easily verified that for this setting the equilibria take the form of

$$x_d = \begin{bmatrix} w_d \\ \psi_d \\ p_{w,d} \\ p_{\psi,d} \end{bmatrix} = \begin{bmatrix} aX^1 + b \\ a \\ 0 \\ 0 \end{bmatrix} \quad (5.42)$$

with $a, b \in \mathbb{R}$ implying $u_{1,d}^\partial = u_{2,d}^\partial = 0$ at the equilibrium, of course.

With regard to control purposes the objective can be stated as follows; design a boundary controller via the proposed approach on the basis of structural invariants in consideration of the parameterisation (5.31) in order that a *desired* equilibrium x_d for *certain* values of $a, b \in \mathbb{R}$ can be stabilised. According to Proposition 5.5 we confine ourselves to the (simple) choice of $\mathcal{K}_\eta^{\partial,\xi} = \delta_\eta^\xi$ concerning the components of the (power conserving) feedback interconnection represented by the map \mathcal{K}^∂ . Furthermore, due to the form of (5.40) we intend to shape the potential energy of the beam only, i.e., the structural invariants of the closed-loop system should not depend on the temporal momenta as well as their derivative coordinates.

Proposition 5.7 *A possible choice for the structural invariants of the closed-loop system which satisfy the conditions (5.35)–(5.38) with respect to $\mathcal{K}_\eta^{\partial,\xi} = \delta_\eta^\xi$ and $\mathcal{G}_\alpha^{\partial,\xi} = \delta_\alpha^\xi$ is given by*

$$\mathfrak{E}^1 = x_c^1 - \frac{1}{L} \int_0^L (j^1\Phi)^* (w + X^1 w_1) dX = x_c^1 - \iota_L^* (w \circ \Phi), \quad (5.43)$$

$$\mathfrak{E}^2 = x_c^2 - \frac{1}{L} \int_0^L (j^1\Phi)^* (\psi + X^1 \psi_1) dX = x_c^2 - \iota_L^* (\psi \circ \Phi) \quad (5.44)$$

and for $n_c = 4$ and $m = 2$ the controller maps are chosen to

$$[J_c^{\alpha_c\beta_c} - R_c^{\alpha_c\beta_c}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R_c^{33} & J_c^{34} - R_c^{34} \\ 0 & 0 & -J_c^{34} - R_c^{34} & -R_c^{44} \end{bmatrix}, \quad [G_{c,\xi}^{\alpha_c}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ G_{c,1}^3 & G_{c,2}^3 \\ G_{c,1}^4 & G_{c,2}^4 \end{bmatrix}$$

with the (simple) choice $J_c^{\alpha_c\beta_c}, R_c^{\alpha_c\beta_c}, G_{c,\xi}^{\alpha_c} \in \mathbb{R}$ for the additional design parameters.

In order to show that (5.43) as well as (5.44) together with the choices of the controller maps fulfil the relations (5.35) – (5.38) we have to consider the condition (5.38) first of all. This condition is clearly fulfilled since for $\lambda = 1, 2$ and $\beta_c = 1, \dots, n_c = 4$ the components are given by $J_c^{\lambda\beta_c} = R_c^{\lambda\beta_c} = 0$. The conditions (5.36) and (5.37) take the form of

$$G_{c,\xi}^\lambda \delta_\alpha^\xi = -\partial_\alpha^1 \mathcal{C}^\lambda \circ \iota_L, \quad \partial_\alpha^1 \mathcal{C}^\lambda \circ \iota_0 = 0$$

with respect to $\mathcal{G}_\alpha^{\partial,\xi} = \delta_\alpha^\xi$, the choice $\mathcal{K}_\eta^{\partial,\xi} = \delta_\eta^\xi$ and the boundary conditions at the free end. Since (5.43) depends only on the deflection w and its derivative coordinates and (5.44) only on the angle of rotation ψ and its derivative coordinates, the corresponding components of G_c are chosen such that the former conditions can be rewritten as

$$G_{c,1}^1 = 1 = -\partial_w^1 \mathcal{C}^1 \circ \iota_L, \quad G_{c,2}^2 = 1 = -\partial_\psi^1 \mathcal{C}^2 \circ \iota_L, \quad \partial_w^1 \mathcal{C}^1 \circ \iota_0 = 0, \quad \partial_\psi^1 \mathcal{C}^2 \circ \iota_0 = 0$$

with respect to $G_{c,2}^1 = G_{c,1}^2 = 0$. These conditions represent the boundary conditions for (5.35). Due to the fact that the interconnection map \mathcal{J} has full rank we deduce that \mathcal{C}^λ are total derivatives only and, thus, we are able to end up with

$$\mathcal{C}^1 = -\frac{1}{L} (w + X^1 w_1) = -\frac{1}{L} d_1 (X^1 w), \quad \mathcal{C}^2 = -\frac{1}{L} (\psi + X^1 \psi_1) = -\frac{1}{L} d_1 (X^1 \psi)$$

which clearly satisfy the relations (5.35) – (5.37).

In consideration of (5.39) the structural invariants (5.43), (5.44) enable us to obtain for the first two controller coordinates the identities

$$x_c^1 = \iota_L^* (w \circ \Phi) + \kappa^1, \quad x_c^2 = \iota_L^* (\psi \circ \Phi) + \kappa^2 \quad (5.45)$$

with the constants $\kappa^1, \kappa^2 \in \mathbb{R}$ depending on the initial conditions of the plant and the controller.

Remark 5.2 *If we take the influence of the gravity field into account, cf. Proposition 4.1, it is worth mentioning that the choice of the structural invariants (5.43), (5.44) is still valid.*

In the last step the controller Hamiltonian H_c must be chosen with regard to the requirements. Therefore, we make the (simple) choice

$$H_c = \frac{1}{2} M_{\mu\nu} x_c^\mu x_c^\nu + \frac{1}{2} c_1 (x_c^1 - x_{c,d}^1)^2 + \frac{1}{2} c_2 (x_c^2 - x_{c,d}^2)^2, \quad \mu, \nu = 3, 4, \quad (5.46)$$

with

$$M_{\mu\nu} = M_{\nu\mu} \in \mathbb{R}, \quad [M_{\mu\nu}] > 0, \quad c_1 > 0, \quad c_2 > 0$$

as well as

$$x_{c,d}^1 = \iota_L^* (w_d \circ \Phi) + \kappa^1 = aL + b + \kappa^1, \quad x_{c,d}^2 = \iota_L^* (\psi_d \circ \Phi) + \kappa^2 = a + \kappa^2.$$

Therefore, the Hamiltonian functional of the closed-loop system \mathfrak{H}_d is given by the sum of (5.40) and (5.46) and it is easily verified that it possesses a minimum at the desired

equilibrium (5.42) together with $x_{c,d}^3 = x_{c,d}^4 = 0$. In order to show that \mathfrak{H}_d is positive definite we consider a bundle morphism $\mathcal{M}_c \times \mathcal{X} \rightarrow \bar{\mathcal{M}}_c \times \bar{\mathcal{X}}$ of the form³

$$\begin{bmatrix} \bar{x}_c^3 \\ \bar{x}_c^4 \\ \bar{w} \\ \bar{\psi} \\ \bar{p}_w \\ \bar{p}_\psi \end{bmatrix} = \begin{bmatrix} x_c^3 - x_{c,d}^3 \\ x_c^4 - x_{c,d}^4 \\ w - w_d \\ \psi - \psi_d \\ p_w - p_{w,d} \\ p_\psi - p_{\psi,d} \end{bmatrix} = \begin{bmatrix} x_c^3 \\ x_c^4 \\ w - aX^1 - b \\ \psi - a \\ p_w \\ p_\psi \end{bmatrix} \quad (5.47)$$

with respect to $x_{c,d}^3 = x_{c,d}^4 = 0$, where this bundle morphism clearly transforms the desired equilibrium of the closed-loop system into the origin according to

$$\bar{x}_d = \begin{bmatrix} \bar{w}_d \\ \bar{\psi}_d \\ \bar{p}_{w,d} \\ \bar{p}_{\psi,d} \end{bmatrix} = 0, \quad \bar{x}_{c,d}^3 = \bar{x}_{c,d}^4 = 0 \quad (5.48)$$

in consideration of (5.47). Additionally, it is worth noting that from this bundle morphism we deduce

$$\iota_L^* (\bar{w} \circ \bar{\Phi}) = x_c^1 - x_{c,d}^1, \quad \iota_L^* (\bar{\psi} \circ \bar{\Phi}) = x_c^2 - x_{c,d}^2$$

with respect to $\bar{\Phi} : \mathcal{D} \rightarrow \bar{\mathcal{X}}$. Thus, in consideration of (5.47) the closed-loop Hamiltonian functional reads as

$$\begin{aligned} \mathfrak{H}_d = \frac{1}{2} \int_0^L (j^1 \bar{\Phi})^* & \left(\frac{1}{\rho} (\bar{p}_w)^2 + \frac{1}{I_m} (\bar{p}_\psi)^2 + EI_a (\bar{\psi}_1)^2 + kGA (\bar{w}_1 - \bar{\psi})^2 \right) dX \\ & + \frac{1}{2} M_{\mu\nu} \bar{x}_c^\mu \bar{x}_c^\nu + \frac{1}{2} c_1 (\iota_L^* (\bar{w} \circ \bar{\Phi}))^2 + \frac{1}{2} c_2 (\iota_L^* (\bar{\psi} \circ \bar{\Phi}))^2 \end{aligned} \quad (5.49)$$

which is positive definite and its formal change takes the form of

$$v_d(\mathfrak{H}_d) = -\bar{x}_c^\mu M_{\mu\nu} R_c^{\nu\zeta} M_{\zeta\chi} \bar{x}_c^\chi \leq 0, \quad \mu, \nu, \zeta, \chi = 3, 4, \quad (5.50)$$

due to (5.25).

Proof of Stability

As mentioned before, for the proof concerning the stability of the desired equilibrium (of the closed-loop system) we have to investigate the well-posedness of the solution of the closed-loop system and the admissibility of all the applied operations. Therefore, we intend to perform the stability analysis analogous to [Luo et al., 1999, Morgül, 1998, Thull, 2010, Zhang, 2007] (and references therein), where we first define the function space \mathcal{Z} as⁴

$$\mathcal{Z} = \left\{ z = (\bar{x}_c^3, \bar{x}_c^4, \bar{w}, \bar{\psi}, \bar{p}_w, \bar{p}_\psi) \mid \bar{x}_c^3, \bar{x}_c^4 \in \mathbb{R}, \bar{w}, \bar{\psi} \in H^1(0, L), \bar{p}_w, \bar{p}_\psi \in L^2(0, L) \right\}$$

³More precisely, we consider a bundle morphism on a submanifold of $\mathcal{M}_c \times \mathcal{X}$ due to the identities of (5.45).

⁴In fact, the particular sections of the infinite dimensional subsystem and their prolongations are limited to this function space; at this stage and in the sequel they are not explicitly stated in order to enhance the readability.

with respect to the spaces

$$\begin{aligned} L^2(0, L) &= \left\{ h \in \Gamma(\pi) \mid \int_0^L (h)^2 dX < \infty \right\}, \\ H^k(0, L) &= \left\{ h \in \Gamma(\pi), h \in L^2(0, L) \mid \partial_J h \in L^2(0, L), 1 \leq \#J \leq k \right\}. \end{aligned}$$

In \mathcal{Z} we introduce the inner product

$$\begin{aligned} \langle z, \check{z} \rangle_{\mathcal{Z}} &= M_{\mu\nu} \bar{x}_c^\mu \check{x}_c^\nu + c_1 \iota_L^* ((\bar{w} \circ \bar{\Phi}) (\check{w} \circ \bar{\Phi})) + c_2 \iota_L^* ((\bar{\psi} \circ \bar{\Phi}) (\check{\psi} \circ \bar{\Phi})) \\ &+ \int_0^L (j^1 \bar{\Phi})^* \left(\frac{1}{\rho} \bar{p}_w \check{p}_w + \frac{1}{I_m} \bar{p}_\psi \check{p}_\psi + EI_a \bar{\psi}_1 \check{\psi}_1 + kGA (\bar{w}_1 - \bar{\psi}) (\check{w}_1 - \check{\psi}) \right) dX \end{aligned} \quad (5.51)$$

which induces the (equivalent) norm

$$\begin{aligned} \|z\|_{\mathcal{Z}}^2 &= \langle z, z \rangle_{\mathcal{Z}} = M_{\mu\nu} \bar{x}_c^\mu \bar{x}_c^\nu + c_1 \iota_L^* (\bar{w} \circ \bar{\Phi})^2 + c_2 \iota_L^* (\bar{\psi} \circ \bar{\Phi})^2 \\ &+ \int_0^L (j^1 \bar{\Phi})^* \left(\frac{1}{\rho} (\bar{p}_w)^2 + \frac{1}{I_m} (\bar{p}_\psi)^2 + EI_a (\bar{\psi}_1)^2 + kGA (\bar{w}_1 - \bar{\psi})^2 \right) dX \end{aligned} \quad (5.52)$$

where, obviously, the relation

$$\|z\|_{\mathcal{Z}}^2 = \langle z, z \rangle_{\mathcal{Z}} = 2 \mathfrak{H}_d \quad (5.53)$$

is met, i.e., the square of the norm just corresponds to the positive definite Hamiltonian functional of the closed-loop system (5.49) aside from a numerical factor. It can be shown that \mathcal{Z} serves as a proper Hilbert space with the equivalent inner product (5.51), see Appendix A.8. In addition, we are able to rewrite the closed-loop system (5.33), (5.34) as a so-called Cauchy Problem of the form $\dot{z} = \mathcal{A}z$ with the initial condition $z(0) = z_0 \in \mathcal{Z}$ for $t_0 = 0$ and the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{Z}$ given by

$$\mathcal{A} \begin{bmatrix} \bar{x}_c^3 \\ \bar{x}_c^4 \\ \bar{w} \\ \bar{\psi} \\ \bar{p}_w \\ \bar{p}_\psi \end{bmatrix} = \begin{bmatrix} -R_c^{33} (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) + (J_c^{34} - R_c^{34}) (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) + G_{c,1}^3 y^{\partial,1} + G_{c,2}^3 y^{\partial,2} \\ (-J_c^{34} - R_c^{34}) (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) - R_c^{44} (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) + G_{c,1}^4 y^{\partial,1} + G_{c,2}^4 y^{\partial,2} \\ \frac{1}{\rho} \bar{p}_w \\ \frac{1}{I_m} \bar{p}_\psi \\ kAG (\bar{w}_1 - \bar{\psi}_1) \\ EI_a \bar{\psi}_{11} + kAG (\bar{w}_1 - \bar{\psi}) \end{bmatrix}$$

including the boundary outputs (5.41), where the domain $\mathcal{D}(\mathcal{A})$ of the operator \mathcal{A} is defined as

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \{ z \in \mathcal{Z} \mid \bar{x}_c^3, \bar{x}_c^4 \in \mathbb{R}, \bar{p}_w, \bar{p}_\psi \in H^1(0, L), \bar{w}, \bar{\psi} \in H^2(0, L), \\ &kGA (\bar{w}_1 - \bar{\psi}) \circ \iota_L + c_1 (\bar{w} \circ \iota_L) + G_{c,1}^3 (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) + G_{c,1}^4 (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) = 0, \\ &EI_a (\bar{\psi}_1 \circ \iota_L) + c_2 (\bar{\psi} \circ \iota_L) + G_{c,2}^3 (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) + G_{c,2}^4 (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) = 0, \\ &(\bar{w}_1 - \bar{\psi}) \circ \iota_0 = 0, (\bar{\psi}_1 \circ \iota_0) = 0 \}. \end{aligned}$$

The linear operator \mathcal{A} is called dissipative if $\langle z, \mathcal{A}z \rangle_{\mathcal{Z}} \leq 0$ is met, see [Luo et al., 1999]. In fact, due to (5.53) this is just equivalent to (5.50) and, therefore, we conclude that \mathcal{A} is a dissipative operator. Furthermore, it can be shown – analogous to [Zhang, 2007] – that the inverse \mathcal{A}^{-1} exists and it is bounded, see Appendix A.9. Due to the fact that \mathcal{Z} is a Hilbert space we are able to directly apply a form of the Lumer-Phillips Theorem (see [Zhang, 2007] and [Liu and Zheng, 1999], Theorem 1.2.4., and in this context [Luo et al., 1999], page 35, as well as [Curtain and Zwart, 1995] page 592) from which it can be deduced that \mathcal{A} is the infinitesimal generator of a contractive C_0 -semigroup $T(t)$. Hence, the solution of the closed-loop system corresponds to $z(t) = \gamma_t(z_0) = T(t)z_0$, cf. (3.7), where for the induced operator norm the relation

$$\|T(t)\| = \sup_{z_0 \in \mathcal{Z} \setminus \{0\}} \frac{\|T(t)z_0\|_{\mathcal{Z}}}{\|z_0\|_{\mathcal{Z}}} \leq 1$$

is met, see [Curtain and Zwart, 1995, Luo et al., 1999], for instance. Therefore, this relation directly implies the stability of the desired equilibrium in the sense of Lyapunov with respect to the norm $\|\cdot\|_{\mathcal{Z}}$ due to

$$\|z(t)\|_{\mathcal{Z}} = \|T(t)z_0\|_{\mathcal{Z}} \leq \|T(t)\| \|z_0\|_{\mathcal{Z}} \leq \|z_0\|_{\mathcal{Z}}.$$

For all further constructions concerning the asymptotic or exponential stability see, e.g., [Liu and Zheng, 1999, Luo et al., 1999, Michel et al., 2007, Zhang, 2007].

Simulation Results

Finally, some simulation results are presented, where we consider the simple but demonstrative case $kGA = EI_a = 1$, $I_m = 1$, $\rho = 1$ and $L = 1$. In Figure 5.3 the initial conditions for the beam at $t = t_0 = 0$ are given by the zero equilibrium, defined by $w = \psi = 0$, and the objective is to stabilise a desired equilibrium characterised by $a = b = 0.1$. The initial conditions for the controller are also set to zero and, thus, we choose $\kappa^1 = \kappa^2 = 0$. The design parameters for the controller are chosen as⁵ $c_1 = 2.9$, $c_2 = 1$, $J_c^{34} = 0.85$ as well as

$$[M_{\mu\nu}] = \begin{bmatrix} 3075 & 300 \\ 300 & 2800 \end{bmatrix} > 0, \quad [R_c^{\alpha_c\beta_c}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2.3 & -1 \\ 0 & 0 & -1 & 2.2 \end{bmatrix}, \quad [G_{c,\xi}^{\alpha_c}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2.2 & 0 \\ 0 & 1.2 \end{bmatrix}.$$

Obviously, the proposed control scheme stabilises the desired equilibrium and the excellent reference reaction of the closed-loop system is demonstrated in the plots of Figure 5.3 for the deflection w and the angle of rotation ψ . Furthermore, the boundary controller also plays the role of a dissipative element since the beam vibration is completely damped and, in addition, the closed-loop Hamiltonian functional \mathfrak{H}_d in Figure 5.3 possesses a minimum

⁵The controller parameters are obtained by minimising the quadratic error for the displacement coordinates w and ψ with respect to the desired values w_d as well as ψ_d and with respect to the initial conditions (parameter optimisation programme for the components of the controller maps J_c , R_c and G_c).

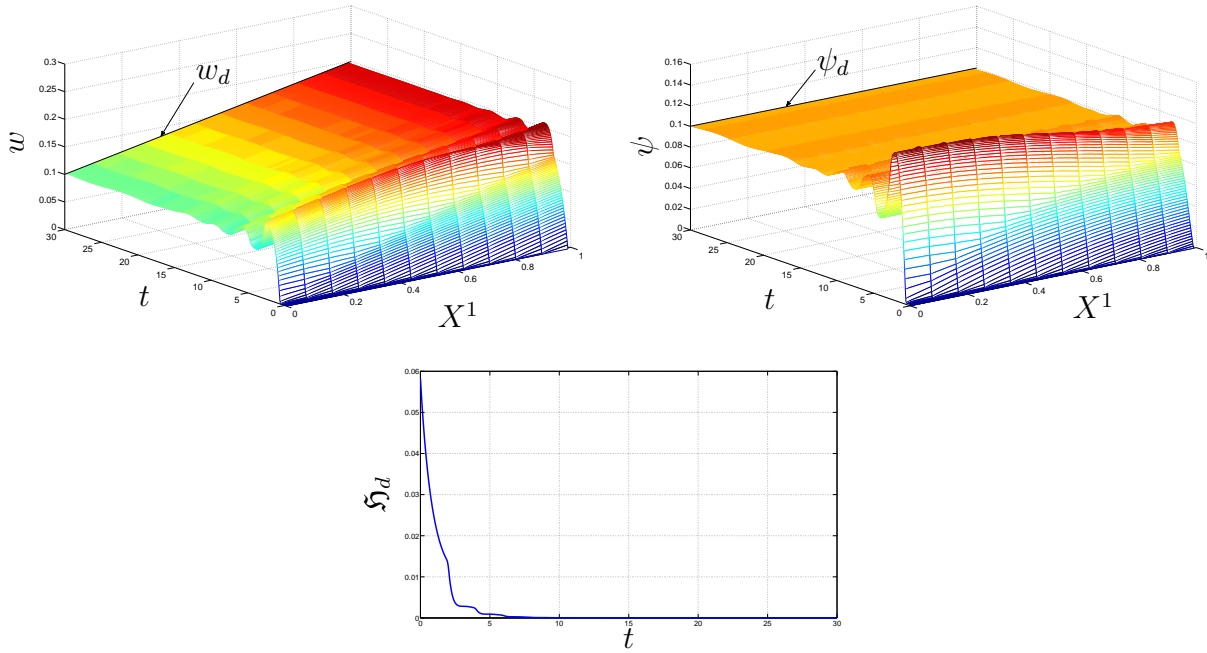


Figure 5.3: Simulation results for the deflection w , the angle of rotation ψ and the closed-loop Hamiltonian functional \mathfrak{H}_d with respect to the desired equilibrium characterised by w_d and ψ_d .

at the desired equilibrium. In Figure 5.4 the controller coefficients are chosen as $c_1 = 2.5$, $c_2 = 1.3$, $J_c^{34} = 0$ as well as

$$[M_{\mu\nu}] = \begin{bmatrix} 2425 & 310 \\ 310 & 3360 \end{bmatrix} > 0, \quad [R_c^{\alpha_c\beta_c}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2.55 & -0.75 \\ 0 & 0 & -0.75 & 1.5 \end{bmatrix}, \quad [G_{c,\xi}^{\alpha_c}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2.8 & 0 \\ 0 & 1.5 \end{bmatrix}$$

in order to stabilise the zero equilibrium ($a = b = 0$) with respect to a good disturbance rejection which is demonstrated for the case of an external disturbance force impulse (amplitude $1 \frac{\text{N}}{\text{m}}$, pulse width 0.1 s) acting on the middle of the beam after $t = 5$ s. Again, the control objective is fulfilled very well and the excellent performance of the closed-loop system concerning the effect of the disturbance force impulse can be seen in the plots of Figure 5.4 for the deflection and the angle of rotation⁶. It is worth noting that both obtained parameter sets for the controller lead to a satisfactory reference reaction and disturbance rejection. Exemplary, in Figure 5.5 the later obtained parameter set for the controller is chosen in order to stress out this fact, where both considered cases are combined, i.e., the initial conditions for the beam are given by the zero equilibrium (defined by $w = \psi = 0$)

⁶At this stage it must be emphasised that we have considered the two cases depicted in Figure 5.3 and 5.4, where the two different sets of controller parameters are derived by minimising two different optimisation criteria with respect to the desired equilibria and the initial conditions in order to demonstrate the usability and the efficiency of the presented approach. Again, a quadratic error function has been considered as described before.

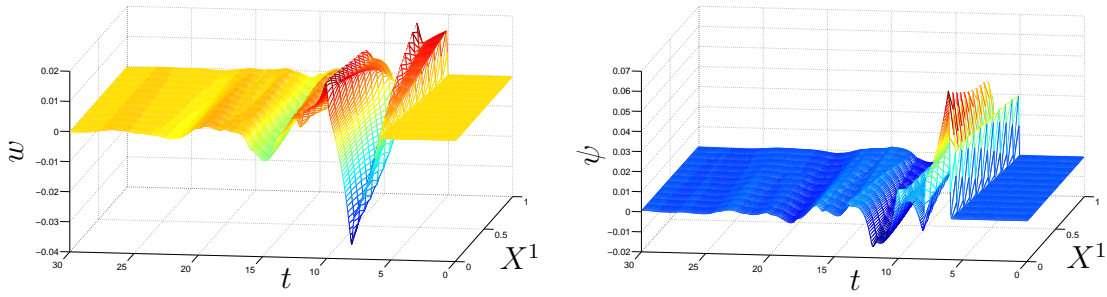


Figure 5.4: Simulation results for the deflection w and the angle of rotation ψ with respect to the impact of an external disturbance force impulse acting on the middle of the beam after $t = 5$ s (amplitude $1 \frac{\text{N}}{\text{m}}$, pulse width 0.1 s).

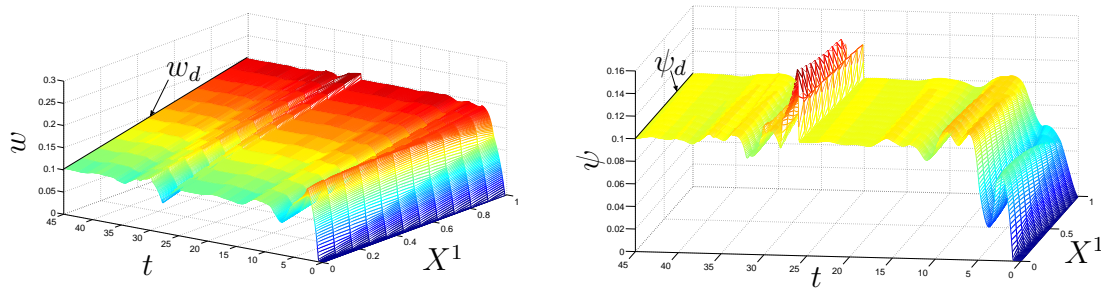


Figure 5.5: Simulation results for the deflection w and the angle of rotation ψ (external disturbance force impulse acting on the middle of the beam after $t = 25$ s with pulse width 0.1 s). The desired equilibrium is characterised by w_d and ψ_d .

and the objective is to stabilise a desired equilibrium characterised by $a = b = 0.1$ (with the choice $\kappa^1 = \kappa^2 = 0$) with respect to an external disturbance force impulse (amplitude $1 \frac{\text{N}}{\text{m}}$, pulse width 0.1 s) acting on the middle of the beam after $t = 25$ s.

Remark 5.3 For the case of a beam clamped at $X^1 = 0$, i.e., $w \circ \iota_0 = 0$ and $\psi \circ \iota_0 = 0$, it also is possible to apply the proposed control scheme for stabilising the zero equilibrium ($a = b = 0$) since these geometric boundary conditions represent an unactuated boundary. In fact, the choice of the structural invariants is still valid and the same controller can be applied with respect to $a = b = 0$. It is remarkable that in contrast to the usage of a pure (non-dynamic) damping injection controller at $X^1 = L$, see, e.g., [Kim and Renardy, 1987], the proposed dynamical boundary controller provides more degrees of freedom in order to adjust a satisfactory reference reaction and disturbance rejection of the closed-loop system.

Remark 5.4 It is worth mentioning that the considered control problem presented in this section (based on the proposed Port-Hamiltonian modelling of the Timoshenko beam) can not be handled within the context of the Port-Hamiltonian framework on the basis of the Stokes-Dirac structures as in [Macchelli and Melchiorri, 2004a,b] due to the usage of the deformations and the momenta for the dependent coordinates (so-called energy variables). Since we consider a beam with a free end, where position control is the main objective with

respect to the control via structural invariants methodology, a Port-Hamiltonian formulation of the beam with respect to the usage of the displacement coordinates is indispensable.

Summary and Outlook

This thesis is mainly dedicated to the geometry, modelling and control of infinite dimensional Port-Hamiltonian systems (called iPCHD systems). It has turned out that the investigation of the formal change of a Hamiltonian functional along a(n) (prolonged) evolutionary vector field which characterises a certain set of evolution equations serves as the crucial aspect for the extension of the Port-Hamiltonian framework to the distributed-parameter case. Then, by a certain choice of these evolution equations it is possible to propose the infinite dimensional Port-Hamiltonian system representation, where two main system classes can be introduced; namely the iPCHD system representations concerning the non-differential and the differential operator case which allow the analogous physical interpretation known from the lumped-parameter PCHD systems such as the characterisation of the dissipative effects (inside the domain) and the definition of the (energy) ports acting inside the domain as well as through the boundary. Particularly, with regard to the (Port-) Hamiltonian formulation of field theories the combination of these two system representations has turned out to be an adequate tool. In fact, the proposed framework is applied to the Port-Hamiltonian modelling of the Timoshenko beam and to the Port-Hamiltonian formulation of fluid dynamics including the Navier-Stokes equations and of (inductionless) magnetohydrodynamics – which incorporates electrically conducting fluids in the presence of (quasi-)stationary electromagnetic fields – in a Lagrangian setting. The last part of this thesis deals with the extension of the *control via structural invariants* method which is a well-known method for the control of lumped-parameter PCHD systems with regard to the proposed iPCHD systems restricted to the non-differential operator case, where specific criteria and conditions analogous to the lumped-parameter case which allow a systematic (boundary) controller design are derived. This control approach is applied to the energy based boundary control of the Timoshenko beam.

With regard to further investigations and future work based on this thesis it will be of interest to apply and extend the proposed framework to the Port-Hamiltonian formulation of fully coupled field theories such as the full equations of magnetohydrodynamics including – aside from the governing equations of fluid dynamics – the Maxwell's equations. Particularly, with regard to control purposes it should be possible to extend the *control via structural invariants* method to iPCHD systems with higher-dimensional domains (such as plates, etc.), where the demand concerning finite dimensional control laws will become

a crucial aspect. Moreover, another main objective should be dedicated to the extension of the (from the finite-dimensional case well-known) control method *Interconnection and Damping Assignment - Passivity based Control (IDA-PBC)*, see, e.g., [Ortega et al., 2002], to the proposed iPCHD system class, where this method is of main interest concerning practical applications. In particular, the application of non-linear control laws or non-linear boundary controllers should be investigated and analysed in detail with respect to the derivation of effective control concepts for iPCHD systems, where, especially for the (formal) stability analysis, the non-linear semigroup theory must be taken into account.

Proofs and Detailed Computations

Appendix A contains a few proofs and some exact computations which are omitted in the previous chapters concerning the readability.

A.1 The Application of the Horizontal Differential

Exemplary, for all applications involving horizontal differentials we intend to proof the relation

$$\int_{\mathcal{D}} (j^{r+1}\Phi)^* (d_i (v^\alpha \partial_\alpha^i \mathcal{H} dX)) = \int_{\mathcal{D}} (j^{r+1}\Phi)^* (d_h (v^\alpha \partial_\alpha^i \mathcal{H} \partial_i \lfloor dX)) .$$

Therefore, we compute

$$\begin{aligned} \int_{\mathcal{D}} (j^{r+1}\Phi)^* (d_i (v^\alpha \partial_\alpha^i \mathcal{H} dX)) &= \int_{\mathcal{D}} \partial_i ((v^\alpha \partial_\alpha^i \mathcal{H}) \circ (j^r \Phi) dX) \\ &= \int_{\mathcal{D}} d((v^\alpha \partial_\alpha^i \mathcal{H}) \circ (j^r \Phi) \partial_i \lfloor dX) \end{aligned}$$

due to the fact $d((v^\alpha \partial_\alpha^i \mathcal{H}) \circ (j^r \Phi) dX) = 0$. Furthermore, we obtain

$$\begin{aligned} \int_{\mathcal{D}} (j^{r+1}\Phi)^* (d_i (v^\alpha \partial_\alpha^i \mathcal{H} dX)) &= \int_{\mathcal{D}} dX^j \wedge \partial_j ((v^\alpha \partial_\alpha^i \mathcal{H}) \circ (j^r \Phi) \partial_i \lfloor dX) \\ &= \int_{\mathcal{D}} (j^{r+1}\Phi)^* (dX^j \wedge d_j (v^\alpha \partial_\alpha^i \mathcal{H} \partial_i \lfloor dX)) , \end{aligned}$$

from which the desired result follows by considering the definition of the horizontal differential.

A.2 The Rate of Deformation Tensors

We intend to compute the material rate of deformation tensor according to

$$D = \frac{1}{2} \partial_0 (C) = \frac{1}{2} \partial_0 \left((g_{\alpha\beta} \circ \Phi) F_i^\alpha F_j^\beta \right) dX^i \otimes dX^j = D_{ij} dX^i \otimes dX^j .$$

Therefore, we conclude

$$D = \frac{1}{2} \left(\partial_\gamma g_{\alpha\beta} V_0^\gamma F_i^\alpha F_j^\beta + g_{\alpha\beta} \partial_i V_0^\alpha F_j^\beta + g_{\alpha\beta} F_i^\alpha \partial_j V_0^\beta \right) \circ \Phi \, dX^i \otimes dX^j$$

with respect to $\partial_0 (g_{\alpha\beta} \circ \Phi) = \partial_\gamma (g_{\alpha\beta} \circ \Phi) V_0^\gamma$ as well as $\partial_0 F_i^\alpha = \partial_i V_0^\alpha$. With the help of the relation

$$\partial_\gamma g_{\alpha\beta} = g_{\delta\beta} \gamma_{\alpha\gamma}^\delta + g_{\delta\alpha} \gamma_{\beta\gamma}^\delta, \quad (\text{A.1})$$

cf. [Marsden and Hughes, 1994], we derive the desired result

$$\begin{aligned} D &= \frac{1}{2} \left(g_{\alpha\beta} \partial_i V_0^\alpha F_j^\beta + g_{\delta\beta} \gamma_{\alpha\gamma}^\delta V_0^\gamma F_i^\alpha F_j^\beta + g_{\alpha\beta} F_i^\alpha \partial_j V_0^\beta + g_{\delta\alpha} \gamma_{\beta\gamma}^\delta V_0^\gamma F_i^\alpha F_j^\beta \right) \circ \Phi \, dX^i \otimes dX^j \\ &= \frac{1}{2} (g_{\alpha\beta} \circ \Phi) \left(F_j^\beta (\partial_i V_0^\alpha + \gamma_{\delta\gamma}^\alpha V_0^\gamma F_i^\delta) + F_i^\alpha (\partial_j V_0^\beta + \gamma_{\delta\gamma}^\beta V_0^\gamma F_j^\delta) \right) \circ \Phi \, dX^i \otimes dX^j. \end{aligned}$$

For the derivation of the spatial rate of deformation tensor we evaluate the expression

$$d = \frac{1}{2} v_\Phi (g_{\alpha\beta} dq^\alpha \otimes dq^\beta) = d_{\alpha\beta} dq^\alpha \otimes dq^\beta, \quad v_\Phi = \partial_0 + v^\alpha \partial_\alpha,$$

with $g_{\alpha\beta} \in C^\infty(\mathcal{Q})$ leading to

$$d = \frac{1}{2} v_\Phi (g_{\alpha\beta}) dq^\alpha \otimes dq^\beta + \frac{1}{2} g_{\alpha\beta} v_\Phi (dq^\alpha) \otimes dq^\beta + \frac{1}{2} g_{\alpha\beta} dq^\alpha \otimes v_\Phi (dq^\beta).$$

This expression includes the relations

$$v_\Phi (g_{\alpha\beta}) = v^\gamma \partial_\gamma g_{\alpha\beta}, \quad v_\Phi (dq^\beta) = v^\gamma \partial_\gamma (dq^\beta) = dv^\beta = \partial_\gamma v^\beta dq^\gamma$$

and, therefore, we are able to write

$$d = \frac{1}{2} (v^\gamma \partial_\gamma g_{\alpha\beta} + g_{\delta\beta} \partial_\alpha v^\delta + g_{\alpha\delta} \partial_\beta v^\delta) dq^\alpha \otimes dq^\beta$$

Using (A.1) we obtain the desired result

$$d = \frac{1}{2} (g_{\alpha\delta} (\partial_\beta v^\delta + \gamma_{\beta\gamma}^\delta v^\gamma) + g_{\beta\delta} (\partial_\alpha v^\delta + \gamma_{\alpha\gamma}^\delta v^\gamma)) dq^\alpha \otimes dq^\beta.$$

The pull-back of this expression by the motion leads to

$$D = \Phi^* (d) = (d_{\alpha\beta} \circ \Phi) F_i^\alpha F_j^\beta dX^i \otimes dX^j = D_{ij} dX^i \otimes dX^j$$

(by neglecting the terms involving dt^0) with the components

$$\begin{aligned} D_{ij} &= \frac{1}{2} \left(g_{\alpha\delta} (\partial_j V_0^\delta + \gamma_{\beta\gamma}^\delta V_0^\gamma F_j^\beta) F_i^\alpha + g_{\beta\delta} (\partial_i V_0^\delta + \gamma_{\alpha\gamma}^\delta V_0^\gamma F_i^\alpha) F_j^\beta \right) \circ \Phi \\ &= \frac{1}{2} (g_{\alpha\beta} \circ \Phi) \left(F_i^\alpha (\partial_j V_0^\beta + \gamma_{\delta\gamma}^\beta V_0^\gamma F_j^\delta) + F_j^\beta (\partial_i V_0^\alpha + \gamma_{\delta\gamma}^\alpha V_0^\gamma F_i^\delta) \right) \circ \Phi. \end{aligned}$$

A.3 The Stored Energy Relation

Based on the result

$$S^{ij} - \bar{S}^{ij} = -J \left(\mathcal{P} \hat{F}_\alpha^i \hat{F}_\beta^j g^{\alpha\beta} \right) \circ \Phi = 2\rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial C_{ij}}$$

we want to prove the relation

$$\mathcal{P} \circ \Phi = -\rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial J}.$$

Therefore, it is easily seen that in consideration of the components of the Cauchy Green tensor $C_{ij} = (g_{\alpha\beta} \circ \Phi) F_i^\alpha F_j^\beta$ the relation

$$\det [C_{ij}] = (\det [g_{\alpha\beta}] \circ \Phi) (\det [F_i^\alpha])^2$$

which allows to reparameterise the Jacobian (4.9) leading to

$$J = \det [F_i^\alpha] \sqrt{\frac{\det [g_{\alpha\beta}] \circ \Phi}{\det [G_{ij}]}} = \sqrt{\frac{\det [C_{ij}]}{\det [G_{ij}]}}$$

is met. Due to the fact that the stored energy only depends on the Jacobian we are able to derive

$$S^{ij} - \bar{S}^{ij} = 2\rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial J} \frac{\partial J}{\partial C_{ij}} = \rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial J} J C^{ij}$$

since

$$\frac{\partial J}{\partial C_{ij}} = \frac{1}{2J} \frac{1}{\det [G_{ij}]} \frac{\partial (\det [C_{ij}])}{\partial C_{ij}} = \frac{1}{2} J C^{ij},$$

where we have used the components of the inverse Cauchy Green tensor which follow from

$$\hat{C} = \Phi^* (\hat{g}) = C^{ij} \partial_i \otimes \partial_j = \left(g^{\alpha\beta} \hat{F}_\alpha^i \hat{F}_\beta^j \circ \Phi \right) \partial_i \otimes \partial_j, \quad C^{ij} C_{jk} = \delta_k^i,$$

and the well-known relation

$$\frac{\partial (\det [M_{ik}])}{\partial M_{ik}} M_{jk} = \delta_j^i \det [M_{ik}] \tag{A.2}$$

for an invertible matrix $M = [M_{ik}]$ with components M_{ik} . Furthermore, we conclude

$$-J \left(\mathcal{P} \hat{F}_\alpha^i \hat{F}_\beta^j g^{\alpha\beta} \right) \circ \Phi = \rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial J} J \left(g^{\alpha\beta} \hat{F}_\alpha^i \hat{F}_\beta^j \circ \Phi \right)$$

and, consequently, we obtain the desired result.

A.4 Hamiltonian Formulation of the Ideal Fluid

We intend to prove the equations

$$\begin{aligned}\partial_0 \Phi^\alpha &= \delta^\alpha \mathcal{H} \circ j^1 \Phi = \frac{(g^{\alpha\beta} \circ \Phi)}{\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} P_\alpha, \\ \partial_0 P_\alpha &= -\delta_\alpha \mathcal{H} \circ j^2 \Phi = -\frac{(\partial_\alpha g^{\beta\gamma}) \circ \Phi}{2\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} P_\beta P_\gamma - \sqrt{\det [G_{ij}]} J \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i (\mathcal{P} \circ \Phi)\end{aligned}$$

with respect to the first-order Hamiltonian density

$$\mathcal{H} \, dX = \left(\frac{1}{2\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} g^{\alpha\beta} p_\alpha p_\beta + \rho_{\mathcal{R}} E_{st} \sqrt{\det [G_{ij}]} \right) dX.$$

The first part of the equations is easily verified since we obtain

$$\dot{q}^\alpha = \delta^\alpha \mathcal{H} = \partial^\alpha \mathcal{H} = \frac{1}{\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} g^{\alpha\beta} p_\beta$$

and by plugging in the motion the desired result follows directly. For the second part of the equations we compute the variational derivative and obtain

$$\dot{p}_\alpha = -\partial_\alpha \mathcal{H} + d_i (\partial_\alpha^i \mathcal{H}) \tag{A.3}$$

with

$$\begin{aligned}\partial_\alpha \mathcal{H} &= \frac{1}{2\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} \partial_\alpha (g^{\gamma\beta}) p_\gamma p_\beta + \rho_{\mathcal{R}} \partial_\alpha (E_{st}) \sqrt{\det [G_{ij}]}, \\ d_i (\partial_\alpha^i \mathcal{H}) &= d_i \left(\rho_{\mathcal{R}} \partial_\alpha^i (E_{st}) \sqrt{\det [G_{ij}]} \right).\end{aligned}$$

Since the stored energy is a function of the Jacobian we are able to state

$$\begin{aligned}d_i (\partial_\alpha^i \mathcal{H}) &= d_i \left(\rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial \check{J}} \partial_\alpha^i (\check{J}) \sqrt{\det [G_{ij}]} \right) \\ &= -d_i \left(\check{\mathcal{P}} \partial_\alpha^i (\det [F_i^\alpha]) \sqrt{\det [g_{\alpha\beta}]} \right) \\ &= -d_i \left(\check{\mathcal{P}} \det [F_i^\alpha] \hat{F}_\alpha^i \sqrt{\det [g_{\alpha\beta}]} \right)\end{aligned}$$

with $\hat{F}_\alpha^i F_j^\alpha = \delta_j^i$, where we have used the relation (A.2). Before we evaluate the expression

$$\begin{aligned}d_i (\partial_\alpha^i \mathcal{H}) &= -\check{J} \hat{F}_\alpha^i d_i (\check{\mathcal{P}}) \sqrt{\det [G_{ij}]} - \check{\mathcal{P}} \sqrt{\det [g_{\alpha\beta}]} d_i \left(\det [F_i^\alpha] \hat{F}_\alpha^i \right) \\ &\quad - \check{\mathcal{P}} \det [F_i^\alpha] \hat{F}_\alpha^i d_i \left(\sqrt{\det [g_{\alpha\beta}]} \right)\end{aligned}$$

we investigate the relation

$$\begin{aligned}\partial_i \left(\left(\hat{F}_\alpha^i \circ \Phi \right) F_j^\alpha \right) &= 0 \\ \partial_i \left(\hat{F}_\alpha^i \circ \Phi \right) F_j^\alpha &= - \left(\hat{F}_\alpha^i \circ \Phi \right) \partial_i \left(F_j^\alpha \right) \\ \partial_i \left(\hat{F}_\alpha^i \circ \Phi \right) &= - \left(\hat{F}_\beta^i \circ \Phi \right) \partial_i \left(F_j^\beta \right) \left(\hat{F}_\alpha^j \circ \Phi \right) \\ &= - \left(\hat{F}_\beta^i \circ \Phi \right) \partial_j \left(F_i^\beta \right) \left(\hat{F}_\alpha^j \circ \Phi \right)\end{aligned}$$

since

$$\partial_i \left(F_j^\beta \right) = \partial_i \left(\partial_j \Phi^\beta \right) = \partial_j \left(\partial_i \Phi^\beta \right) = \partial_j \left(F_i^\beta \right) .$$

Therefore, we conclude

$$\begin{aligned}d_i \left(\det [F_i^\alpha] \hat{F}_\alpha^i \right) &= d_i \left(\det [F_i^\alpha] \right) \hat{F}_\alpha^i - \det [F_i^\alpha] \hat{F}_\beta^i d_j \left(F_i^\beta \right) \hat{F}_\alpha^j \\ &= \det [F_j^\beta] \hat{F}_\beta^j d_i \left(F_j^\beta \right) \hat{F}_\alpha^i - \det [F_i^\alpha] \hat{F}_\beta^i d_j \left(F_i^\beta \right) \hat{F}_\alpha^j \\ &= \det [F_i^\alpha] \left(\hat{F}_\beta^j d_i \left(F_j^\beta \right) \hat{F}_\alpha^i - \hat{F}_\beta^j d_i \left(F_j^\beta \right) \hat{F}_\alpha^i \right) \\ &= 0\end{aligned}\tag{A.4}$$

and, furthermore, we obtain

$$\begin{aligned}d_i \left(\sqrt{\det [g_{\alpha\beta}]} \right) &= q_i^\alpha \partial_\alpha \left(\sqrt{\det [g_{\alpha\beta}]} \right) = q_i^\alpha \frac{1}{2} \frac{\partial_\alpha \left(\det [g_{\alpha\beta}] \right)}{\sqrt{\det [g_{\alpha\beta}]}} \\ &= q_i^\alpha \frac{1}{2} \frac{1}{\sqrt{\det [g_{\alpha\beta}]}} \frac{\partial \left(\det [g_{\alpha\beta}] \right)}{\partial g_{\beta\gamma}} \partial_\alpha \left(g_{\beta\gamma} \right) \\ &= q_i^\alpha \frac{1}{2} \sqrt{\det [g_{\alpha\beta}]} g^{\beta\gamma} \partial_\alpha \left(g_{\beta\gamma} \right) .\end{aligned}$$

Consequently, (A.3) takes the form of

$$\begin{aligned}\dot{p}_\alpha &= - \frac{1}{2\rho_{\mathcal{R}} \sqrt{\det [G_{ij}]}} \partial_\alpha \left(g^{\gamma\beta} \right) p_\gamma p_\beta - \rho_{\mathcal{R}} \partial_\alpha \left(E_{st} \right) \sqrt{\det [G_{ij}]} \\ &\quad - \check{J} \hat{F}_\alpha^i d_i \left(\check{\mathcal{P}} \right) \sqrt{\det [G_{ij}]} - \frac{1}{2} \check{\mathcal{P}} \check{J} \sqrt{\det [G_{ij}]} g^{\beta\gamma} \partial_\alpha \left(g_{\beta\gamma} \right) .\end{aligned}$$

The term which involves the stored energy reads as

$$\begin{aligned}\rho_{\mathcal{R}} \partial_\alpha \left(E_{st} \right) \sqrt{\det [G_{ij}]} &= \rho_{\mathcal{R}} \frac{\partial E_{st}}{\partial \check{J}} \partial_\alpha \left(\check{J} \right) \sqrt{\det [G_{ij}]} \\ &= - \check{\mathcal{P}} \det [F_i^\alpha] \partial_\alpha \left(\sqrt{\det [g_{\alpha\beta}]} \right) \\ &= - \frac{1}{2} \check{\mathcal{P}} \det [F_i^\alpha] \sqrt{\det [g_{\alpha\beta}]} g^{\gamma\beta} \partial_\alpha \left(g_{\gamma\beta} \right) \\ &= - \frac{1}{2} \check{\mathcal{P}} \check{J} \sqrt{\det [G_{ij}]} g^{\gamma\beta} \partial_\alpha \left(g_{\gamma\beta} \right)\end{aligned}$$

and, thus, we obtain

$$\dot{p}_\alpha = -\frac{1}{2\rho_{\mathcal{R}}\sqrt{\det[G_{ij}]}}\partial_\alpha(g^{\gamma\beta})p_\gamma p_\beta - \check{J}\hat{F}_\alpha^i d_i(\check{\mathcal{P}})\sqrt{\det[G_{ij}]}.$$

By plugging in the motion we have

$$\partial_0 P_\alpha = -\frac{\partial_\alpha(g^{\gamma\beta})\circ\Phi}{2\rho_{\mathcal{R}}\sqrt{\det[G_{ij}]}}P_\gamma P_\beta - J\left(\hat{F}_\alpha^i\circ\Phi\right)\left(d_i(\check{\mathcal{P}})\circ j^2\Phi\right)\sqrt{\det[G_{ij}]}$$

and, due to

$$d_i(\check{\mathcal{P}})\circ j^2\Phi = \partial_i(\check{\mathcal{P}}\circ j^1\Phi) = \partial_i(\mathcal{P}\circ\Phi),$$

we obtain the final result

$$\partial_0 P_\alpha = -\frac{\partial_\alpha(g^{\gamma\beta})\circ\Phi}{2\rho_{\mathcal{R}}\sqrt{\det[G_{ij}]}}P_\gamma P_\beta - J\left(\hat{F}_\alpha^i\circ\Phi\right)\partial_i(\mathcal{P}\circ\Phi)\sqrt{\det[G_{ij}]}.$$

A.5 The first Viscous Piola-Kirchhoff Stress Form

For the case of a Newtonian fluid the viscous stress form in Cartesian coordinates reads as

$$\bar{\sigma} = \mathcal{K}]d = (\lambda\delta^{\alpha\beta}\partial_\gamma v^\gamma + \mu(\delta^{\alpha\gamma}\partial_\gamma v^\beta + \delta^{\beta\delta}\partial_\delta v^\alpha))\partial_\alpha] \text{vol} \otimes \partial_\beta$$

and, therefore, we obtain

$$d_{\Lambda_c}(\bar{\sigma}) \wedge dt^0 = ((\lambda + \mu)\partial_\alpha(\delta^{\alpha\beta}\partial_\gamma v^\gamma) + \mu\partial_\alpha(\delta^{\alpha\gamma}\partial_\gamma v^\beta)) \text{vol} \otimes \partial_\beta.$$

In order to obtain the Lagrangian counterpart we compute the first viscous Piola-Kirchhoff stress form (4.21) which follows in Cartesian coordinates to

$$\begin{aligned} \bar{P} &= \frac{1}{2}J\left(\hat{F}_\alpha^i\mathcal{K}^{\alpha\beta\gamma\delta}\hat{F}_\gamma^k\hat{F}_\delta^l\right)\circ\Phi\left(\delta_{\varepsilon\tau}F_l^\tau\partial_k V_0^\varepsilon + \delta_{\varepsilon\tau}F_k^\varepsilon\partial_l V_0^\tau\right)\partial_i] \text{VOL} \otimes \partial_\beta \\ &= J\left(\hat{F}_\alpha^i\mathcal{K}^{\alpha\beta\gamma\delta}\hat{F}_\gamma^k\delta_{\varepsilon\delta}(\partial_k V_0^\varepsilon)\right)\circ\Phi\partial_i] \text{VOL} \otimes \partial_\beta. \end{aligned}$$

With regard to

$$\mathcal{K}^{\alpha\beta\gamma\delta} = \lambda\delta^{\alpha\beta}\delta^{\gamma\delta} + \mu\delta^{\alpha\gamma}\delta^{\beta\delta} + \mu\delta^{\alpha\delta}\delta^{\beta\gamma}$$

we are able to conclude

$$\bar{P} = J\left(\hat{F}_\alpha^i\hat{F}_\gamma^k\circ\Phi\right)\left(\lambda\delta^{\alpha\beta}\partial_k V_0^\gamma + \mu\delta^{\alpha\gamma}\partial_k V_0^\beta + \mu\delta^{\beta\gamma}\partial_k V_0^\alpha\right)\partial_i] \text{VOL} \otimes \partial_\beta$$

and, therefore, we obtain

$$d_{\Lambda_c}^\Phi(\bar{P}) \wedge dt^0 = \partial_i\left(J\left(\hat{F}_\alpha^i\mathcal{K}^{\alpha\beta\gamma\delta}\hat{F}_\gamma^k\delta_{\varepsilon\delta}(\partial_k V_0^\varepsilon)\right)\circ\Phi\right) \text{VOL} \otimes \partial_\beta \quad (\text{A.5})$$

resulting in

$$\begin{aligned} d_{\Lambda_c}^{\Phi}(\bar{P}) \wedge dt^0 = J \left[\lambda \delta^{\alpha\beta} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\gamma} \right) + \mu \delta^{\alpha\gamma} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\beta} \right) \right. \\ \left. + \mu \delta^{\beta\gamma} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\alpha} \right) \right] \text{VOL} \otimes \partial_{\beta} \end{aligned}$$

in consideration of (A.4). Before we proceed we inspect the expression

$$\begin{aligned} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\alpha} \right) &= \left(\hat{F}_{\alpha}^i \hat{F}_{\gamma}^k \circ \Phi \right) \partial_i \left(\partial_k V_0^{\alpha} \right) + \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\alpha} \\ &= \left(\hat{F}_{\alpha}^i \hat{F}_{\gamma}^k \circ \Phi \right) \partial_k \left(\partial_i V_0^{\alpha} \right) - \left(\hat{F}_{\alpha}^i \hat{F}_{\delta}^k \circ \Phi \right) \partial_i \left(F_j^{\delta} \right) \left(\hat{F}_{\gamma}^j \circ \Phi \right) \partial_k V_0^{\alpha} \end{aligned}$$

by considering the relation

$$\begin{aligned} \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) F_j^{\gamma} \right) &= 0 \\ \partial_i \left(\hat{F}_{\gamma}^k \circ \Phi \right) F_j^{\gamma} &= - \left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_i \left(F_j^{\gamma} \right) \\ \partial_i \left(\hat{F}_{\gamma}^k \circ \Phi \right) &= - \left(\hat{F}_{\delta}^k \circ \Phi \right) \partial_i \left(F_j^{\delta} \right) \left(\hat{F}_{\gamma}^j \circ \Phi \right). \end{aligned}$$

Furthermore, we end up with the result

$$\begin{aligned} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\alpha} \right) &= \left(\hat{F}_{\alpha}^i \hat{F}_{\gamma}^k \circ \Phi \right) \partial_k \left(\partial_i V_0^{\alpha} \right) - \left(\hat{F}_{\alpha}^k \hat{F}_{\delta}^i \circ \Phi \right) \partial_k \left(F_j^{\delta} \right) \left(\hat{F}_{\gamma}^j \circ \Phi \right) \partial_i V_0^{\alpha} \\ &= \left(\hat{F}_{\alpha}^i \hat{F}_{\gamma}^k \circ \Phi \right) \partial_k \left(\partial_i V_0^{\alpha} \right) - \left(\hat{F}_{\gamma}^j \hat{F}_{\delta}^i \circ \Phi \right) \partial_j \left(F_k^{\delta} \right) \left(\hat{F}_{\alpha}^k \circ \Phi \right) \partial_i V_0^{\alpha} \\ &= \left(\hat{F}_{\alpha}^i \hat{F}_{\gamma}^k \circ \Phi \right) \partial_k \left(\partial_i V_0^{\alpha} \right) + \left(\hat{F}_{\gamma}^j \circ \Phi \right) \partial_j \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i V_0^{\alpha} \\ &= \left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k \left(\left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i V_0^{\alpha} \right). \end{aligned}$$

With the help of this relation we are able to state

$$\begin{aligned} d_{\Lambda_c}^{\Phi}(\bar{P}) \wedge dt^0 = J \left[\lambda \delta^{\alpha\beta} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\gamma} \right) + \mu \delta^{\alpha\gamma} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\beta} \right) \right. \\ \left. + \mu \delta^{\beta\gamma} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\alpha} \right) \right] \text{VOL} \otimes \partial_{\beta} \end{aligned}$$

leading to the final result

$$\begin{aligned} d_{\Lambda_c}^{\Phi}(\bar{P}) \wedge dt^0 = J \left[(\lambda + \mu) \delta^{\alpha\beta} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\gamma} \right) \right. \\ \left. + \mu \delta^{\alpha\gamma} \left(\hat{F}_{\alpha}^i \circ \Phi \right) \partial_i \left(\left(\hat{F}_{\gamma}^k \circ \Phi \right) \partial_k V_0^{\beta} \right) \right] \text{VOL} \otimes \partial_{\beta}. \end{aligned}$$

A.6 The Damping Tensor in iMHD

We want to prove that the matrix representation of

$$\mathcal{R}_{\alpha\beta} = \check{J} \kappa^{\gamma\delta} B_{\alpha\gamma} B_{\beta\delta} \sqrt{\det [G_{ij}]} = \mathcal{R}_{\beta\alpha},$$

which is denoted by $[\mathcal{R}_{\alpha\beta}]$, is positive semidefinite provided that the matrix $[\kappa^{\gamma\delta}]$ is symmetric and positive definite. Since $\check{J}\sqrt{\det[G_{ij}]}$ is always positive we intend to investigate the definiteness of the remaining expression $[\kappa^{\gamma\delta}B_{\alpha\gamma}B_{\beta\delta}]$.

First of all, we introduce the matrix representations

$$[B_{\alpha\beta}] = \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix} = [b_1 \ b_2 \ b_3] , \quad [\kappa^{\alpha\beta}] = \begin{bmatrix} \kappa^{11} & \kappa^{12} & \kappa^{13} \\ \kappa^{21} & \kappa^{22} & \kappa^{23} \\ \kappa^{31} & \kappa^{32} & \kappa^{33} \end{bmatrix} ,$$

where b_α denotes the α^{th} -column of $[B_{\alpha\beta}]$. Therefore, we obtain

$$[\kappa^{\gamma\delta}B_{\alpha\gamma}B_{\beta\delta}] = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \end{bmatrix} [\kappa^{\alpha\beta}] [b_1 \ b_2 \ b_3] = \begin{bmatrix} b_1^T [\kappa^{\alpha\beta}] b_1 & b_1^T [\kappa^{\alpha\beta}] b_2 & b_1^T [\kappa^{\alpha\beta}] b_3 \\ b_2^T [\kappa^{\alpha\beta}] b_1 & b_2^T [\kappa^{\alpha\beta}] b_2 & b_2^T [\kappa^{\alpha\beta}] b_3 \\ b_3^T [\kappa^{\alpha\beta}] b_1 & b_3^T [\kappa^{\alpha\beta}] b_2 & b_3^T [\kappa^{\alpha\beta}] b_3 \end{bmatrix} .$$

According to [Swamy, 1973], a matrix is positive semidefinite if and only if all principle minors of the matrix are non-negative. By applying this result we have to investigate

$$b_1^T [\kappa^{\alpha\beta}] b_1 \geq 0, \quad b_2^T [\kappa^{\alpha\beta}] b_2 \geq 0, \quad b_3^T [\kappa^{\alpha\beta}] b_3 \geq 0, \quad \det [\kappa^{\gamma\delta}B_{\alpha\gamma}B_{\beta\delta}] \geq 0$$

as well as

$$\begin{aligned} \det \begin{bmatrix} b_1^T [\kappa^{\alpha\beta}] b_1 & b_1^T [\kappa^{\alpha\beta}] b_2 \\ b_2^T [\kappa^{\alpha\beta}] b_1 & b_2^T [\kappa^{\alpha\beta}] b_2 \end{bmatrix} &\geq 0, \\ \det \begin{bmatrix} b_1^T [\kappa^{\alpha\beta}] b_1 & b_1^T [\kappa^{\alpha\beta}] b_3 \\ b_3^T [\kappa^{\alpha\beta}] b_1 & b_3^T [\kappa^{\alpha\beta}] b_3 \end{bmatrix} &\geq 0, \\ \det \begin{bmatrix} b_2^T [\kappa^{\alpha\beta}] b_2 & b_2^T [\kappa^{\alpha\beta}] b_3 \\ b_3^T [\kappa^{\alpha\beta}] b_2 & b_3^T [\kappa^{\alpha\beta}] b_3 \end{bmatrix} &\geq 0. \end{aligned}$$

In particular, from the matrix representation $[B_{\alpha\beta}]$ we deduce that $\det[B_{\alpha\beta}] = 0$ is met resulting in

$$\det [\kappa^{\gamma\delta}B_{\alpha\gamma}B_{\beta\delta}] = 0$$

and, therefore, the fourth condition from above is already fulfilled. With regard to the remaining conditions we make the assumption

$$[\kappa^{\alpha\beta}] = [\kappa^{\beta\alpha}] > 0. \quad (\text{A.6})$$

In order to show that this assumption guarantees the positive semidefiniteness of $[\mathcal{R}_{\alpha\beta}]$ it is clear that the first three conditions are already fulfilled since these are pure quadratic forms. In consideration of the analysis of the last three conditions we rewrite these expressions in the form

$$\begin{aligned} \begin{bmatrix} b_1^T [\kappa^{\alpha\beta}] b_1 & b_1^T [\kappa^{\alpha\beta}] b_2 \\ b_2^T [\kappa^{\alpha\beta}] b_1 & b_2^T [\kappa^{\alpha\beta}] b_2 \end{bmatrix} &= \begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix} [\kappa^{\alpha\beta}] [b_1 \ b_2] , \\ \begin{bmatrix} b_1^T [\kappa^{\alpha\beta}] b_1 & b_1^T [\kappa^{\alpha\beta}] b_3 \\ b_3^T [\kappa^{\alpha\beta}] b_1 & b_3^T [\kappa^{\alpha\beta}] b_3 \end{bmatrix} &= \begin{bmatrix} b_1^T \\ b_3^T \end{bmatrix} [\kappa^{\alpha\beta}] [b_1 \ b_3] , \\ \begin{bmatrix} b_2^T [\kappa^{\alpha\beta}] b_2 & b_2^T [\kappa^{\alpha\beta}] b_3 \\ b_3^T [\kappa^{\alpha\beta}] b_2 & b_3^T [\kappa^{\alpha\beta}] b_3 \end{bmatrix} &= \begin{bmatrix} b_2^T \\ b_3^T \end{bmatrix} [\kappa^{\alpha\beta}] [b_2 \ b_3] . \end{aligned}$$

Since the matrices

$$\begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix}, \quad \begin{bmatrix} b_1^T \\ b_3^T \end{bmatrix}, \quad \begin{bmatrix} b_2^T \\ b_3^T \end{bmatrix}$$

possess linearly independent rows it can be deduced that the relations

$$\begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix} [\kappa^{\alpha\beta}] [b_1 \ b_2] > 0, \quad \begin{bmatrix} b_1^T \\ b_3^T \end{bmatrix} [\kappa^{\alpha\beta}] [b_1 \ b_3] > 0, \quad \begin{bmatrix} b_2^T \\ b_3^T \end{bmatrix} [\kappa^{\alpha\beta}] [b_2 \ b_3] > 0$$

are met due to (A.6) and, therefore, the former conditions are clearly fulfilled.

A.7 Control of finite dimensional PCHD Systems

We intend to prove the conditions of Proposition 5.2, where the forthcoming computations are mainly based on [van der Schaft, 2000]. Since we consider \bar{n} structural invariants of the specific form (5.10) the set of the PDEs (5.9) takes the form of

$$\begin{bmatrix} \partial_\alpha C^\lambda & \delta_{\alpha c}^\lambda \end{bmatrix} \begin{bmatrix} J^{\alpha\beta} - R^{\alpha\beta} & -G_\xi^\alpha K^{\eta\xi} G_{c,\eta}^{\beta c} \\ G_{c,\xi}^{\alpha c} K^{\xi\eta} G_\eta^\beta & J_c^{\lambda\beta c} - R_c^{\alpha\beta c} \end{bmatrix} = 0$$

from which we obtain

$$\partial_\alpha C^\lambda (J^{\alpha\beta} - R^{\alpha\beta}) + G_{c,\xi}^\lambda K^{\xi\eta} G_\eta^\beta = 0 \quad (\text{A.7})$$

as well as

$$- (\partial_\alpha C^\lambda) G_\xi^\alpha K^{\eta\xi} G_{c,\eta}^{\beta c} + J_c^{\lambda\beta c} - R_c^{\lambda\beta c} = 0. \quad (\text{A.8})$$

Multiplication of (A.7) with $\partial_\beta C^\rho$, $\rho = 1, \dots, \bar{n}$, leads to

$$\partial_\alpha C^\lambda (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta C^\rho = -G_{c,\xi}^\lambda K^{\xi\eta} G_\eta^\beta (\partial_\beta C^\rho) = -G_{c,\eta}^\lambda K^{\eta\xi} G_\xi^\beta (\partial_\beta C^\rho) = -(\partial_\alpha C^\rho) G_\xi^\alpha K^{\eta\xi} G_{c,\eta}^\lambda.$$

If we compare this result with (A.8) it can be deduced that

$$- (\partial_\alpha C^\rho) G_\xi^\alpha K^{\eta\xi} G_{c,\eta}^\lambda = -J_c^{\rho\lambda} + R_c^{\rho\lambda} = J_c^{\lambda\rho} + R_c^{\lambda\rho},$$

is met which – together with the former result – leads to

$$\partial_\alpha C^\lambda (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta C^\rho = J_c^{\lambda\rho} + R_c^{\lambda\rho}.$$

In consideration of the skew-symmetric and the symmetric parts we obtain

$$\begin{aligned} (\partial_\alpha C^\lambda) J^{\alpha\beta} (\partial_\beta C^\rho) &= J_c^{\lambda\rho}, \\ - (\partial_\alpha C^\lambda) R^{\alpha\beta} (\partial_\beta C^\rho) &= R_c^{\lambda\rho}, \end{aligned} \quad (\text{A.9})$$

and due to the fact that R and R_c are (by definition) positive semidefinite maps – note that $(\partial_\alpha C^\lambda) R^{\alpha\beta} (\partial_\beta C^\rho) \geq 0$ is met – it can be deduced that the conditions

$$R_c^{\lambda\rho} = 0, \quad (\text{A.10})$$

$$R^{\alpha\beta} (\partial_\beta C^\rho) = 0 \quad (\text{A.11})$$

must be fulfilled. Therefore, it is easily seen that (A.9), (A.10) as well as (A.11) correspond to the first three conditions of Proposition 5.2, namely (5.12) – (5.14). Furthermore, (A.11) equals

$$R^{\alpha\beta} (\partial_\beta C^\rho) = (\partial_\beta C^\rho) R^{\beta\alpha} = (\partial_\alpha C^\rho) R^{\alpha\beta} = 0 = (\partial_\alpha C^\lambda) R^{\alpha\beta}$$

and, thus, (A.7) simplifies to

$$\partial_\alpha C^\lambda (J^{\alpha\beta} - R^{\alpha\beta}) + G_{c,\xi}^\lambda K^{\xi\eta} G_\eta^\beta = (\partial_\alpha C^\lambda) J^{\alpha\beta} + G_{c,\xi}^\lambda K^{\xi\eta} G_\eta^\beta = 0$$

from which (5.15) – which reads as

$$(\partial_\alpha C^\lambda) J^{\alpha\beta} = -G_{c,\xi}^\lambda K^{\xi\eta} G_\eta^\beta \quad (\text{A.12})$$

– follows directly.

A.8 The Equivalent Norm on \mathcal{Z}

We intend to show that the function space \mathcal{Z} given by

$$\mathcal{Z} = \left\{ z = (\bar{x}_c^3, \bar{x}_c^4, \bar{w}, \bar{\psi}, \bar{p}_w, \bar{p}_\psi) \mid \bar{x}_c^3, \bar{x}_c^4 \in \mathbb{R}, \bar{w}, \bar{\psi} \in H^1(0, L), \bar{p}_w, \bar{p}_\psi \in L^2(0, L) \right\}$$

equipped with the natural inner product

$$\langle z, \check{z} \rangle_n = \bar{x}_c^3 \check{x}_c^3 + \bar{x}_c^4 \check{x}_c^4 + \int_0^L (j^1 \bar{\Phi})^* \left(\bar{\psi} \check{\psi} + \bar{\psi}_1 \check{\psi}_1 + \bar{w} \check{w} + \bar{w}_1 \check{w}_1 + \bar{p}_w \check{p}_w + \bar{p}_\psi \check{p}_\psi \right) dX$$

and the natural norm

$$\|z\|_n^2 = \langle z, z \rangle_n = (\bar{x}_c^3)^2 + (\bar{x}_c^4)^2 + \int_0^L (j^1 \bar{\Phi})^* \left((\bar{\psi})^2 + (\bar{\psi}_1)^2 + (\bar{w})^2 + (\bar{w}_1)^2 + (\bar{p}_w)^2 + (\bar{p}_\psi)^2 \right) dX$$

defines a proper Hilbert space with the equivalent inner product $\langle z, \check{z} \rangle_{\mathcal{Z}}$ of (5.51) and the induced equivalent norm

$$\begin{aligned} \|z\|_{\mathcal{Z}}^2 = \langle z, z \rangle_{\mathcal{Z}} &= M_{\mu\nu} \bar{x}_c^\mu \bar{x}_c^\nu + c_1 \iota_L^* (\bar{w} \circ \bar{\Phi})^2 + c_2 \iota_L^* (\bar{\psi} \circ \bar{\Phi})^2 \\ &+ \int_0^L (j^1 \bar{\Phi})^* \left(\frac{1}{\rho} (\bar{p}_w)^2 + \frac{1}{I_m} (\bar{p}_\psi)^2 + EI_a (\bar{\psi}_1)^2 + kGA (\bar{w}_1 - \bar{\psi})^2 \right) dX. \end{aligned}$$

In fact, the equivalence of the norms must be shown according to

$$k_1 \|z\|_n^2 \leq \|z\|_{\mathcal{Z}}^2 \leq k_2 \|z\|_n^2, \quad k_1, k_2 > 0, \quad (\text{A.13})$$

see, e.g., [Zeidler, 1990] as well as [Thull, 2010] and references therein. First of all, we intend to investigate the lower bound. Before we proceed we analyse the term (the pull-backs are omitted in order to enhance the readability)

$$\int_0^L (\bar{w}_1)^2 = \int_0^L (\bar{w}_1 - \bar{\psi})^2 dX + 2 \int_0^L (\bar{w}_1 \bar{\psi}) dX - \int_0^L (\bar{\psi})^2 dX.$$

In consideration of

$$0 \leq \int_0^L \left(\frac{\bar{\psi}}{\sqrt{\varepsilon_1}} \pm \bar{w}_1 \sqrt{\varepsilon_1} \right)^2 dX = \frac{1}{\varepsilon_1} \int_0^L (\bar{\psi})^2 dX \pm 2 \int_0^L (\bar{w}_1 \bar{\psi}) dX + \varepsilon_1 \int_0^L (\bar{w}_1)^2 dX$$

with $\varepsilon_1 > 0$ leading to

$$\mp 2 \int_0^L (\bar{w}_1 \bar{\psi}) dX \leq \frac{1}{\varepsilon_1} \int_0^L (\bar{\psi})^2 dX + \varepsilon_1 \int_0^L (\bar{w}_1)^2 dX \quad (\text{A.14})$$

we obtain the relation

$$(1 - \varepsilon_1) \int_0^L (\bar{w}_1)^2 dX \leq \int_0^L (\bar{w}_1 - \bar{\psi})^2 dX + \left(\frac{1}{\varepsilon_1} - 1 \right) \int_0^L (\bar{\psi})^2 dX. \quad (\text{A.15})$$

Before we proceed we investigate the relation

$$\int_0^L (\bar{\psi})^2 dX = \iota^* \left(X^1 (\bar{\psi})^2 \right) - 2 \int_0^L X^1 \bar{\psi} \bar{\psi}_1 dX = L \iota_L^* (\bar{\psi})^2 - 2 \int_0^L X \bar{\psi} \bar{\psi}_1 dX$$

which enables us to conclude

$$\int_0^L (\bar{\psi})^2 dX \leq L \iota_L^* (\bar{\psi})^2 + \frac{1}{2} \int_0^L (\bar{\psi})^2 dX + 2 \int_0^L (X^1 \bar{\psi}_1)^2 dX \quad (\text{A.16})$$

– by considering a similar result as (A.14) for $\varepsilon_1 = 2$ – resulting in

$$\int_0^L (\bar{\psi})^2 dX \leq 2L \iota_L^* (\bar{\psi})^2 + 4 \int_0^L (X^1 \bar{\psi}_1)^2 dX \leq 2L \iota_L^* (\bar{\psi})^2 + 4L^2 \int_0^L (\bar{\psi}_1)^2 dX$$

It is worth noting that this inequality corresponds to the well-known Poincaré inequality, see, e.g., [Vazquez and Krstic, 2008] and references therein. Therefore, from (A.15) we deduce

$$(1 - \varepsilon_1) \int_0^L (\bar{w}_1)^2 dX \leq \int_0^L (\bar{w}_1 - \bar{\psi})^2 dX + 2L \left(\frac{1}{\varepsilon_1} - 1 \right) \iota_L^* (\bar{\psi})^2 + 4L^2 \left(\frac{1}{\varepsilon_1} - 1 \right) \int_0^L (\bar{\psi}_1)^2 dX$$

which enables us to finally conclude

$$\begin{aligned} \|z\|_{\mathcal{Z}}^2 &\geq M_{\mu\nu} \bar{x}_c^\mu \bar{x}_c^\nu + c_1 \iota_L^* (\bar{w})^2 + \left(c_2 - 2LkGA \left(\frac{1}{\varepsilon_1} - 1 \right) \right) \iota_L^* (\bar{\psi})^2 \\ &\quad + \int_0^L \left(\left(EI_a - 4L^2kGA \left(\frac{1}{\varepsilon_1} - 1 \right) \right) (\bar{\psi}_1)^2 + kGA(1 - \varepsilon_1) (\bar{w}_1)^2 \right) dX \\ &\quad + \int_0^L \left(\frac{1}{\rho} (\bar{p}_w)^2 + \frac{1}{I_m} (\bar{p}_\psi)^2 \right) dX, \end{aligned}$$

where the inequalities

$$\begin{aligned} c_2 - 2LkGA \left(\frac{1}{\varepsilon_1} - 1 \right) &> 0 \\ EI_a - 4L^2kGA \left(\frac{1}{\varepsilon_1} - 1 \right) &> 0 \\ 1 - \varepsilon_1 &> 0 \end{aligned}$$

must be satisfied. These inequalities can be subsumed as

$$\max \left\{ \frac{1}{\frac{c_2}{2LkGA} + 1}, \frac{1}{\frac{EI_a}{4L^2kGA} + 1} \right\} < \varepsilon_1 < 1, \quad (\text{A.17})$$

where it is always guaranteed that such a constant exists due to

$$\frac{c_2}{2LkGA} > 0, \quad \frac{EI_a}{4L^2kGA} > 0.$$

Furthermore, as in [Thull, 2010] it is also possible to find non-negative constants \bar{c}_1, \bar{c}_2 such that the relations

$$\begin{aligned} c_1 \iota_L^* (\bar{w})^2 &\geq \bar{c}_1 \int_0^L (\bar{w})^2 dX, \\ \left(c_2 - 2LkGA \left(\frac{1}{\varepsilon_1} - 1 \right) \right) \iota_L^* (\bar{\psi})^2 &\geq \bar{c}_2 \int_0^L (\bar{\psi})^2 dX \end{aligned}$$

are (pointwise) met and, moreover, we are able to state

$$\lambda_{\min}([M_{\mu\nu}]) \left((\bar{x}_c^3)^2 + (\bar{x}_c^4)^2 \right) \leq M_{\mu\nu} \bar{x}_c^\mu \bar{x}_c^\nu \leq \lambda_{\max}([M_{\mu\nu}]) \left((\bar{x}_c^3)^2 + (\bar{x}_c^4)^2 \right), \quad (\text{A.18})$$

where $\lambda_{\min}([M_{\mu\nu}])$ as well as $\lambda_{\max}([M_{\mu\nu}])$ denote the smallest and largest positive real eigenvalue of the symmetric and positive definite matrix $[M_{\mu\nu}]$. Therefore, it is guaranteed that a positive constant k_1 with respect to (A.17) exists such that

$$\|z\|_{\mathcal{Z}}^2 \geq k_1 \|z\|_n^2 \quad (\text{A.19})$$

is (pointwise) met.

Next, we intend to investigate the upper bound of (A.13). By considering (A.14) with $\varepsilon_2 > 0$ as well as (A.18) we are able to state

$$\begin{aligned} \|z\|_{\mathcal{Z}}^2 &\leq \lambda_{\max}([M_{\mu\nu}]) \left((\bar{x}_c^3)^2 + (\bar{x}_c^4)^2 \right) + c_1 \iota_L^* (\bar{w})^2 + c_2 \iota_L^* (\bar{\psi})^2 \\ &\quad + \int_0^L \left(EI_a (\bar{\psi}_1)^2 + kGA (1 + \varepsilon_2) (\bar{w}_1)^2 + kGA \left(1 + \frac{1}{\varepsilon_2} \right) (\bar{\psi})^2 \right) dX \\ &\quad + \int_0^L \left(\frac{1}{\rho} (\bar{p}_w)^2 + \frac{1}{I_m} (\bar{p}_\psi)^2 \right) dX. \end{aligned}$$

Furthermore, it is also possible to find positive constants \tilde{c}_1, \tilde{c}_2 such that the relations

$$c_1 \iota_L^* (\bar{w})^2 \leq \tilde{c}_1 \int_0^L ((\bar{w})^2 + (\bar{w}_1)^2) dX, \quad c_2 \iota_L^* (\bar{\psi})^2 \leq \tilde{c}_2 \int_0^L ((\bar{\psi})^2 + (\bar{\psi}_1)^2) dX$$

are met by applying the Sobolev embedding theorem, see [Zeidler, 1990], for instance. Therefore, it is guaranteed that a positive constant k_2 for (arbitrary) $\varepsilon_2 > 0$ exists such that

$$\|z\|_{\mathcal{Z}}^2 \leq k_2 \|z\|_n^2 \quad (\text{A.20})$$

is (pointwise) met.

Finally, from the relations (A.19) as well as (A.20) it can be concluded that the norms $\|z\|_{\mathcal{Z}}$ and $\|z\|_n$ are equivalent and, thus, the function space \mathcal{Z} defines a proper Hilbert space with respect to the inner product $\langle z, \check{z} \rangle_{\mathcal{Z}}$ of (5.51).

A.9 The Existence of the Inverse Operator \mathcal{A}^{-1}

The existence of the inverse operator \mathcal{A}^{-1} can be shown by solving the equations $\tilde{z} = \mathcal{A}z$ for (arbitrary values of) $\tilde{z} = (\tilde{x}_c^3, \tilde{x}_c^4, \tilde{w}, \tilde{\psi}, \tilde{p}_w, \tilde{p}_\psi) \in \mathcal{Z}$. To enhance the readability we consider sections instead of the coordinate representations and for the restriction of the relevant terms to the boundary we suppress the inclusion mappings notation and directly plug in the boundary points. In fact, we consider the set of equations

$$\begin{bmatrix} \tilde{x}_c^3 \\ \tilde{x}_c^4 \\ \tilde{w} \\ \tilde{\psi} \\ \tilde{p}_w \\ \tilde{p}_\psi \end{bmatrix} = \begin{bmatrix} -R_c^{33}(M_{33}\tilde{x}_c^3 + M_{34}\tilde{x}_c^4) + (J_c^{34} - R_c^{34})(M_{34}\tilde{x}_c^3 + M_{44}\tilde{x}_c^4) + G_{c,1}^3 \frac{1}{\rho} \tilde{p}_w(L) + G_{c,2}^3 \frac{1}{I_m} \tilde{p}_\psi(L) \\ (-J_c^{34} - R_c^{34})(M_{33}\tilde{x}_c^3 + M_{34}\tilde{x}_c^4) - R_c^{44}(M_{34}\tilde{x}_c^3 + M_{44}\tilde{x}_c^4) + G_{c,1}^4 \frac{1}{\rho} \tilde{p}_w(L) + G_{c,2}^4 \frac{1}{I_m} \tilde{p}_\psi(L) \\ \frac{1}{\rho} \tilde{p}_w(X^1) \\ \frac{1}{I_m} \tilde{p}_\psi(X^1) \\ kAG(\partial_{11}\tilde{w}(X^1) - \partial_1\tilde{\psi}(X^1)) \\ EI_a\partial_{11}\tilde{\psi}(X^1) + kAG(\partial_1\tilde{w}(X^1) - \tilde{\psi}(X^1)) \end{bmatrix}$$

with respect to $\partial_{11} = \partial_1 \circ \partial_1$. From the third and fourth equation we conclude

$$\tilde{p}_w(X^1) = \rho\tilde{w}(X^1), \quad \tilde{p}_\psi(X^1) = I_m\tilde{\psi}(X^1). \quad (\text{A.21})$$

Therefore, from the first and the second equation we directly obtain

$$\begin{bmatrix} \tilde{x}_c^3 \\ \tilde{x}_c^4 \end{bmatrix} = \left(\begin{bmatrix} -R_c^{33} & J_c^{34} - R_c^{34} \\ -J_c^{34} - R_c^{34} & -R_c^{44} \end{bmatrix} \begin{bmatrix} M_{33} & M_{34} \\ M_{34} & M_{44} \end{bmatrix} \right)^{-1} \begin{bmatrix} \tilde{x}_c^3 - G_{c,1}^3 \tilde{w}(L) - G_{c,2}^3 \tilde{\psi}(L) \\ \tilde{x}_c^4 - G_{c,1}^4 \tilde{w}(L) - G_{c,2}^4 \tilde{\psi}(L) \end{bmatrix}, \quad (\text{A.22})$$

provided that – since $[M_{\mu\nu}] > 0$ – the matrix $[J_c^{\mu\nu} - R_c^{\mu\nu}]$ with $\mu, \nu = 3, 4$ is invertible.

From now on, the forthcoming part is mainly based on [Zhang, 2007], where a Timoshenko beam with a free end and a standard PD control law at the actuated boundary is considered for stabilising the zero equilibrium. Integration of the fifth equation from 0 to X^1 yields

$$kAG(\partial_1\tilde{w}(X^1) - \tilde{\psi}(X^1)) = \underbrace{kAG(\partial_1\tilde{w}(0) - \tilde{\psi}(0))}_0 + \int_0^{X^1} \tilde{p}_w(Z_1) dZ_1 \quad (\text{A.23})$$

in consideration of the free end condition at $X^1 = 0$ and, consequently, the sixth equation can be written as

$$EI_a\partial_{11}\tilde{\psi}(X^1) = - \int_0^{X^1} \tilde{p}_w(Z_1) dZ_1 + \tilde{p}_\psi(X^1).$$

Integration of this expression from 0 to X^1 results in

$$EI_a\partial_1\tilde{\psi}(X^1) = \underbrace{EI_a\partial_1\tilde{\psi}(0)}_0 - \int_0^{X^1} \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2 + \int_0^{X^1} \tilde{p}_\psi(Z_1) dZ_1 \quad (\text{A.24})$$

in consideration of the free end at $X^1 = 0$ or, equivalently,

$$\partial_1\tilde{\psi}(X^1) = - \frac{1}{EI_a} \int_0^{X^1} \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2 + \frac{1}{EI_a} \int_0^{X^1} \tilde{p}_\psi(Z_1) dZ_1.$$

Integration of the last expression from X^1 to L yields

$$\bar{\psi}(X^1) = \bar{\psi}(L) + \frac{1}{EI_a} \int_{X^1}^L \int_0^{Z_3} \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2 dZ_3 - \frac{1}{EI_a} \int_{X^1}^L \int_0^{Z_2} \tilde{p}_\psi(Z_1) dZ_1 dZ_2. \quad (\text{A.25})$$

Before we proceed we evaluate (A.24) at $X^1 = L$ which reads as

$$\begin{aligned} EI_a \partial_1 \bar{\psi}(L) &= - \int_0^L \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2 + \int_0^L \tilde{p}_\psi(Z_1) dZ_1 \\ &= -c_2 \bar{\psi}(L) - G_{c,2}^3 (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) - G_{c,2}^4 (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) \end{aligned}$$

from which we obtain

$$\begin{aligned} \bar{\psi}(L) &= -\frac{G_{c,2}^3}{c_2} (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) - \frac{G_{c,2}^4}{c_2} (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) \\ &\quad + \frac{1}{c_2} \int_0^L \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2 - \frac{1}{c_2} \int_0^L \tilde{p}_\psi(Z_1) dZ_1. \end{aligned}$$

Therefore, (A.25) finally takes the form of

$$\begin{aligned} \bar{\psi}(X^1) &= -\frac{G_{c,2}^3}{c_2} (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) - \frac{G_{c,2}^4}{c_2} (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) - \frac{1}{c_2} \int_0^L \tilde{p}_\psi(Z_1) dZ_1 \\ &\quad + \frac{1}{c_2} \int_0^L \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2 - \frac{1}{EI_a} \int_{X^1}^L \int_0^{Z_2} \tilde{p}_\psi(Z_1) dZ_1 dZ_2 \\ &\quad + \frac{1}{EI_a} \int_{X^1}^L \int_0^{Z_3} \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2 dZ_3. \quad (\text{A.26}) \end{aligned}$$

Furthermore, from (A.23) we conclude

$$kAG \partial_1 \bar{w}(X^1) = kAG \bar{\psi}(X^1) + \int_0^{X^1} \tilde{p}_w(Z_1) dZ_1$$

and integration of this expression from X^1 to L yields

$$\bar{w}(X^1) = \bar{w}(L) - \int_{X^1}^L \bar{\psi}(Z_1) dZ_1 - \frac{1}{kAG} \int_{X^1}^L \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2. \quad (\text{A.27})$$

Evaluation of (A.23) at $X^1 = L$ results in

$$\begin{aligned} kAG (\partial_1 \bar{w}(L) - \bar{\psi}(L)) &= \int_0^L \tilde{p}_w(Z_1) dZ_1 \\ &= -c_1 \bar{w}(L) - G_{c,1}^3 (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) - G_{c,1}^4 (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) \end{aligned}$$

which leads to

$$\bar{w}(L) = -\frac{G_{c,1}^3}{c_1} (M_{33} \bar{x}_c^3 + M_{34} \bar{x}_c^4) - \frac{G_{c,1}^4}{c_1} (M_{34} \bar{x}_c^3 + M_{44} \bar{x}_c^4) - \frac{1}{c_1} \int_0^L \tilde{p}_w(Z_1) dZ_1.$$

Substituting this result in (A.27) enables us to finally obtain

$$\begin{aligned} \bar{w}(X^1) = & -\frac{G_{c,1}^3}{c_1} (M_{33}\bar{x}_c^3 + M_{34}\bar{x}_c^4) - \frac{G_{c,1}^4}{c_1} (M_{34}\bar{x}_c^3 + M_{44}\bar{x}_c^4) - \int_{X^1}^L \bar{\psi}(Z_1) dZ_1 \\ & - \frac{1}{c_1} \int_0^L \tilde{p}_w(Z_1) dZ_1 - \frac{1}{kAG} \int_{X^1}^L \int_0^{Z_2} \tilde{p}_w(Z_1) dZ_1 dZ_2 \quad (\text{A.28}) \end{aligned}$$

which is clearly a function of the \tilde{z} -components by plugging in the relations (A.22) and (A.26). Therefore, the relations (A.21), (A.22), (A.26) as well as (A.28) determine the components of z for a given \tilde{z} and, thus, we have shown the existence of the inverse operator \mathcal{A}^{-1} . Furthermore, it may be deduced that for a bounded \tilde{z} the components of z are also bounded.

Bibliography

- R. Aris. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Publications, New York, 1989.
- A. Bennett. *Lagrangian Fluid Dynamics*. Cambridge University Press, 2006.
- W. M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press Inc., Orlando, 1986.
- W. L. Burke. *Applied Differential Geometry*. Cambridge University Press, 1994.
- A. J. Chorin and J. E. Marsden. *A Mathematical Introduction to Fluid Mechanics*. Springer-Verlag, New York, 1990.
- R. F. Curtain and H. J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, New York, 1995.
- M. Dalsmo and A. J. van der Schaft. On representation and integrability of mathematical structures in energy-conserving physical systems. *SIAM Journal on Control and Optimization*, 37:54–91, 1999.
- P. A. Davidson. *An Introduction to Magnetohydrodynamics*. Cambridge University Press, 2001.
- H. Ennsbrunner. *Infinite-dimensional Euler-Lagrange and Port Hamiltonian Systems*. PhD thesis, Johannes Kepler University, Linz, Austria, 2006.
- H. Ennsbrunner and K. Schlacher. On the Geometrical Representation and Interconnection of Infinite Dimensional Port Controlled Hamiltonian Systems. In *Proceedings, 44th IEEE Conference on Decision and Control and the European Control Conference, Seville, Spain, 2005*.
- A. C. Eringen and G. A. Maugin. *Electrodynamics of Continua II: Fluids and Complex Media*. Springer-Verlag, New York, 1990.
- T. Frankel. *The Geometry of Physics*. Cambridge University Press, 2nd ed., 2004.

- G. Giachetta, L. Mangiarotti, and G. Sardanashvily. *New Lagrangian and Hamiltonian Methods in Field Theory*. World Scientific, Singapore, 1997.
- F. Gómez-Estern, R. Ortega, F. R. Rubio, and J. Aracil. Stabilization of a Class of Underactuated Mechanical Systems via Total Energy Shaping. In *Proceedings, 40th IEEE Conference on Decision and Control, Orlando, Florida, USA, 2001*.
- A. Jadczyk, J. Janyska, and M. Modugno. Galilei general relativistic quantum mechanics revisited. "Geometria, Física-Matemática e outros Ensaio", *Homenagem a António Ribeiro Gomes*, pages 253–313, 1998.
- I. V. Kanatchikov. Canonical structure of classical field theory in the polymomentum phase space. *Reports on Mathematical Physics*, 41(1):49–90, 1998.
- H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd ed., 2002.
- J. U. Kim and Y. Renardy. Boundary Control of the Timoshenko Beam. *SIAM Journal on Control and Optimization*, 25(6):1417–1429, 1987.
- A. Kugi. *Non-linear Control Based on Physical Models*. Springer-Verlag, London, 2001.
- Z. Liu and S. Zheng. *Semigroups associated with dissipative systems*. Chapman & Hall/CRC, 1999.
- Z. H. Luo, B. Z. Guo, and Ö. Morgül. *Stability and Stabilization of Infinite Dimensional Systems with Applications*. Springer-Verlag, London, 1999.
- A. Macchelli and C. Melchiorri. Modeling and Control of the Timoshenko Beam: The Distributed Port Hamiltonian Approach. *SIAM Journal on Control and Optimization*, 43(2):743–767, 2004a.
- A. Macchelli and C. Melchiorri. Control by Interconnection and Energy Shaping of the Timoshenko Beam. *Mathematical and Computer Modelling of Dynamical Systems*, 10(3-4):231–251, 2004b.
- A. Macchelli and C. Melchiorri. Control by Interconnection of Mixed Port Hamiltonian Systems. *IEEE Transactions on Automatic Control*, 50(11):1839–1844, 2005.
- A. Macchelli, A. J. van der Schaft, and C. Melchiorri. Port Hamiltonian Formulation of Infinite Dimensional Systems i. Modelling. In *Proceedings, 43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas, 2004c*.
- A. Macchelli, A. J. van der Schaft, and C. Melchiorri. Port Hamiltonian Formulation of Infinite Dimensional Systems ii. Boundary Control by Interconnection. In *Proceedings, 43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas, 2004d*.
- J. E. Marsden and T. J. R. Hughes. *Mathematical Foundations of Elasticity*. Dover Publications, New York, 1994.

- J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry*. Springer-Verlag, New York, 1994.
- J. E. Marsden, S. Pekarsky, S. Shkoller, and M. West. Variational methods, multisymplectic geometry and continuum mechanics. *Journal of Geometry and Physics*, 38:253–284, 2001.
- L. Meirovitch. *Principles and Techniques of Vibrations*. Prentice Hall, 1997.
- A. N. Michel, L. Hou, and D. Liu. *Stability of Dynamical Systems - Continuous, Discontinuous and Discrete Systems*. Birkhäuser, Boston, 2007.
- Ö. Morgül. Stabilization and Disturbance Rejection for the Wave Equation. *IEEE Transactions on Automatic Control*, 43(1):89–95, 1998.
- P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag, New York, 2nd ed., 1993.
- R. Ortega, A. J. van der Schaft, I. Mareels, and B. Maschke. Putting energy back in control. *IEEE Control Systems Magazine*, 21(2):18–33, 2001.
- R. Ortega, A. J. van der Schaft, B. Maschke, and G. Escobar. Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, 38:585–596, 2002.
- H. Rodriguez, A. J. van der Schaft, and R. Ortega. On Stabilization of Nonlinear Distributed Parameter Port-Controlled Hamiltonian Systems via Energy Shaping. In *Proceedings, 40th IEEE Conference on Decision and Control, Orlando, Florida, USA*, 2001.
- D. J. Saunders. *The Geometry of Jet Bundles*. Cambridge University Press, 1989.
- K. Schlacher. *Distributed PCHD-Systems, from the Lumped to the Distributed Parameter Case*, pages 239–255. *Advances in Control Theory and Applications*, in Bonivento C., Isidori A., Marconi L., Rossi C., Serie Lecture Notes in Control and Information Sciences 353, Springer Verlag, 2007.
- K. Schlacher. Mathematical Modeling for Nonlinear Control: A Hamiltonian Approach. *Mathematics and Computers in Simulation*, 79(4):829–849, 2008.
- K. Schlacher, G. Grabmair, H. Ennsbrunner, and R. Stadlmayr. *Some Applications of Differential Geometry in Mechanics*, pages 261–281. CISM Courses and Lectures No. 444, Springer Verlag, Wien, 2004.
- M. Schöberl. *Geometry and Control of Mechanical Systems - An Eulerian, Lagrangian and Hamiltonian Approach*. PhD thesis, Johannes Kepler University, Linz, Austria, 2007.
- M. Schöberl and K. Schlacher. Covariant formulation of the governing equations of continuum mechanics in an Eulerian description. *Journal of Mathematical Physics*, 48(5): 052902–1 – 052902–15, 2007a.

- M. Schöberl and K. Schlacher. Geometric Analysis of Time Variant Hamiltonian Control Systems. In *Proceedings, IFAC Symposium on Nonlinear Control Systems, Pretoria, South Africa*, 2007b.
- M. Schöberl and K. Schlacher. First order Hamiltonian Field Theory and Mechanics. *Mathematical and Computer Modelling of Dynamical Systems*, 17(1):105–121, 2011.
- M. Schöberl, H. Ennsbrunner, and K. Schlacher. Modelling of piezoelectric structures - a Hamiltonian approach. *Mathematical and Computer Modelling of Dynamical Systems*, 14(3):179–193, 2008.
- M. Schöberl, A. Siuka, and K. Schlacher. Geometric Aspects of First Order Field Theories in Piezoelectricity and Magnetohydrodynamics. In *Proceedings, International Conference on Electromagnetics in Advanced Applications, Sydney, Australia*, 2010.
- J. C. Simo, J. E. Marsden, and P. S. Krishnaprasad. The Hamiltonian structure of nonlinear elasticity: The material and convective representations of solids, rods, and plates. *Archive for Rational Mechanics and Analysis*, 104(2):125–183, 1988.
- A. Siuka, M. Schöberl, and K. Schlacher. Hamiltonian Evolution Equations of inductionless Magnetohydrodynamics. In *Proceedings, 19th International Symposium on Mathematical Theory of Networks & Systems, Budapest, Hungary*, 2010.
- A. Siuka, M. Schöberl, K. Rieger, and K. Schlacher. Regelung verteilt-parametrischer Hamiltonscher Systeme auf Basis struktureller Invarianten. *at - Automatisierungstechnik*, accepted for publication, 2011.
- S. Stramigioli, B. Maschke, and A. J. van der Schaft. Passive Output Feedback and Port Interconnection. In *Proceedings, 4th IFAC Nonlinear Control Systems Design Symposium, Enschede, Netherlands*, 1998.
- G. W. Sutton and A. Sherman. *Engineering Magnetohydrodynamics*. Dover Publications, New York, 2006.
- K. N. Swamy. On Sylvester's Criterion for Positive-Semidefinite Matrices. *IEEE Transactions on Automatic Control*, page 306, 1973.
- D. Thull. *Tracking Control of Mechanical Distributed Parameter Systems with Applications*. Shaker Verlag, 2010.
- E. Truckenbrodt. *Fluidmechanik Band 1: Grundlagen und elementare Strömungsvorgänge dichtebeständiger Fluide*. Springer-Verlag, Berlin, 3rd ed., 1989.
- A. J. van der Schaft. *L2-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, London, 2000.
- A. J. van der Schaft and B. Maschke. Hamiltonian formulations of distributed parameter systems with boundary energy flow. *Journal of Geometry and Physics*, 42:166–194, 2002.

- R. Vazquez and M. Krstic. *Control of Turbulent and Magnetohydrodynamic Channel Flows: Boundary Stabilization and State Estimation*. Birkhäuser, Boston, 2008.
- E. Zeidler. *Nonlinear Functional Analysis and its Applications II/B*. Springer-Verlag, New York, 1990.
- C.-G. Zhang. Boundary feedback stabilization of the undamped Timoshenko beam with both ends free. *Journal of Mathematical Analysis and Applications*, 326(1):488–499, 2007.
- F. Ziegler. *Mechanics of Solids and Fluids*. Springer-Verlag, Vienna, New York, 2nd ed., 1998.

Curriculum Vitae

Personal Data

Name	Andreas Siuka
Address	Am Auring 14, 4533 Piberbach
Date of Birth	March 22nd, 1984
Place of Birth	Linz
Nationality	Austria
Email	andreas.siuka@jku.at

Education

since 11/2008	Work on Ph.-D. thesis. Supervisor: o.Univ.-Prof. Dr. Kurt Schlacher, Institute of Automatic Control and Control Systems Technology, Johannes Kepler University Linz.
10/2003 – 10/2008	Study Mechatronics at the Johannes Kepler University Linz (finished with distinction). Specialisation on Automatic Control. Diploma thesis: <i>Analysis and Control of Underactuated Mechanical Systems by Energy Shaping Methods</i> , Supervisor: o.Univ.-Prof. Dr. Kurt Schlacher.
07/2002 – 08/2003	Military Service, Militärmusik Oberösterreich, Linz.
09/1994 – 06/2002	Grammar School Stift Schlierbach, Schlierbach.
09/1990 – 17/1994	Elementary School, Neuhofen an der Krens

Professional Experience

- since 10/2009 Johannes Kepler University Linz, lecturer at the Institute of Automatic Control and Control Systems Technology.
- since 11/2008 Johannes Kepler University Linz, temporary research assistant at the Institute of Automatic Control and Control Systems Technology.

Further Education, Summer Schools and Workshops

- 05/2011 Workshop on *Nonlinear Output Regulation* held by Prof. Lorenzo Marconi, Linz, Austria.
- 07/2009 Summer School on *Algebraic Analysis and Computer Algebra - New Perspectives for Applications (AACA '09)* held by Prof. Jean-Francois Pommaret, Hagenberg, Austria.
- 12/2008 Workshop on *Identification and Model Free Control* held by Prof. Michel Fliess, Linz, Austria.

Publications

Journal Articles

- Siuka A., Schöberl M., Rieger K., Schlacher K.: *Regelung verteilt-parametrischer Hamiltonscher Systeme auf Basis struktureller Invarianten*, In: *at - Automatisierungstechnik*, accepted for publication, 2011.
- Siuka A., Schöberl M.: *Applications of energy based control methods for the inverted pendulum on a cart*, In: *Robotics and Autonomous Systems*, Vol. 57, No. 10, Elsevier, pp. 1012-1017, ISSN: 0921-8890, 2009.

Conference Papers

- Schöberl M., Siuka A., Schlacher K.: *Geometric Aspects of First Order Field Theories in Piezoelectricity and Magnetohydrodynamics*, In: *Proceedings of the International Conference on Electromagnetics in Advanced Applications (ICEAA '10)*, pp. 55–58, 2010, Sydney, Australia.
- Siuka A., Schöberl M., Schlacher K.: *Hamiltonian Evolution Equations of inductionless Magnetohydrodynamics*, In: *CD Proceedings of the 19th International Symposium on Mathematical Theory of Networks & Systems (MTNS 2010)*, pp. 1889-1896, 2010, Budapest, Hungary.

- Siuka A., Schöberl M.: *Applications of Energy based Control Methods for the Inverted Pendulum on a Cart*, In: Proceedings 5th International Conference on Computational Intelligence, Robotics and Autonomous Systems (CIRAS 2008), pp. 187-192, 2008, Linz, Austria.

Invited Talks

- Siuka A., Schöberl M., Schlacher K.: *Modellierung und Regelung verteilt-parametrischer Hamiltonscher Systeme*, Control Colloquium at the Institute of Automatic Control (Technical University Munich), October 27, 2010, Garching/Munich, Germany.
- Rieger K., Siuka A.: *An Introduction to PDE Control - The Infinite-Dimensional Backstepping Approach*, ACCM Lecture Series: Research and Education in Advanced Dynamics and Model Based Control of Structures and Machines, June 2, 2010, Linz, Austria.

Talks

- Siuka A., Schöberl M., Schlacher K.: *Zur Formulierung von Feldtheorien: Ein Torbasierter Hamiltonscher Ansatz*, GAMM Fachausschuss Dynamik und Regelungstheorie, March 19, 2011, Linz, Austria.
- Siuka A., Schöberl M., Schlacher K.: *Regelung Hamiltonscher Systeme in evolutionärer Darstellung mittels strukturellen Invarianten*, GMA Fachausschuss 1.40 (Theoretische Verfahren der Regelungstechnik), September 22, 2010, Anif/Salzburg, Austria.
- Siuka A., Schöberl M., Schlacher K.: *Hamiltonian Evolution Equations of inductionless Magnetohydrodynamics*, 19th International Symposium on Mathematical Theory of Networks & Systems (MTNS 2010), July 9, 2010, Budapest, Hungary.
- Siuka A., Schöberl M., Schlacher K.: *Hamiltonsche Formulierung der Grundgleichungen der Magnetohydrodynamik*, GAMM Fachausschuss Dynamik und Regelungstheorie, October 13, 2009, Magdeburg, Germany.
- Siuka A., Schöberl M.: *Applications of Energy based Control Methods for the Inverted Pendulum on a Cart*, International Conference on Computational Intelligence, Robotics and Autonomous Systems (CIRAS 2008), June 20, 2008, Linz, Austria.

Linz, May 2011

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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